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Teze doktorské disertační práce
k získání vědeckého titulu „doktor věd“
ve skupině věd fyzikálně-matematických

Natural operators and their applications

(Přirozené operátory a jejich aplikace)

Komise pro obhajoby doktorských disertací v oboru
„Matematické struktury“

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Resumé

Předložená disertační práce je souborem 11 původních vědeckých prací [**1**] – [**11**] a úvodu, který je psán jako přehled teorie přirozených a kalibračně přirozených bandlů a operátorů a jejich využití v diferenciální geometrii a matematické fyzice.

Invariantnost geometrických operací s poli geometrických objektů na dané varietě vzhledem k lokálním difeomorfismům či volbě lokálních souřadnic je jedním ze základních požadavků nejen moderní diferenciální geometrie na diferencovatelných varietách, ale i řady oborů teoretické fyziky, jako je např. obecná relativita či klasická teorie pole. Požadavek invariantnosti je efektivně řešen pomocí teorie přirozených bandlů a přirozených operátorů. Pojem *přirozený bandl*, který zavedel v roce 1972 A. Nijenhuis, si velmi rychle vydobyl významné místo v moderní diferenciální geometrii.

První kapitola tezí je věnována popisu hlavních vlastností přirozených bandlů a přirozených operátorů a také přínosu autora předkládané disertace v této oblasti [**1, 3**].

Hlavním nástrojem při studiu přirozených operátorů na přirozených bandlech je jejich jednoznačná reprezentace pomocí zobrazení mezi typovými fibry přirozených bandlů, které jsou ekvivariantní vzhledem k akcím diferenciální grupy jistého konečného řádu. To umožuje v celé řadě případů úplný popis invariantních operací. Ve druhé kapitole jsou jako příklady přirozených operátorů uvedeny původní výsledky autora. Je provedena úplná klasifikace přirozených operátorů typu Frölicher-Nijenhuisovy závorky projektabilních tečně hodnotových forem na fibrované varietě [**2**]. V obecné relativitě na prostoročase s absolutním časem (Galileův prostoročas) je důležitý vzájemně jednoznačný vztah mezi čas zachovávajícími konexemi na prostoročase a fázovými konexemi na prvním jetovém prodloužení prostoročasu. Tento vztah je v práci [**5**] zobecněn pro libovolnou fibrovanou varietu a je klasifikován operátor, který transformuje lineární konexi na fibrované varietě do konexe na prvním jetovém prodloužení této variety. Významným nástrojem při studiu přirozených operátorů na lineárních konexích a tenzorových polích jsou redukční věty, které říkají, že takovéto operátory je možno vyjádřit jako operátory na kovariantních derivacích daných tenzorových polí a tenzoru křivosti. Tyto redukční věty jsou v [**10**] zobecněny pro operátory s hodnotami v přirozených bandlech vyšších řádů a jako aplikace je provedena úplná klasifikace tensorových polí typu $(0, 2)$ na kotečném bandlu variety s konexí. Jako aplikace v obecné teorii relativity jsou uvedeny klasifikace symplektických a Poissonových struktur na tečném bandlu prostoročasu bez absolutního času (Einsteinův prostoročas) [**4, 6**]

a klasifikace kosymplektické struktury na fázovém prostoru Galileovského prostoročasu [**4**].

V kalibračně invariantních fyzikálních teoriích se mimo invariantnosti vzhledem k lokálním difeomorfismům vyžaduje také invariantnost vzhledem ke změně kalibrace. První geometrickou interpretací kalibračně přirozených operací je práce R. Utiyamy z roku 1956. Geometricky se kalibračně invariantní teorie dají popsat pomocí *kalibračně přirozených bandlů* a přirozených operátorů na kalibračně přirozených bandlech, které zavedl v roce 1981 D. Eck. Ve třetí kapitole je uveden přehled vlastností kalibračně přirozených bandlů a přirozených operátorů a jejich infinitesimální vlastnosti [**3**].

Jako aplikace v kalibračně invariantní teorii pole uvádíme klasifikaci přirozených kvantových Lagrangianů a přirozených Schrödingerových operátorů na kvantovém bandlu nad Galileovským prostoročasem [**7**]. Utiyamovy výsledky pro kalibrační grupu $Gl(n,\mathbb{R})$ jsou zobecněny jako redukční věty pro obecnou lineární konexi na vektorovém bandlu [**8**], včetně operátorů s hodnotami v bandlech vyšších řadů [**9**]. Konečně v [**11**] je zobecněna Utiyamova věta pro libovolný řád pro libovolnou kalibrační grupu G.

Introduction

The term "geometric invariant" has been used in differential geometry since the end of the 19th century. In the 1930's Schouten and his collaborators, [81], used the notion of "geometric object". A modern functorial approach to the theory of geometrical objects and invariant operations with geometric objects was introduced by Nijenhuis, [73], in the 1950's. Starting from the famous paper by Nijenhuis, [76], geometrical objects and invariant operations with geometrical objects have been very intensively studied by using the concepts of natural bundles and natural differential operators. Nijenhuis defined natural bundles as lifting functors on the category $\mathcal{M}_m$ of m-dimensional manifolds and their local embeddings. Lifting functors are supposed to satisfy three conditions: prolongation, localization and regularity (continuity).

The following main four types of problems have been studied in the last 34 years:

1. finiteness of order of natural bundles and operators;
2. extension of lift functors on further categories and study of special types of functors;
3. regularity conditions;
4. properties and classifications of natural differential operators.

1. Palais and Terng first proved, [77], that the order k of a natural bundle is finite $k < 2^{n+1}$ where n is the dimension of the standard fiber. Later Epstein and Thurston, [44], gave much better bound. They proved $k \leq 2n + 1$ and that this bound is sharp for $m = 1$. Finally Zajtz, [92], proved $k \leq \{\frac{n}{n-1}; \frac{n}{m} + 1\}$. Krupka, [63], and Terng, [85], have proved independently that a k-order natural bundle is a bundle associated with the frame bundle of order k.

2. Kolář, [57], generalized lift functors on the category $\mathcal{M}$ of all differentiable manifolds and their smooth mappings. Such functors are called prolongation functors. Some geometric properties of prolongation functors were studied in [**1**][1]. Mikulski, [70], has shown that a prolongation functor with infinite order exists. Later various prolongation functors on subcategories of $\mathcal{M}$ were studied. The special attention was devoted to product preserving functors (Weil functors) which were studied by Eck, [43], Kainz and Michor, [55], and Luciano, [67]. Constructions used on Weil functors can be generalized also for infinite dimensional functional bundles, [31, 34].

3. In definitions of lift and prolongation functors there is the regularity condition saying that a smoothly parameterized family of diffeomorphisms

[1] The references marked by bold numbers refer to the papers of the author which are included to the thesis.

is prolonged into a smoothly parameterized family of diffeomorphisms. But this condition turns out to be a consequence of remaining prolongation and localization properties. This was proved by Epstein and Thurston, [44], for lifting functors and by Kolář and Slovák, [61], for prolongation functors.

4. Main problem of the theory of natural differential operators is to give a complete classification of them for concrete underlying geometric structures. Such classification is based on the one–to–one correspondence of natural operators and equivariant maps between standard fibres. To classify equivariant maps we can use several methods. Formerly the method of Lie equations was used, [16], recently we use the algebraic method described in [60]. In literature it is possible to find many examples and applications of natural operations used in geometry and physics. For wide list of references we recommend to see [16, 60, 45].

In the thesis we use the term "natural operator" on natural bundles in the sense of [16, 60, 85], see Section 1.10. The bundle structure of natural bundles given by natural prolongation functors is studied in Section 1.9 and [**1**]. Infinitesimal properties of natural bundles and natural operators are studied in Sections 1.14 and 1.15, see [**3**]. As applications of natural operators we shall classify the Frölicher-Nijenhuis bracket of projectable tangent valued forms, see Section 2.1 and [**2**], the relations between linear connections on a fibred manifold and connections on the 1st jet prolongation are studied in Section 2.2, see [**5**], higher order valued reduction theorems are studied in Section 2.5 and [**10**]. As applications in classical field theories we study natural symplectic and Poisson structures on the tangent bundle of the Einstein spacetime (a pseudo-Riemannian manifold with a Lorentzian metric) given by a metric and a linear connections, see Section 2.3 and [**4, 6**], and natural cosymplectic structures on the phase space of the Galilei spacetime given by a vertical metric and a phase connection, see Section 2.4 and [**4**].

Natural operators on natural bundles describe the invariance of geometrical or physical theories with respect to changes of local coordinates. But in physical theories another sort of invariance plays an important role, the so called "gauge invariance". Invariant gauge theory has been introduced in the book by H. Weyl, [90], in 1918 as a generalization of the Einstein's general relativity (published in 1915). Weyl considered spacetime metrics invariant not only with respect to isomorphisms of spacetime but also with respect to "gauge transformations" (the term "gauge = Eiche" was used for the first time by H. Weyl). The original invariant physical gauge theories was related with the gauge group $U(1)$ acting on wave functions and electromagnetic potentials. In early 1950's the concept of gauge invariance was generalized for any Lie group G playing the role of the gauge group. The first geometrical interpretation of gauge invariance with respect to a general gauge group can be found in the famous paper by Utiyama, [88]. The geometrical description of the gauge invariance is the following, Drechsler and Mayer, [41]. Let $\pi : P \to M$ be a principal G-bundle over a (usually spacetime) manifold M and $E \to M$ be a bundle associated with P. An automorphism of P, over M, induces a fibred automorphism of E, over M, which is said to be a change of gauge. A physical theory is said to be gauge invariant if it is invariant with respect to changes of gauge and with respect to local diffeomorphisms of M. Gauge invariant theories can be described geometrically by using the concepts of gauge-natural bundle functors and natural operators between gauge-natural bundles, see Section 3 and [42, 60].

Infinitesimal properties of gauge-natural bundles and natural operators of gauge-natural bundles are studied in Sections 3.11 and 3.12, see [**3**]. Gauge-natural theories have wide applications in gauge field theories, see [45]. As concrete applications we shall study natural quantum Lagrangians and natural Schrödinger operators on the quantum bundle over the Galilei spacetime, see Section 4.1 and [**7**]. We shall generalize the Utiyama's results for the gauge group $Gl(n, \mathbb{R})$ and prove the reduction theorems for general linear connections, see Section 4.2 and [**8**]. The higher order valued versions of reduction theorems for general linear connections are presented in Section 4.3, see [**9**]. As application of higher order valued reduction theorems we shall classify all classical connections on the total space of a vector bundle given by a general linear connection and a classical connection on a base manifold, see Section 4.3 and [30]. Finally, in Section 4.4 and [**11**], we present the higher order Utiyama's theorem for any gauge group G.

In what follows we shall use the following notations. $\mathcal{M}$ is the category of all smooth manifolds and smooth mappings, $\mathcal{M}_m$ is the category of all m-dimensional smooth manifolds and local diffeomorphisms, $\mathcal{FM}_m$ is the category of all fibred manifolds with m-dimensional bases and smooth fibred morphisms covering local diffeomorphisms of bases, $\mathcal{VB}_m$ ($\mathcal{AB}_m$) is the category of all vector (affine) bundles with m-dimensional bases and smooth linear (affine) fibred morphisms covering local diffeomorphisms of bases and, finally, $\mathcal{PB}_m(G)$ is the category of all principal G-bundles with m-dimensional bases and smooth principal fibred morphisms covering local diffeomorphisms of bases.

In what follows all manifolds and maps are supposed to be smooth.

1. Natural bundles and operators

We recall here definitions and basic properties of the theory of natural bundles and natural operations, ([**1, 3**]) and [16, 60, 76, 85]. As examples we mention functors and operators which will be used later.

1.1. Natural bundles.

We recall the original definition by A. Nijenhuis, [76],

Definition 1.1. A *natural bundle* is a quadruple $(E, \pi, \mathbf{B}, X)$ where E ("total space") and X ("base space") are C^∞-manifolds, $\pi : E \to X$ a C^∞-map ("projection map") and $\mathbf{B} : \mathbf{C}(X) \to \mathbf{C}(E)$ is a functor ("lifting functor") of the pseudogroups (categories) of local diffeomorphisms of X and E, subject to these three conditions:
(a) If U is an object of $\mathbf{C}(X)$, then $\mathbf{B}(U) = \pi^{-1}(U)$;
(b) If $V \subset U$ is open and $f : U \to X$ is a morphism of $\mathbf{C}(X)$, then

$$\mathbf{B}f|\mathbf{B}(V) = (f|V)\,;$$

(c) If $F : U \to X \times Y$ is a (smooth) family of local diffeomorphisms of X, and f_y is a member of the family determined by $y \in Y$, then $\bar{F} : \bar{U} \to E \times Y$ is a (smooth) family of diffeomorphisms of E, where

$$\bar{U} = \{(z, y) \in E \times Y | (\pi(z), y) \in U\}$$

and

$$\bar{F}(z, y) = (\mathbf{B}f_y(z), y)\,. \qquad \square$$

The last, so called *regularity or continuity condition*, says that a smoothly parametrized family of diffeomorphisms in $\mathbf{C}(X)$ is lifted to smoothly parametrized family of diffeomorphisms in $\mathbf{C}(E)$. But this condition turns out to be a consequence of (a) and (b), [44].

Recently the definition of lifting functors was reformulated as follows, [60],

Definition 1.2. A *natural lift functor* is a covariant functor F from $\mathcal{M}_m$ to $\mathcal{FM}_m$ satisfying
i) for each manifold $M \in \operatorname{Ob}\mathcal{M}_m$,

$$p_M : FM \to M$$

is a fibred manifold over M,
ii) for each embedding $f \in \operatorname{Mor}\mathcal{M}_m$, Ff is a fibred manifold morphism over f, which maps fibres diffeomorphically onto fibres. $\square$

A *natural bundle* is then a triplet (FM, p_M, M).

Later (Theorem 1.4) we shall see that $p_M : FM \to M$ is indeed a bundle.

1.2. **Natural bundle functor.** The concept of natural lift functor was generalized, [**1**] and [58, 60], to the concept of natural bundle functor.

Definition 1.3. A *natural bundle functor* on a subcategory $\mathcal{C}$ of $\mathcal{M}$ is a covariant functor F from $\mathcal{C}$ to the category $\mathcal{FM}$ satisfying

i') for each manifold $M \in \operatorname{Ob}\mathcal{C}$, $p_M : FM \to M$ is a fibred manifold over M,

ii') for each $f \in \operatorname{Mor}\mathcal{C}$, Ff is a fibred manifold map covering f such that $F\iota(U) = \iota(FU)$ for any open subset $\iota : U \hookrightarrow M$. □

A natural bundle functor on the subcategory $\mathcal{M}_m$ of $\mathcal{M}$, for a certain m, is a natural lift functor. In literature natural bundle functors are also called "prolongation functors".

1.3. **Geometrical object.** A *geometrical object* on a manifold M is now an element from FM, where F is a natural bundle functor. A section $\sigma : M \to FM$ is a *field of geometrical objects* on M.

1.4. **Order of natural bundle functors.** We say that a natural lift functor F is of finite order r if r is the smallest number such that

$$j^r_x f = j^r_x g \Rightarrow Ff|F_xM = Fg|F_xM$$

for any $(f, g : M \to \overline{M}) \in \operatorname{Mor}\mathcal{M}_m$ and any $x \in M$.

Any natural lift functor have a finite order, [60, 77, 92], while there exist natural bundle functors of an infinite order, [70].

1.5. **Differential group.** Let us denote by G^r_m the Lie group

$$G^r_m = \operatorname{inv}J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$$

of invertible r-jets (with source and target 0) of diffeomorphisms of $\mathbb{R}^m$ which preserve 0. The group multiplication is given by the jet composition. The canonical coordinates on G^r_m will be denoted by $(a^\lambda_\mu, \dots, a^\lambda_{\mu_1\dots\mu_r})$ and tilde will refer to the inverse element.

1.6. **Standard fiber.** Let F be an r-order natural lift functor and let $F_0 = F_0\mathbb{R}^m$. Because of ii) of Definition 1.2 F_0 is diffeomorphic with F_xM for any $x \in M$, $M \in \operatorname{Ob}\mathcal{M}_m$. F_0 will be called the *standard fiber* of F. Applying F on origin–preserving diffeomorphisms of $\mathbb{R}^m$ we get a left action of G^r_m on F_0 which make F_0 to be a left smooth G^r_m-manifold, [63, 85].

1.7. **Natural fibred coordinate chart.** Local coordinate charts (x^λ) on M and (y^p) on F_0 induce a fibred coordinate chart (x^λ, y^p) on FM, which is said to be the *natural fibred coordinate chart*.

1.8. **Examples. 1.** The *tangent functor* T is a natural bundle functor of order one on the category $\mathcal{M}$ with values in the category $\mathcal{VB}$. In dimension m the corresponding standard fiber is $\mathbb{R}^m$ on which $G^1_m = Gl(m,\mathbb{R})$ acts in the standard way by the matrix multiplication. The natural fibred coordinate chart on TM will be denoted by $(x^\lambda, \dot{x}^\lambda)$.

2. The *cotangent functor* T^* is a natural lift functor of order one with values in the category $\mathcal{VB}_m$. The standard fiber is $\mathbb{R}^{m*}$ with the standard action of G^1_m. The natural fibred coordinate chart on T^*M will be denoted by $(x^\lambda, \dot{x}_\lambda)$.

3. The functor $\wedge^p T^*$ of p–forms is a natural lift functor of order one with values in the category $\mathcal{VB}_m$. The standard fiber is $\wedge^p\mathbb{R}^{m*}$ on which G^1_m acts in the standard tensor way. The natural fibred coordinate chart on $\wedge^p T^*M$ will be denoted by $(x^\lambda, \omega_{\lambda_1\dots\lambda_p})$, $1 \le \lambda_1 < \dots < \lambda_p \le m$.

4. The functor of *pseudo–Riemannian metrics* pRm is a natural lift functor of order one such that pRmM are subbundles of objects of the category $\mathcal{VB}_m$. Its standard fiber $(\mathrm{pRm})_0$ is the subspace in $\odot^2\mathbb{R}^{m*}$ of non-degenerate symmetric matrices and the tensor action of G^1_m. The natural fibred coordinate chart on pRm(M) will be denoted by $(x^\lambda, g_{\lambda\mu})$, $g_{\lambda\mu} = g_{\mu\lambda}$, $\det(g_{\lambda\mu}) \ne 0$.

5. The functor of *k^r-velocities* T^r_k is a natural bundle functor of order r on the category $\mathcal{M}$. For any $M \in \mathrm{Ob}\,\mathcal{M}$, we define $T^r_k M = J^r_0(\mathbb{R}^k, M)$ and, for any $f \in \mathrm{Mor}\,\mathcal{M}$, $f : M \to \overline{M}$, we define $T^r_k f(j^r_0\alpha) = j^r_0(f \circ \alpha)$, where $j^r_0\alpha \in T^r_k M$. The standard fiber of T^r_k in dimension m is $J^r_0(\mathbb{R}^k, \mathbb{R}^m)_0$ and the action of G^r_m on the standard fiber is given by the jet composition.

6. The functor of *r-order frames* F^r is a natural lift functor of order r. For any $M \in \mathrm{Ob}\,\mathcal{M}_m$, we define $F^r M = \mathrm{inv} J^r_0(\mathbb{R}^m, M)$ and, for any $f \in \mathrm{Mor}\,\mathcal{M}_m$, $F^r f$ is defined as in Example 1.8.5. The values of the functor F^r are in the category $\mathcal{PB}_m(G^r_m)$.

7. The functor Cla of *classical (linear) connections* on a given manifold is a natural lift functor of order two with values in the category $\mathcal{AB}_m$. Its standard fiber is $\mathbb{R}^{*m} \otimes \mathbb{R}^m \otimes \mathbb{R}^{m*}$ on which G^2_m acts via the well known transformation relations of the Christoffel symbols

$$\bar{\Lambda}_\mu{}^\lambda{}_\nu = a^\lambda_\rho (\Lambda_\sigma{}^\rho{}_\tau \tilde{a}^\sigma_\mu \tilde{a}^\tau_\nu + \tilde{a}^\rho_{\mu\nu})\,.$$

The natural fibred coordinate chart on Cla M will be denoted by $(x^\lambda, \Lambda_\mu{}^\lambda{}_\nu)$.

By Cla_τ will be denoted the functor of torsion free linear connections. In natural fibred coordinates Cla_τ is characterized by $\Lambda_\mu{}^\lambda{}_\nu = \Lambda_\nu{}^\lambda{}_\mu$.

8. Let F be a natural lift functor of order r and J^s be the functor of s-jet prolongation, [79]. Then $J^s F \equiv J^s \circ F$ is a natural lift functor of order $(r+s)$. If F_0 is the standard fiber of F, then the standard fiber of $J^s F$ is

$(J^sF)_0 = T^s_n F_0$ and the action of G^{r+s}_m on $(J^sF)_0$ is obtained by the jet prolongation of the action of G^r_m on F_0.

1.9. **The bundle structure.** In the theory of natural lift functors the functor of r-order frames, defined in Example 1.8.6, plays a fundamental role. Namely, we have the following theorem, [60, 63, 85].

Theorem 1.4. *Any natural lift functor F of order r, with the standard fiber F_0, is canonically represented by*

$$FM = [F^rM, F_0], \quad Ff = [F^rf, \mathrm{id}_{F_0}],$$

where $M \in \mathrm{Ob}\,\mathcal{M}_m$, $f \in \mathrm{Mor}\,\mathcal{M}_m$, and $[F^rM, F_0] = (F^rM, F_0)/G^r_m$ is the bundle associated with F^rM. □

This theorem implies that there is the one–to–one correspondence between r–order natural lift functors and left G^r_m–manifolds.

Now we shall generalize Theorem 1.4 to prolongation functors, [**1**]. Let us define the category L^r. $\mathrm{Ob}\,L^r$ is the set of natural numbers $1, 2, 3, \dots$, $\mathrm{Mor}\,L^r(m,n) = L^r(m,n) = J^r_0(\mathbb{R}^m, \mathbb{R}^n)_0$ and the composition in L^r is given by the composition of jets. If F is an r-th order prolongation functor, we shall denote by $\mathcal{S} = \{S_1, S_2, \dots\}$, $S_i = F_0\mathbb{R}^i$. Then we have the action λ of the category L^r on $\mathcal{S}$ defined by a system of maps

$$\lambda_{m,n} : L^r(m,n) \times S_m \to S_n$$

given by

$$\lambda_{m,n}(A, s) = Ff(s)$$

for any $A = j^r_0 f \in J^r_0(\mathbb{R}^m, \mathbb{R}^n)_0$, $s \in F_0\mathbb{R}^m$. It is easy to see that λ satisfies the condition

$$\lambda_{m,p}(B \circ A, s) = \lambda_{n,p}(B, \lambda_{m,n}(A, s))\,, \tag{1.1}$$

$A \in L^r(m,n)$, $B \in L^r(n,p)$, $s \in S_m$. On the other hand if an action λ of L^r on $\mathcal{S}$ with the property (1.1) is given, then we define $FM = (F^rM, S_m)$ with the equivalence

$$(u, s) \sim (u \circ A, \lambda_{m,m}(A^{-1}, s))$$

and

$$Ff(u, s) = (v, \lambda_{m,n}(v^{-1} \circ A \circ u, s))\,,$$

where $u \in F^r_x M$, $v \in F^r_y N$, $A \in J^r_x(M,N)_y$. It is easy to see that $F(g \circ f) = (Fg) \circ (Ff)$ and F is a prolongation functor of order r. So we have a generalization of Theorem 1.4

Theorem 1.5. ([**1**]) *There is a bijective correspondence between r-th order prolongation functors and actions of the category L^r on $\mathcal{S}$.* □

1.10. **Natural operator.** Let F be a natural lift functor, $f : M \to \overline{M}$ be a mapping in $\mathrm{Mor}\mathcal{M}_m$ and $\sigma : M \to FM$ be a section. Then we define the section $f^*\sigma : \overline{M} \to F\overline{M}$ by $f^*\sigma = Ff \circ \sigma \circ f^{-1}$.

Definition 1.6. A *natural differential operator* D from a natural lift functor F_1 to a natural lift functor F_2 is a family of differential operators

$$\{D(M) : C^\infty(F_1M) \to C^\infty(F_2\overline{M})\}_{M \in \mathrm{Ob}\mathcal{M}_m}$$

such that

(i) $D(\overline{M})(f^*\sigma) = f^*D(M)(\sigma)$ for every section $\sigma \in C^\infty(F_1M)$ and every $f : M \to \overline{M}$ in $\mathrm{Mor}\mathcal{M}_m$,

(ii) $D_U(\sigma|U) = (D_M\sigma)|U$ for every section $\sigma \in C^\infty(F_1M)$ and every open submanifold $U \subset M$,

(iii) every smoothly parameterized family of sections of F_1M is transformed into a smoothly parametrized family of sections of F_2M. □

1.11. **Order of natural differential operator.** A natural differential operator is of order k, $0 \le k \le \infty$, if all $D(M)$, $M \in \mathrm{Ob}\mathcal{M}_m$, are of order k. Thus, a k-order natural differential operator D from F_1 to F_2 is characterized by the associated fibred manifold morphisms $\mathcal{D}(M) : J^kF_1M \to F_2M$, over M, according to the formula $\mathcal{D}(M)(j^k_x\sigma) = D(M)(\sigma)(x)$. The family $\mathcal{D} = \{\mathcal{D}(M)\}_{M\in \mathrm{Ob}\mathcal{M}_m}$ defines a natural transformation of the functors J^kF_1 and F_2.

1.12. **Equivariant mappings given by natural operators.** Coordinate independent geometrical constructions are in fact natural differential operators between natural lift functors. The study of natural differential operators is based on relations between natural differential operators and equivariant mappings. The basic tool is the following theorem, [16, 60, 85],

Theorem 1.7. *There is a bijective correspondence between the set of k-order natural differential operators from a natural lift functor F_1 to a natural lift functor F_2 and equivariant mappings from the standard fiber of J^kF_1 to the standard fiber of F_2.* □

Theorem 1.7 can be generalized to prolongation functors as follows

Theorem 1.8. ([**1**]) *There is a bijective correspondence between the set of natural transformations of two r-th order prolongation functors and the set of covariant maps of actions of L^r given by these prolongation functors.* □

1.13. **Examples. 1.** The *exterior derivative* d is a first order natural operator from $\wedge^pT^*$, $p \ge 0$, to $\wedge^{p+1}T^*$. The corresponding G^2_n-equivariant mapping from $J^1(\wedge^pT^*)_0 = T^1_n(\wedge^p\mathbb{R}^{m*})$ to $(\wedge^{p+1}T^*)_0 = \wedge^{p+1}\mathbb{R}^{m*}$ is given, in the canonical coordinate chart $(\omega_{\lambda_1\dots\lambda_p})$, $1 \le \lambda_1 < \dots < \lambda_p \le m$, on $(\wedge^p \mathbb{R}^{m*})$, by

$$\omega_{\lambda_1\dots\lambda_{p+1}} \circ d = \omega_{[\lambda_1\dots\lambda_p,\lambda_{p+1}]}\,,$$

where [...] denotes the antisymmetrization. For $p \ge 1$, the naturality determines d up to a constant multiple, [16, 58], while in classical proofs the linearity was supposed.

2. The *Levi-Civita connection* is a first order natural differential operator from pRm to Cla_τ. The corresponding G^2_m-equivariant mapping from $J^1(\mathrm{pRm})_0$ to Cla_0 is given by the *formal Christoffel symbols*

$$\Lambda_\mu{}^\lambda{}_\nu = -\tfrac{1}{2} g^{\lambda\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}),$$

where $(g^{\lambda\mu})$ is the inverse matrix of $(g_{\lambda\mu})$. The uniqueness of the Levi–Civita connection is the classical geometrical problem. The proof of the uniqueness by using natural technics can be found in [16, 64, 84].

3. The *curvature tensor* of a classical connection is a first order natural differential operator from Cla to $T^* \otimes T \otimes (\wedge^2 T^*)$. The corresponding G^3_m-equivariant mapping from $J^1\,\mathrm{Cla}_0$ to $(T^* \otimes T \otimes (\wedge^2 T^*))_0 = \mathbb{R}^{*m} \otimes \mathbb{R}^m \otimes (\wedge^2 \mathbb{R}^{m*})$ is given by

$$w_\nu{}^\kappa{}_{\lambda\mu} = \Lambda_\nu{}^\kappa{}_{\lambda,\mu} - \Lambda_\nu{}^\kappa{}_{\mu,\lambda} + \Lambda_{\rho}{}^\kappa_{\mu}\Lambda_\nu{}^\rho_\lambda - \Lambda_\rho{}^\kappa{}_\lambda \Lambda_\nu{}^\rho{}_\mu\,.$$

The curvature tensor is not unique operator of this type and plays an important role in classification of natural operators defined on classical connections, see Section 2.5 and [**10**], [58, 60, 80].

1.14. **Infinitesimal properties of natural lift functors.** The regularity property of lift functors allows us to lift vector fields on a manifold M to projectable vector fields on the natural bundle FM by using flows. Namely if $\exp(t\xi)$ is the flow of a vector field ξ on M then

$$F(\exp(t\xi)) = \exp(t\mathcal{F}\xi)$$

is the flow of the vector field $\mathcal{F}\xi$ on FM which is said the *flow lift* of ξ. Moreover, if F is of order r, then $\mathcal{F}\xi$ depends on r-jets of ξ. Properties of the flow lift are used in [**3**] to define on a fibred manifold $p : E \to M$ infinitesimal natural structure of order r by a rule transforming vector fields on M into projectable vector fields on E. This transformation can be described by using the notion of *systems*, see [**3**] and [71]. Let us recall that a *projectable, linear, regular system of vector fields* on a fibred manifold E is a pair (H, η), where $q_H : H \to M$ is a vector bundle, called the *space* of the system, and $\eta : H \times_M E \to TE$ is a linear fibred morphism over E, called the *evaluation morphism* of the system, which is projectable over a linear fibred morphism over M, of maximum rank, $\bar{\eta} : H \to TM$. Any (local) section $h : M \to H$ induces the distinguished vector field $\tilde{\eta}(h)$ on E by $\tilde{\eta}(h)(y) =$

$\eta(h(p(y)), y)$, $y \in E$. The system is *monic* if the construction $h \mapsto \tilde{\eta}(h)$ is injective and is *almost involutive* if on H there is a bracket $[,]$ such that $\tilde{\eta}([k,k]) = [\tilde{\eta}(h), \tilde{\eta}(k)]$. A projectable, linear, regular, canonical, monic and almost involutive system is called *strong*. If (H, η) be a projectable, linear and regular system of vector fields on E, we can define the Lie derivative of sections $\sigma \in C^\infty(E)$ with respect to sections $h \in C^\infty(H)$ by $\mathcal{L}_h\sigma = T\sigma \circ \bar{\eta}(h) - \tilde{\eta}(h) \circ \sigma : M \to VE$.

Definition 1.9. ([**3**]) An *infinitesimal natural lift* of order r is a fibred manifold $p : E \to M$ together with a system (J^rTM, μ) of vector fields of E which is linear, regular, canonical, projectable over $\pi_0^r : J^rTM \to TM$ and almost involutive with respect to the subsheaf of integrable sections of $J^rTM \to TM$. The system (J^rTM, μ) is called the *natural system*. □

1.15. **Infinitesimal properties of natural operators.** If $\sigma : M \to FM$ is a section of an r-th order natural bundle (a field of geometrical objects) then we can define the Lie derivative of σ with respect to a vector field ξ by the formula

$$\mathcal{L}_\xi\sigma = \frac{d}{dt}|_0\{\exp(-t\xi)^*\sigma\}\,.$$

$\mathcal{L}_\xi\sigma$ is a section of VFM. Natural differential operators D from a natural lift functor F to a natural lift functor G are infinitesimally characterized by the commutativity with the Lie derivatives, [56], in the sense that

$$\mathcal{L}_\xi D(M)(\sigma) = TD(M)(\mathcal{L}_\xi\sigma)\,,$$

for any vector field ξ on M and any section $\sigma \in C^\infty(FM)$. This property can be used to define natural operators between infinitesimal natural lifts defined by Definition 1.9.

Definition 1.10. ([**3**]) Let E_1, E_2 be two fibred manifolds over M and assume that a structure of infinitesimal r-order natural lift is given on E_1 by a natural system (J^rTM, μ_1) and a structure of infinitesimal s-order natural lift is given on E_2 by a natural system (J^sTM, μ_2). A k-order operator D from $C^\infty(E_1)$ to $C^\infty(E_2)$ is said to be (infinitesimally) natural if

$$TD(\mathcal{L}_{j^r\xi}\sigma) = \mathcal{L}_{j^s\xi}D(\sigma)\,,$$

for any section $\sigma : M \to E_1$ and any vector field $\xi : M \to TM$. □

2. Applications of natural operators on natural bundles

In this Section we shall mention some applications of natural operators on natural bundles. First, we shall classify the Frölicher-Nijenhuis bracket of projectable tangent valued forms, Section 2.1 and [**2**], the relations between linear connections on a fibred manifold and connections on the 1st jet prolongation of this fibred manifold are studied in Section 2.2 and [**5**], higher order valued reduction theorems are studied in Section 2.5 and [**10**]. As application in classical field theories we study natural symplectic and Poisson structures on the tangent bundle of a pseudo-Riemannian manifold given by a metric and a linear connection, see Section 2.3 and [**4, 6**], and natural cosymplectic structures on the phase space of the Galilei spacetime given by a vertical metric and a phase connection, see Section 2.4 and [**4**]. For other examples of natural operators see [16, 60].

2.1. **Natural operations with tangent valued forms.** Frölicher and Nijenhuis (F-N), [46, 74], introduced a bracket $[\,,\,]$ in the sheaf $\Omega(M,TM) = \bigoplus_{0\le r\le m} \Omega^r(M,TM)$, $m = \dim M$, of (local) tangent valued differential forms on a manifold M and proved that it gives rise to a graded Lie algebra (F-N algebra), i.e., the bracket is $\mathbb{R}$-bilinear and satisfies the graded anticommutativity and the graded Jacobi identity. This algebra has been widely applied to the study of complex, almost complex, almost tangent and other structures on a manifold, see [39, 47, 75]. The F-N algebra can be linked with the theory of connections, [38, 40, 51, 72], in the sense that the differential calculus associated with a classical connection can be expressed in terms of the F-N algebra. Mangiarotti and Modugno, [68], showed that the F-N algebra $\mathcal{P}(E) = \bigoplus_{0\le r\le m+n} \mathcal{P}^r(E)$, $m+n = \dim E$, of projectable tangent valued forms on a fibred manifold $p : E \to M$ is the natural framework for the study of (general) Ehresmann connections on fibred manifolds and that the F-N bracket yields a generalization of the standard differential calculus associated with general connections. In particular the exterior covariant differential, the curvature tensor and the Bianchi identity can be expressed by the F-N bracket.

The F-N bracket on $\Omega(M,TM)$ satisfy the naturality condition, [48]. Kolář and Michor, [59], gave the full classification of natural $\mathbb{R}$-bilinear natural operators $\Omega^r(M,TM) \times \Omega^s(M,TM) \to \Omega^{r+s}(M,TM)$. They proved that, for $r,s \ge 2, r+s < \dim M - 1$, there exists a ten parameter family of such operators.

In [**2**] all $\mathbb{R}$-bilinear natural operators $\mathcal{P}^r(E) \times \mathcal{P}^s(E) \to \mathcal{P}^{r+s}(E)$ are classified. We have

Theorem 2.1. ([**2**]) *All natural $\mathbb{R}$-bilinear operators*

$$\mathcal{P}^r(E) \times \mathcal{P}^s(E) \to \mathcal{P}^{r+s}(E)\,, \qquad \dim M > r+s, r, s \ge 1\,,$$

form a vector space over $\mathbb{R}$ generated by the following three operators

$$[\phi,\psi]\,, \quad p^* dC\underline{\phi} \wedge \psi\,, \quad \phi \wedge p^* dC\underline{\psi}\,,$$

where C is the contracting operator, d is the exterior derivative and $\wedge$ is the exterior product of base differential forms with tangent valued forms on E.

It is easy to see that only scalar multiples of the F-N bracket satisfy the graded anticommutativity and the graded Jacobi identity. So we have

Corollary 2.2. ([**2**]) *The F-N bracket is the only (up to a multiplicative constant) natural graded $\mathbb{R}$-bilinear operator $\mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{P}(E)$.* □

The Theorem 2.1 has direct consequences for the theory of general connections on a fibred manifold. Namely we have

Theorem 2.3. ([**2**]) *Let Γ be a general connection on E. Then, (exterior) covariant differential $D_\Gamma = [\Gamma, .]$ is the only (up to a multiplicative constant) derivation $D_\Gamma : \mathcal{P}(E) \to \mathcal{P}(E)$ of degree 1 which satisfies the naturality condition $f^*(D_\Gamma \phi) = D_{f^*\Gamma}(f^*\phi)$.* □

Theorem 2.4. ([**2**]) *The curvature tensor is the only (up to a multiplicative constant) vertical valued 2-form associated naturally with a given connection Γ.* □

2.2. **Relations between classical connections on the tangent bundle and connections on the 1-jet bundle of a fibred manifold.** In general relativistic theories over spacetime with absolute time (the Galilei spacetime, see Section 2.4) there is the bijective relation between time preserving linear connections on spacetime and affine connections on 1-jet bundle of spacetime (phase space), [35]. This result can be generalized for a general fibred manifold and so we classify all natural operations transforming classical connections on the tangent bundle of a fibred manifold to connections on the 1-jet bundle.

Let $p : Y \to M$ be a fibred manifold with a local fibred coordinate chart $(x^\lambda, x^i) = (x^a)$, $\lambda = 1, \dots, \dim M = m$, $i = 1, \dots, \dim Y - \dim M = n$, $a = 1, \dots, \dim Y = n + m$. A classical (linear) connection Λ on the bundle $\pi_M : TM \to M$ and a classical (linear) connection K on the bundle $\pi_Y : TY \to Y$ can be expressed by the vertical projections $\nu_\Lambda : TTM \to TM$ and $\nu_K : TTY \to TY$, respectively. a pair of classical connections (K, Λ) is said to be *fibre preserving* if $Tp \circ \nu_K = TTp \circ \nu_\Lambda$.

Let us consider the complementary contact maps of 1-jet prolongation of Y

$$\text{д} : J^1Y \to T^*M \underset{Y}{\otimes} TY\,, \quad \vartheta : J^1Y \to T^*Y \underset{Y}{\otimes} VY\,.$$

Then we have

Theorem 2.5. ([**5**]) *Let Λ be a classical connection on M and K a classical connection on Y. The map*

$$\nu_\Gamma = \vartheta \circ \nu_{(K \otimes \Lambda^*)} \circ T\text{д}$$

turns out to be a connection on the bundle $\pi^1_0 : J^1Y \to Y$. *Moreover, we have the coordinate expression*

$$\Gamma_a{}^i_\lambda = K_a{}^i{}_j x^j_\lambda + K_a{}^i{}_\lambda - x^i_\mu (K_a{}^\mu{}_j x^j_\lambda + K_a{}^\mu{}_\lambda)\,,$$

i.e., the connection Γ *is independent of* Λ.

Thus, we have obtained a natural operator

$$\chi : K \mapsto \Gamma$$

transforming classical connections on TY *into connections on* J^1Y. □

The connection $\chi(K)$ is not generally affine. We have

Lemma 2.6. ([**5**]) *If* (K, Λ) *are fibre preserving, then the induced connection* $\chi(K)$ *on* J^1Y *is affine.* □

The connection $\chi(K)$ is not unique connection on J^1Y constructed naturally from K. We have the following classification theorem

Theorem 2.7. ([**5**]) *All natural operations transforming a classical connection* K *on* Y *into connections on* J^1Y *form the following 2–parameter family*

$$\chi(K) + (\mathrm{id} \otimes д^* \otimes \vartheta)(k_1 T_K + k_2 I \otimes \hat{T}_K)\,,$$

where $k_1, k_2 \in \mathbb{R}$, T_K *is the torsion tensor of* K, $\hat{}$ *denotes the contraction and* I *is the identity tensor on* TY. □

Corollary 2.8. ([**5**]) *For a torsion free connection* K *the connection* $\chi(K)$ *is the unique natural connection on* J^1Y *given by* K. □

2.3. **Natural symplectic and Poisson structures on the tangent bundle of a pseudo-Riemannian manifold.** Let (M, g) be a pseudo-Riemannian manifold and $h(u) = \frac{1}{2} g(u, u)$, $u \in TM$. If $\dim M = 4$ and g is a Lorentzian metric (M, g) is said to be the Einstein spacetime. Then $\Omega(g) = d_v h$, d_v being the vertical differential, is the canonical natural metric symplectic 2-form on TM. From the point of view of natural geometry $\Omega(g)$ is a natural operator $J^1(\mathrm{pRm}M) \times_M TM \to \wedge^2 T^*(TM)$ over the identity of TM. Let us note that $\Omega(g)$ can be defined as the lift of the metric g with respect to the Levi Civita connection $K(g)$ as $\Omega(g, K(g)) = \nu_K \bar{\wedge} \vartheta$, where $\bar{\wedge}$ is the wedge product followed by the contraction through the metric g and $\vartheta = d^\lambda \otimes \partial_\lambda$ is the identity form on TM. The following natural question arises: to classify all natural operators of the above type. This problem was solved for symmetric $(0, 2)$-tensor fields on TM by Kowalski and Sekizawa, [62], and for natural 2-forms in [19]. The classification, in both symmetric and antisymmetric situations, is based on the classification of natural F-metrics on TM. We have, see [62] for Riemannian metrics and [19] for pseudo-Riemannian metrics,

Lemma 2.9. *Let (M, g) be a pseudo-Riemannian manifold of dimension ≥ 3. Then all natural F-metrics on M derived from g are symmetric and are of the form*

$$\beta_u(g)(\xi,\eta) = \mu(h(u))g_x(\xi,\eta) + \nu(h(u))g_x(\xi,u)g_x(\eta,u) \tag{2.1}$$

where μ, ν are arbitrary smooth functions of one real variable and $u, \xi, \eta \in T_xM$. □

Now, by using the natural F-metrics (2.1), we have

Theorem 2.10. ([19]) *All natural operators from $J^1(\mathrm{pRm}M) \times TM$ to $\wedge^2 T^*(TM)$ over the identity of TM are lifts $\Omega(\beta, K)$ of natural F-metrics with respect to the Levi-Civita connection $K(g)$.* □

The result of Theorem 2.10 can be generalized for any classical connection K and we obtain natural 2-form $\Omega(\beta, K)$ on TM. We have

Theorem 2.11. ([19]) *All natural operators $\mathrm{pRm}M \times \mathrm{Cla}\,M \times TM \to \wedge^2 T^*(TM)$ over the identity of TM are lifts of natural F-metrics with respect to classical connections.* □

If (M, g) is a Lorentz manifold (the Einstein spacetime) then natural 2-forms on TM plays the fundamental role for geometrical quantisation, [91]. So it is very important to know under which conditions $\Omega(\beta, K)$ is symplectic, [65]. We have

Theorem 2.12. ([**4**]) *Let K be a classical connection on M. Then $\Omega(\beta, K)$ is a symplectic 2-form on TM if and only if*

$$\beta_u(g)(\xi,\eta) = \mu(h(u))g_x((\xi,\eta) + \frac{d\mu(h(u))}{dt} g_x(\xi,u)g_x(\eta,u)\,,$$

$u, \xi, \eta \in T_xM$, where the real smooth function μ satisfies

$$\mu(t) \neq 0, \quad \mu(t) + 2t\frac{d\mu(t)}{dt} \neq 0$$

for all $t \in \mathbb{R}$. Moreover g and K have to satisfy

(A) $d_K g = 0$,

(B) $g \otimes \nabla g$ *is the symmetric (0,5)-tensor field,*

where $d_K g$ is the exterior covariant differential defined in [60]. □

Remark 2.13. In [32] conditions for $\Omega(g, K)$ to be symplectic was found for a general (non linear) connection on TM. □

The geometric quantization on TM can be considered also with respect to a Poisson structure, [89], given by a natural 2-vector field Λ on TM such that the Schouten-Nijenhius bracket satisfies $[\Lambda, \Lambda] = 0$. First, for a classical connection K on M and the metric g, we have the canonical 2-vector field $\Lambda(g,K)$ given in coordinates by

$$\Lambda(g,K) = g^{\lambda\mu}(\partial_\lambda + K_\lambda{}^\rho{}_\sigma \dot{x}^\sigma \dot{\partial}_\rho) \wedge \dot{\partial}_\mu\,. \tag{2.2}$$

Then we have the following classification theorem given by a natural F-metric γ

Theorem 2.14. ([**6**]) *Let (M,g) ($\dim M > 3$) be an oriented pseudo-Riemannian manifold endowed with a symmetric classical connection K. Then all natural 2-vector fields on TM are of the form*

$$\Lambda(\gamma,K) = \gamma_1(h(u))\,\Lambda(g,K) + \gamma_2(h(u))\,u^H \wedge u^V$$

where γ_1,γ_2 are smooth real functions defined on $\mathbb{R}$ and u^H or u^V are horizontal or vertical lifts, respectively. □

Lemma 2.15. ([**6**]) *The 2-vector field $\Lambda(\gamma,K)$ is of maximal rank if and only if $\gamma_1(t)\neq 0$ and $\gamma_1(t)+2t\gamma_2(t)\neq 0$ for any $t\in\mathbb{R}$.* □

Theorem 2.16. ([**6**]) *The nondegenerate 2-vector field $\Lambda(\gamma,K)$ defines a Poisson structure on TM if and only if the conditions* (A), (B) *and*

$$\gamma_1(t)\gamma_2(t) - \gamma_1(t)\dot{\gamma}_1(t) - 2t\gamma_2(t)\dot{\gamma}_1(t) = 0 \tag{C}$$

are satisfied for any $t\in\mathbb{R}$. □

Let us note that the conditions of Theorem 2.16 for nondegenerate natural Poisson structures are equivalent with conditions for natural symplectic structures given by Theorem 2.12, i.e., the Poisson structure given by $\Lambda(\gamma,K)$ is dual to the symplectic structure given by $\Omega(\beta,K)$.

2.4. **Natural cosymplectic structures on the phase space of the Galilei spacetime.** The Galilei spacetime is assumed to be a 4-dimensional manifold $t: E\to B$ fibred over 1-dimensional affine orientable manifold B (time) and endowed with a vertical Riemannian metric g. Typical fibred coordinate charts will be denoted by (x^0,y^i). In what follows the index 0 will refer to the base space and Latin indices $i,j,k,...=1,2,3$ will refer to the fibres, while Greek indices $\varphi,\psi,...=0,1,2,3$ will refer both to the base space and the fibres.

On the Galilei background a motion is defined to be a section of $t: E\to B$. This implies that the 1-jet bundle of motions (the Galilei phase bundle) is the usual 1-jet bundle $\pi^1_0 : J^1E\to E$.

We consider the 1-jet bundle J^1E as the affine subbundle $J^1E\subset T^*B\otimes_E TE$ which is constituted by the vectors which project on $1\in T^*B\otimes_B TB$. The induced fibred coordinate charts on J^1E will be denoted (x^0,y^i,y^i_0). The canonical local bases of the modules of vector fields and forms on J^1E will be denoted by $(\partial_\varphi,\partial^0_i)$ and (d^φ,d^i_0).

A (phase) connection on $J^1E \to E$ is defined to be a tangent valued 1-form $\Gamma : J^1E \to T^*E \underset{J^1E}{\otimes} TJ^1E$, which projects on $1_E : E \to T^*E \otimes_E TE$. Its coordinate expression is

$$\Gamma = d^\varphi \otimes (\partial_\varphi + \Gamma^i_{0\varphi} \partial^0_i), \quad \Gamma^i_{0\varphi} \in C^\infty(J^1E).$$

The connection Γ is said to be affine if $\Gamma^i_{0\lambda} = \Gamma^i_{j\lambda} y^j_0 + \Gamma^i_{0\lambda}$, $\quad \Gamma^i_{\lambda\mu} \in C^\infty(E)$.

In [54] it is proved

Theorem 2.17. *There is a canonical bijection between time-preserving connections on $TE \to E$ and affine phase connections on $J^1E \to E$. In coordinates this bijection reads as $K_\varphi{}^i{}_\psi \mapsto \Gamma^i_{\varphi\psi}$.* □

According to Theorem 2.17 a *spacetime connection* is a torsion free time-preserving connection on TE or equivalently an affine torsion free phase connection on $J^1E \to E$.

The spacetime connection Γ can be characterized by the associated vertical-valued 1-form $\nu_\Gamma : J^1E \to T^*J^1E \otimes_E (T^*B \otimes_E VE)$ with the coordinate expression

$$\nu_\Gamma = (d^i_0 - (\Gamma^i_{j\varphi} y^j_0 + \Gamma^i_{0\varphi}) d^\varphi) \otimes d^0 \otimes \partial_i.$$

The *contact* 2-form on J^1E derived from g and Γ is then the T^*B-valued 2-form

$$\Omega(g, \Gamma) = \nu_\Gamma \bar{\wedge} \vartheta : J^1E \to T^*B \underset{J^1E}{\otimes} \wedge^2 T^*J^1E,$$

where $\bar{\wedge}$ denotes the wedge product followed by the contraction through the metric g. In coordinates we have

$$\Omega(g, \Gamma) = g_{ij} d^0 \otimes (d^i_0 - (\Gamma^i_{k\varphi} y^k_0 + \Gamma^i_{0\varphi}) d^\varphi) \wedge (d^j - y^j_0 d^0).$$

In [54] it is proved that the contact 2-form $\Omega(g, \Gamma)$ is a non-degenerate 2-form in the sense that $dt \wedge \Omega \wedge \Omega \wedge \Omega$ is a volume form on J^1E.

Moreover we have the following, [54],

Theorem 2.18. *The contact 2-form $\Omega(g, \Gamma)$ is closed if and only if $R^i{}_\varphi{}^j{}_\psi = R^j{}_\psi{}^i{}_\varphi$ and $\nabla \tilde{g} = 0$, where $\tilde{g}$ is the contravariant metric.* □

Remark 2.19. The closed contact 2-form $\Omega[g, \Gamma]$ plays a distinguished role in the theory built by Jadczyk and Modugno, [54], and is used for geometric quantization in the Galilei background. □

Let us denote by $Q_\tau(J^1E) \to E$ the bundle of space-time connections. From the viewpoint of natural geometry $\Omega(g, \Gamma)$ is a natural operator from $S^2V^*E \times_E Q_\tau(J^1E) \times_E J^1E$ to $T^*B \otimes_{J^1E} \wedge^2 T^*J^1E$ over the identity of J^1E.

Theorem 2.20. ([**4**]) *All natural operators from* $S^2 V^* E \times_E Q_\tau(J^1 E) \times_E J^1 E$ *to* $T^* B \otimes_{J^1 E} \wedge^2 T^* J^1 E$ *over the identity of* $J^1 E$ *are scalar multiples of the contact* 2*-form* $\Omega(g, \Gamma)$. □

Remark 2.21. According to Theorem 2.20 dt and $\Omega(g, \Gamma)$ define the unique (up to a constant multiple) cosymplectic structure on the Galilei phase space. □

2.5. **Higher order valued reduction theorems for classical connections.** It is well known that natural operators of linear symmetric connections on manifolds and of tensor fields which have values in bundles of geometrical objects of order one can be factorized through the curvature tensors, the tensor fields and their covariant differentials. These results are known as the first (the operators of connections only) and the second reduction theorems (RT). The history of the first RT goes back to the paper by Christoffel, [37], and the history of the second RT goes back to the paper by Ricci and Levi Civita, [78]. For further references see [60, 66, 80, 87]. In [80] the proof for algebraic operators (concomitants) is given. In [60] the first and the second RTs are proved for all natural differential operators by using the modern approach of natural bundles and natural differential operators, [60, 16, 76, 85]. The local version of the first RT is known also as the *replacement theorem*, [86, 87]. The RTs play very important role in theoretical physics. Namely, if we represent linear connections on manifolds as principal connections on the principal bundles of first order frames, then the RTs are in fact higher order versions of the Utiyama's theorem (the first RT) and Utiyama's invariant interaction (the second RT), [88].

In [**10**] we generalize the RTs for natural operators which have values in higher order natural bundles. For these theorems we shall use the name *higher order valued reduction theorems* for classical symmetric connections. Let us denote by $\nabla^{(k,r)}$, $k \leq r$, the sequence of operators $(\nabla^k, \dots, \nabla^r)$ and by $C_C^{(k,r)} M$ the (k,r)-order *curvature bundle* of classical symmetric connections obtained as the image of the operator $\nabla^{(k,r)} R$ on $\mathrm{Cla}_\tau M$. Then we obtain the *first k-order valued reduction theorem* for classical symmetric connections.

Theorem 2.22. ([**10**]) *Let F be a natural bundle functor of order $k \geq 1$ and let $r+2 \geq k$. All natural differential operators $f : C^\infty(\mathrm{Cla}_\tau M) \to C^\infty(FM)$ which are of order r are of the form*

$$f(j^r \Lambda) = g(j^{k-2} \Lambda, \nabla^{(k-2,r-1)} R[\Lambda])$$

where g is a unique natural operator

$$g : J^{k-2} \mathrm{Cla}_\tau M \underset{M}{\times} C_C^{(k-2,r-1)} M \to FM\,. \qquad \square$$

Similarly, we define the (k,r)-order *Ricci bundle* $Z^{(k,r)}M$ as the image of the pair of the operators $(\nabla^{(k-2,r-2)}R, \nabla^{(k,r)})$ applied on classical symmetric connections and sections of a 1st order natural bundle VM. Then the *second k-order valued reduction theorem* can be formulated as follows.

Theorem 2.23. ([**10**]) *Let F be a natural bundle of order $k \geq 1$ and let $r+1 \geq k$. All natural differential operators $f : C^\infty(\mathrm{Cla}_\tau M \underset{M}{\times} VM) \to C^\infty(FM)$ of order r with respect sections of VM are of the form*

$$f(j^{r-1}\Lambda, j^r\Phi) = g(j^{k-2}\Lambda, j^{k-1}\Phi, \nabla^{(k-2,r-2)}R[\Lambda], \nabla^{(k,r)}\Phi)$$

where g is a unique natural operator

$$g : J^{k-2}\,\mathrm{Cla}_\tau M \underset{M}{\times} J^{k-1}VM \underset{M}{\times} Z^{(k,r)}M \to FM\,. \qquad \square$$

Remark 2.24. The order $(r-1)$ of the above operators with respect to classical symmetric connections is the minimal order we have to use. The second reduction theorem can be easily generalized for any operator of order $s \geq r-1$ with respect to connections. Then

$$f(j^s\Lambda, j^r\Phi) = g(j^{k-2}\Lambda, j^{k-1}\Phi, \nabla^{(k-2,s-1)}R[\Lambda], \nabla^{(k,r)}\Phi)\,. \qquad \square$$

Remark 2.25. If Λ is a linear non-symmetric connection on M, then there exists its splitting $\Lambda = \widetilde{\Lambda} + T$, where $\widetilde{\Lambda}$ is the classical connection obtained by the symmetrization of Λ and T is the torsion tensor of Λ. Then all natural operators of order r defined on Λ are of the form

$$f(j^r\Lambda) = f(j^r\widetilde{\Lambda}, j^rT) = g(j^{k-2}\widetilde{\Lambda}, j^{k-1}T, \widetilde{\nabla}^{(k-2,r-1)}R[\widetilde{\Lambda}], \widetilde{\nabla}^{(k,r)}T)\,. \qquad \square$$

Remark 2.26. If g is a metric field on M, then there exists the unique classical Levi Civita connection Λ given by the metric field g. Then, applying the second reduction theorem, we get that all natural operators of order $r \geq 1$ defined on g are of the form

$$\begin{aligned} f(j^r g) &= f(j^{r-1}\Lambda, j^r g) = h(j^{k-2}\Lambda, j^{k-1}g, \nabla^{(k-2,r-2)}R[\Lambda]) \\ &= h(j^{k-1}g, \nabla^{(k-2,r-2)}R[\Lambda])\,. \qquad \square \end{aligned}$$

Typical applications of of higher order valued reduction theorems are classifications of natural tensor fields on the tangent (or cotangent) bundle of a manifold endowed with a classical connection or lifts of tensor fields to the tangent (or cotangent) bundle by means of a classical connection, see [22, 24, 62, 82, 83]. As a concrete example let us classify all (0,2)-tensor fields on T^*M given by a linear (non-symmetric) connection Λ.

Theorem 2.27. ([**10**]) *Let (M,Λ) be a manifold endowed with a linear (non-symmetric) connection Λ. Then all finite order natural (0,2)-tensor* *fields on T^*M are of the maximal order one and they form a 14-parameter family of operators with coordinate expression*

$$\begin{aligned}\Phi = \big(&A\,\dot x_\lambda\,\dot x_\mu + C_1\,\dot x_\lambda\,T_\rho{}^\rho{}_\mu + C_2\,\dot x_\mu\,T_\rho{}^\rho{}_\lambda + C_3\,\dot x_\rho\,T_\lambda{}^\rho{}_\mu\\ &+ F_1\,T_\rho{}^\rho{}_\lambda\,T_\sigma{}^\sigma{}_\mu + F_2\,T_\sigma{}^\rho{}_\lambda\,T_\rho{}^\sigma{}_\mu + F_3\,T_\rho{}^\rho{}_\sigma\,T_\lambda{}^\sigma{}_\mu\\ &+ G_1\,T_\rho{}^\rho{}_{\lambda;\mu} + G_2\,T_\rho{}^\rho{}_{\mu;\lambda} + G_3\,T_\lambda{}^\rho{}_{\mu;\rho} + H_1\,R_\rho{}^\rho{}_{\lambda\mu} + H_2\,R_\lambda{}^\rho{}_{\rho\mu}\big)d^\lambda\otimes d^\mu\\ &+ B\,d^\lambda\otimes(\dot d_\lambda + \Lambda_\lambda{}^\rho{}_\mu\,\dot x_\rho\,d^\mu) + C\,(\dot d_\lambda + \Lambda_\lambda{}^\rho{}_\mu\,\dot x_\rho\,d^\mu)\otimes d^\lambda\,,\end{aligned}$$

where $A, B, C, C_i, F_i, G_i, H_j$, $i = 1,2,3$, $j = 1,2$, are real constants. □

3. Gauge–natural bundles

In this Section we recall basic definitions and properties of gauge–natural bundle functors, [42, 58], and infinitesimal gauge–natural structures, [**3**].

3.1. **Gauge–natural bundle functors.** Gauge–natural bundle functors was introduced by D. Eck, [42]. We recall here the definition of [60]. Let us recall that B is the base functor from the category $\mathcal{FM}$ to the category $\mathcal{M}$.

Definition 3.1. A *gauge–natural bundle* over m-dimensional manifolds is a functor $F : \mathcal{PB}_m(G) \to \mathcal{FM}$ such that

(a) every $\mathcal{PB}_m(G)$-object $\pi : P \to BP$ is transformed into a fibered manifold $q_P : FP \to BP$ over BP,

(b) every $\mathcal{PB}_m(G)$-morphism $f : P \to \bar P$ is transformed into a fibered morphism $Ff : FP \to F\bar P$ over Bf,

(c) for every open subset $U \subset BP$, the inclusion $i : \pi^{-1}(U) \to P$ is transformed into the inclusion $Fi : q_P^{-1}(U) \to FP$. □

A *gauge–natural bundle* is then a quadruple $(FP, \pi_P, M, \pi : P \to BP)$.

Later (Theorem 3.3) we shall see that FP is actually a bundle.

In the original definition, [42], there is one more continuity condition which says that a smoothly parametrized family of diffeomorphisms of P is "transformed" into a smoothly parameterized family of isomorphisms of FP. But this condition is a consequence of i), ii) and iii), [60].

3.2. **Functor W^r.** Let $(\pi : P \to M) \in \operatorname{Ob}\mathcal{PB}_m(G)$, let W^rP be the space of all r-jets $j^r_{(0,e)}\varphi$, where $\varphi : \mathbb{R}^m \times G \to P$ is in $\operatorname{Mor}\mathcal{PB}_m(G)$, $0 \in \mathbb{R}^m$ and e is the unity in G. The space W^rP is a principal fibre bundle over M with structure group $W^r_mG = J^r_{(0,e)}(\mathbb{R}^m\times G, \mathbb{R}^m\times G)$ of all r-jets of principal fibre bundle isomorphisms $\Psi : \mathbb{R}^m \times G \to \mathbb{R}^m \times G$ covering the diffeomorphisms $\psi : \mathbb{R}^m \to \mathbb{R}^m$ such that $\psi(0) = 0$. The group W^r_nG is the semidirect product $G^r_m \rtimes T^r_mG$ of G^r_m and T^r_mG with respect to the action of G^r_m on T^r_mG given by the jet composition. Let $(\varphi : P \to \overline{P}) \in \operatorname{Mor}\mathcal{PB}_m(G)$, then we can define the principal bundle morphism $W^r\varphi : W^rP \to W^r\bar P$ by the jet composition. The rule transforming any $P \in \mathrm{Ob}\,\mathcal{P}B_m(G)$ into $W^r P \in \mathrm{Ob}\,\mathcal{PB}_m(W^r_m G)$ and any $\varphi \in \mathrm{Mor}\,\mathcal{PB}_m(G)$ into $W^r\varphi \in \mathrm{Mor}\,\mathcal{PB}_m(W^r_m G)$ is a gauge–natural bundle functor, [58].

Let us note that the first prolongation $W^1_m G$ can be expressed as the product $G^1_m \times G \times (\mathcal{G} \otimes \mathbb{R}^{m*})$ with the following composition, [58],

$$(X, g, Z)(\bar{X}, \bar{g}, \bar{Z}) = (X\bar{X}, g\bar{g}, ad(\bar{g}^{-1})Z\bar{X} + \bar{Z}).$$

3.3. **Bundle structure.** The gauge–natural bundle functor W^r described in Paragraph 3.2 plays a fundamental role in the theory of gauge–natural bundle functors. We have, [42, 58],

Theorem 3.2. *Every gauge–natural bundle FP is a fibred bundle associated with the gauge–natural bundle $W^r P$ for a certain order r.* □

3.4. **Order of gauge–natural bundle functors.** The number r from Theorem 3.2 is called *order* of the gauge–natural bundle functor F. So if F is an r-order gauge–natural bundle functor then

$$FP = [W^r P, F_0], \quad F\varphi = [W^r\varphi, \mathrm{id}_{F_0}],$$

where F_0 is a $W^r_m G$-manifold called the *standard fibre* of F.

3.5. **Gauge and total order of gauge–natural functors.** Let F be an s-order gauge–natural bundle functor and let $r \leq s$ be the minimal number such that the action of $W^s_m G = G^s_m \rtimes T^s_m G$ on F_0 can be factorized through the canonical projection $\pi^s_r : T^s_m G \to T^r_m G$, $s \geq r$. Then s is said to be the *total order* of F, r is the *gauge order* and we say that F is of order (s, r). In what follows we shall denote by $W^{(s,r)}_m G = G^s_m \rtimes T^r_m G$ and by $W^{s,r}P$ the corresponding principal bundle.

3.6. **Gauge–natural fibred coordinate chart.** A local fibred coordinate chart (x^λ, p^σ) on P and a coordinate chart (y^p) on F_0 induce a fibred coordinate chart (x^λ, y^p) on FP, which is said to be the *gauge–natural fibred coordinate chart*.

3.7. **Examples. 1.** Any r-order natural lift functor in the sense of Definition 1.1 is the $(r,0)$-order gauge–natural bundle functor with the trivial gauge action, i.e., the action $(G^r_m \times G) \times F_0 \longrightarrow F_0$ does not depend on G.

2. Let $(\pi : P \to M) \in \mathrm{Ob}\,\mathcal{PB}_m(G)$ and let us denote by $\mathrm{Pri}\,P \to M$ the bundle of principal connections on P. Then Pri is a (1,1)-order gauge–natural bundle functor with the standard fibre $\mathcal{G} \otimes \mathbb{R}^{m*}$ and with the action of $W^1_m G$ given by, [58],

$$(X, g, Z)(Y) = ad(g)(Y + Z)X^{-1}.$$

In particular, let $G = G^r_n$, then $\mathrm{Pri}\, P$ can be viewed as the bundle $\mathrm{Lin}\, E$ of linear connections on an associated vector bundle $E \to M$ with n-dimensional fibres. The standard fibre of Lin is $\mathrm{Lin}_0 = \mathbb{R}^{*n} \otimes \mathbb{R}^n \otimes \mathbb{R}^{m*}$ with coordinates $(K_j{}^i{}_\lambda)$, $i, j = 1, ..., n$, $\lambda = 1, ..., m$, and the action of $W^{(1,1)}_m G^1_n = G^1_m \rtimes T^1_m G^1_n$ on Lin_0 is given, in the canonical coordinates $(a^\lambda_\mu, a^i_j, a^i_{j\lambda})$ on $G^1_m \rtimes T^1_m G^1_n$, by

$$\bar{K}_j{}^i{}_\lambda = a^i_p\, K_q{}^p{}_\rho\, \tilde{a}^q_j\, \tilde{a}^\rho_\lambda + a^i_{p\rho}\, \tilde{a}^p_j\, \tilde{a}^\rho_\lambda\,,$$

where tilde refers to the inverse element.

3. Let F_0 be a left G–manifold. The *associated gauge–natural bundle functor* is defined by

$$ass_{F_0}(P) = [P, F_0], \quad ass_{F_0}(\varphi) = [\varphi, \mathrm{id}_{F_0}]\,,$$

where $P \in \mathrm{Ob}\mathcal{PB}_m(G)$, $\varphi \in \mathrm{Mor}\mathcal{PB}_m(G)$, is a 0-order gauge–natural bundle. Especially the adjoint bundle $\mathrm{ad}\, P$ is the 0-order gauge–natural bundle given by the adjoint action of G on its Lie algebra $\mathcal{G}$.

4. If F is a gauge–natural bundle functor of order (s, r) then $J^k F$ is a gauge–natural bundle functor of order at most $(s + k, r + k)$. The number $(s + k)$ is exact, but $(r + k)$ may be too big. For instance if F is an s-order natural lift functor, i.e., an $(s$,0)-order gauge–natural bundle functor, then $J^k F$ is an $(s + k)$-order natural lift functor, i.e., an $(s + k$,0)-order gauge–natural bundle functor.

5. $\mathrm{ad}\, P \otimes (\wedge^p T^* M)$ is a (1,0)-order gauge–natural bundle functor.

3.8. **Gauge–natural operators.** Let $(\varphi, f) \in \mathrm{Mor}\, \mathcal{PB}_m(G)$, $\varphi : P \to \overline{P}$, $f : M \to \overline{M}$, F be a gauge–natural bundle functor and $\sigma : M \to FP$ be a section. Then we define the section $\varphi^*\sigma : \overline{M} \to F\overline{P}$ by $\varphi^*\sigma = F\varphi \circ \sigma \circ f^{-1}$.

Definition 3.3. A *natural differential operator* D from a gauge–natural bundle functor F_1 to a gauge–natural bundle functor F_2 is a family of differential operators

$$\{D(P) : C^\infty(F_1 P) \to C^\infty(F_2 P)\}_{P \in \mathrm{Ob}\mathcal{PB}_m(G)}$$

such that

(i) $D(\overline{P})(\varphi^*\sigma) = \varphi^* D(P)(\sigma)$ for every section $\sigma \in C^\infty(F_1 P)$ and every $(\varphi, f) \in \mathrm{Mor}\mathcal{PB}_m(G)$, $\varphi : P \to \overline{P}$ over $f : M \to \overline{M}$,

(ii) $D_{\pi^{-1}(U)}(\sigma|U) = (D_P\sigma)|U$ for every section $\sigma \in C^\infty(F_1 P)$ and every open subset $U \subset M$,

(iii) every smoothly parameterized family of sections of $F_1 P$ is transformed into a smoothly parameterized family of sections of $F_2 P$. $\square$

Definition 3.4. A differential operator D from a gauge–natural bundle functor F_1 to a gauge–natural bundle functor F_2 is said to be *gauge–natural* if

$$D(\overline{P})(F_1\varphi \circ \sigma) = F_2\varphi \circ D(P)(\sigma)$$

for any $\varphi \in \mathrm{Mor}\mathcal{PB}_m(G)$, over the identity, and any section $\sigma \in C^\infty(F_1P)$.

3.9. **Order of natural operators.** A natural differential operator D from F_1 to F_2 is of a finite order k if all $D(P)$, $(\pi : P \to M) \in \mathrm{Ob}\mathcal{PB}_m(G)$, depend on k-order jets of sections of F_1P. Thus, a k-order natural operator from F_1 to F_2 is characterized by the associated fibred manifold morphism $\mathcal{D}(P) : J^kF_1P \to F_2P$, over M, such that the family $\mathcal{D} = \{\mathcal{D}(P)\}_{P\in \mathrm{Ob}\mathcal{PB}_m(G)}$ is a natural transformation of J^kF_1 to F_2.

Theorem 3.5. *Let F_1 and F_2 be gauge–natural bundle functors of order $\leq r$. Then we have a one-to-one correspondence between natural differential operators of order k from F_1 to F_2 and $W^{r+k}_m G$-equivariant mappings from $(J^kF_1)_0$ to $(F_2)_0$.* □

This theorem is due to Eck, [42], see also [60].

Remark 3.6. For the case of gauge–natural operators of order k we obtain that the corresponding equivariant mappings are equivariant with respect to the actions of the group $T^{r+k}_m G \approx \{J^{r+k}_0 \mathrm{id}\} \times T^{r+k}_m G$.

3.10. **Curvature operator.** The curvature operator of principal connections is a 1-order natural operator from Pri to $\mathcal{G}\otimes(\wedge^2T^*)$ with the associated $W^{(2,2)}_m G$-equivariant morphism

$$({u^a}_{\lambda\mu}) \circ R = {\Gamma^a}_{\lambda,\mu} - {\Gamma^a}_{\mu,\lambda} + c^a_{bd}\, {\Gamma^b}_\lambda\, {\Gamma^d}_\mu\,,$$

where c^a_{bd} are the structure constants of G.

3.11. **Infinitesimal properties of gauge–natural bundle functors.** The continuity property of gauge–natural bundle functors allows to transform G-invariant vector fields on a principal G-bundle P to projectable vector fields on the gauge–natural bundle FM by using flows. Namely if $\exp(t\Xi)$ is the flow of a G invariant vector field Ξ on P, projectable on the vector field ξ on M, then $F(\exp(t\Xi)) = \exp(t\mathcal{F}\Xi)$ is the flow of the vector field $\mathcal{F}\Xi$ on FP which is said the *flow transformation* of Ξ. Moreover, if F is of order r, then $\mathcal{F}\Xi$ depends on r-jets of Ξ. Properties of the flow transformation are used in [**3**] to define on a fibred manifold $E \to M$ infinitesimal gauge–natural structure of order r by a rule transforming a strong system of vector fields into projectable vector fields on E. This transformation can be described by using the notion of *systems*, see [**3**] and [71].

Definition 3.7. ([**3**]) Let (H, η) be a strong system on $p : E \to M$. An *infinitesimal gauge–natural transformation* of order r is a fibred manifold $p : E \to M$ together with a system $(J^r H, \mu)$ which is linear, regular, canonical, projectable over $(\pi^r_0 \circ J^r \bar{\eta}) : J^r H \to TM$ and almost involutive with respect to the subsheaf of integrable sections of $J^r H \to M$. □

We say that the system $(J^r H, \eta)$ defines a *structure of an infinitesimal gauge–natural bundle of order* r on E. The system $(J^r H, \eta)$ is called the *gauge–natural system.*

3.12. **Infinitesimal properties of natural operators.** If $\sigma : M \to FP$ is a section of an r-th order gauge–natural bundle then we can define the Lie derivative of σ with respect to a G-invariant vector field Ξ on P, over the vector field ξ on M, by the formula

$$\mathcal{L}_\Xi \sigma = \frac{d}{dt}|_0 \{\exp(-t\Xi)^* \sigma\} \,.$$

$\mathcal{L}_\xi \sigma$ is a section of VFP. Natural differential operators D from a gauge–natural bundle functor F to a gauge–natural functor functor G are infinitesimally characterized by the commutativity with the Lie derivatives, [**3**], in the sense that

$$\mathcal{L}_\Xi D(P)(\sigma) = TD(P)(\mathcal{L}_\Xi \sigma) \,,$$

for any G-invariant vector field Ξ on P and any section $\sigma \in C^\infty(FP)$. This property can be used to define natural operators between infinitesimal gauge–natural transformations defined by Definition 3.7.

Definition 3.8. ([**3**]) Let E_1, E_2 be two fibred manifolds over M and let a structure of r-order infinitesimal gauge–natural transformation be given on E_1 by a gauge–natural system $(J^r H, \mu_1)$ and a structure of s-order infinitesimal gauge–natural transformation is given on E_2 by a gauge–natural system $(J^s H, \mu_2)$. A k-order operator D from $C^\infty(E_1)$ to $C^\infty(E_2)$ is said to be (infinitesimally) natural if

$$TD(\mathcal{L}_{j^r h} \sigma) = \mathcal{L}_{j^s h} D(\sigma) \,,$$

for any section $\sigma : M \to E_1$ and any vector field $h : M \to H$. □

4. Applications of natural operators on gauge–natural bundles

As applications of natural operators on gauge-natural bundles we shall study natural quantum Lagrangians and natural Schödinger operators on the quantum bundle over the Galilei spacetime, [**7**], we shall generalize the Utiyama's reduction method for the gauge group $Gl(n, \mathbb{R})$, [**8, 9**], and for a general Lie group G, [**11**].

4.1. **Natural operators on the quantum bundle over Galilei spacetime.** In Galilei covariant classical and quantum mechanics studied in [**7**] and [21, 25, 26, 27, 28, 53, 54] all objects have their physical dimensions expressed by the fact that they have values in unit spaces. Moreover, the theory is covariant with respect to changes of bases of units. We assume the following fundamental unit spaces, which are positive 1–dimensional "semi–vector spaces" over $\mathbb{R}^+$: the space $\mathbb{T}$ of *time intervals*, the space $\mathbb{L}$ of *lengths* and the space $\mathbb{M}$ of *masses*. a *time unit of measurement* is defined to be an element $u_0 \in \mathbb{T}$, or its dual $u^0 \in \mathbb{T}^*$. Moreover, we assume the *Planck constant* to be an element $\hbar \in \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M}$. We refer to a particle with mass $m \in \mathbb{M}$ and charge $q \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$, where $\mathbb{L}^{p/q} = \otimes^p \mathbb{L} \otimes \otimes^q \mathbb{L}^*$.

We assume the classical *(Galilei) spacetime* to be a 4-dimensional orientable manifold E, the *absolute time* to be a 1-dimensional oriented affine space B associated with the vector space $\mathbb{T} \otimes \mathbb{R}$ and the *time map* to be a surjective map $t : E \to B$ of rank 1. Moreover, we assume the fibres of spacetime to be equipped with a "scaled" Riemannian metric $g : E \to \mathbb{L}^2 \otimes (V^*E \otimes_E V^*E)$ or its inverse $\bar{g} : E \to \mathbb{L}^{*2} \otimes (VE \otimes_E VE)$.

Thus, we have the *time-form* $dt : E \to \mathbb{T} \otimes T^*E$. Given a mass $m \in \mathbb{M}$, it is convenient to introduce the "normalized" metric $G = \frac{m}{\hbar} g : E \to \mathbb{T} \otimes (V^*E \otimes_E V^*E)$ or its inverse $\bar{G} = \frac{\hbar}{m} \bar{g} : E \to \mathbb{T}^* \otimes (VE \otimes_E VE)$. We stress that the normalized metric and all objects which will be derived from it incorporate the chosen mass and the Planck constant.

We choose an orientation of spacetime. We shall refer to *spacetime charts* (x^λ), which are adapted to the fibring t and to the chosen orientation of E, and such that x^0 is a Cartesian chart of B associated with a time unit of measurement u_0. The index 0 will refer to the base space, Latin indices $i, j, \dots = 1, 2, 3$ will refer to the fibres, while Greek indices $\lambda, \mu, \dots = 0, 1, 2, 3$ will refer both to the base space and the fibres.

We have the coordinate expressions $dt = u_0 \otimes d^0$ and $G = G^0_{ij}\, u_0 \otimes \check{d}^i \otimes \check{d}^j$.

The metric g and the spacetime orientation yield the *space-like vertical volume form* $\eta : E \to \mathbb{L}^3 \otimes \Lambda^3 V^*E$ and *spacetime volume form* $\upsilon = dt \wedge \eta : E \to (\mathbb{T} \otimes \mathbb{L}^3) \otimes \Lambda^4 T^*E$, respectively.

The classical *phase space* is defined to be the first jet space $t^1_0 : JE \equiv J_1E \to E$ of sections.

The *spacetime connection* is defined to be a torsion free linear connection K of the bundle $TE \to E$ such that $\nabla dt = 0$ and $\nabla g = 0$. Such connection is characterized by $K_\lambda{}^0{}_\mu = 0$, the vertical restriction of K is the Levi-Civita connection $\varkappa$ given by g and $g_{pi} K_0{}^p{}_j + g_{pi} K_0{}^p{}_i = -\frac{1}{2} \partial_0 g_{ij}$. Let us note that the operator χ, [**5**], identifies spacetime connections with affine connections on the phase space.

We assume the *quantum bundle* to be a Hermitian line bundle over spacetime $\pi : Q \to E$, i.e., $\pi : Q \to E$ is a Hermitian complex vector bundle with one-dimensional fibres. Let us denote by $h : Q \times_E Q \to \mathbb{C} \otimes \Lambda^3 V^* E$ the Hermitian product with values in vertical volume forms. Let $\mathbf{b} : E \to Q$ be a (local) base of Q such that $h(\mathbf{b}, \mathbf{b}) = \eta$. Such a local base is said to be *normal* and the fibred coordinate chart (x^0, x^i, z), $z \in \mathcal{F}(Q, \mathbb{C} \otimes \mathbb{L}^{*3/2})$, induced by a normal base of Q is said to be a *normal coordinate chart* on Q. In any fibred normal coordinate chart $h(\Psi, \Psi) = \bar{\psi}\psi\eta$ for every section $\Psi = \psi\mathbf{b} \in \mathcal{S}(Q)$.

A linear connection Ч on Q is said to be *Hermitian* if it preserves the Hermitian fibred product h. In a normal fibred coordinate chart Hermitian connections are expressed in the form

$$\text{Ч} = d^\lambda \otimes (\partial_\lambda + i\, \text{Ч}_\lambda\, \mathbb{I}), \qquad \text{Ч}_\lambda \in \mathcal{F}(E, \mathbb{R})\,,$$

where $\mathbb{I} = z \otimes \mathbf{b}$ is the Liouville vector field on Q.

Let us consider the pullback bundle $\pi^\uparrow : Q^\uparrow := JE \times_E Q \to JE$ of the quantum bundle $\pi : Q \to E$, with respect to $t^1_0 : JE \to E$. Let us recall that a connection $\text{Ч} : Q^\uparrow \to T^*JE \otimes_{JE} TQ^\uparrow$ is said to be the universal connection of the system of connections $\xi : JE \times_E Q \to T^*E \otimes_E TQ$ if, for every section $o : E \to JE$, the associated connection $\xi(o) : Q \to T^*E \otimes_E TQ$ of the system is obtained from Ч by pullback according to the formula $\xi(o) = o^*\text{Ч}$.

A connection Ч: $Q^\uparrow \to T^*JE \otimes_{JE} TQ^\uparrow$ is said to be a *quantum connection* if, [53], Q1) Ч is Hermitian, Q2) Ч is a universal connection, Q3) the curvature of Ч is given by $R[\text{Ч}] = i\,\Omega \otimes \mathbb{I}$, where $\Omega = \Omega[g, \chi(K)]$ is the cosymplectic 2-form studied in Section 2.4. Let us note that the coefficients of the quantum connections are in fact *quantum potentials* of the quantum theory.

Let us consider a section $\Psi \in \mathcal{S}(Q)$, its pullback on JE (denoted by the same symbol) and a quantum connection Ч. The covariant differential of Ψ with respect to Ч is a fibred morphism over E

$$\nabla[\text{Ч}]\Psi : JE \to T^*E \underset{E}{\otimes} Q\,, \quad \nabla_\lambda = \partial_\lambda \psi - i\ \text{Ч}_\lambda\, \psi\,,$$

and the *time-like* and the *space-like covariant differentials* of Ψ are

$$\overset{\circ}{\nabla}\Psi = \text{д} \lrcorner \nabla\Psi : JE \to \mathbb{T}^* \otimes Q, \quad \check{\nabla}\Psi : JE \to V^*E \underset{E}{\otimes} Q\,.$$

Then, for any section $\Psi \in \mathcal{S}(Q)$, we obtain the following invariant fibred morphisms over E

$$\overset{\circ}{\mathcal{L}}(\Psi) = \frac{1}{2} dt \wedge \Big(h(\Psi, i\overset{\circ}{\nabla}\Psi) + h(i\overset{\circ}{\nabla}\Psi, \Psi) \Big) : JE \to \Lambda^4 T^* E\,,$$

$$\check{\mathcal{L}}(\Psi) = \frac{1}{2} dt \wedge (\bar{G} \otimes h)(\check{\nabla}\Psi, \check{\nabla}\Psi) : JE \to \Lambda^4 T^* E\,,$$

and the *canonical quantum Lagrangian* is a unique (up to a multiplicative factor) linear combination of the above morphisms which projects on E, namely

$$\mathcal{L}_{can}(\Psi) = \overset{\circ}{\mathcal{L}}(\Psi) - \check{\mathcal{L}}(\Psi)$$

with coordinate expression

$$\mathcal{L}_{can}(\Psi) = \frac{1}{2} \left(i(\bar{\psi}\nabla_0\psi - \psi\overline{\nabla_0\psi}) - G_0^{pq}\overline{\nabla_p\psi}\nabla_q\psi \right) \upsilon^0\,,$$

where $\upsilon^0 = \upsilon(u^0) = \sqrt{g}\, d^0 \wedge d^1 \wedge d^2 \wedge d^3$.

The canonical quantum Lagrangian is a natural operator transforming vertical metrics, sections of the quantum bundle and quantum connections into volume forms on E. This operator is of order one with respect to sections of the quantum bundle. Now we shall discuss the classification of natural quantum Lagrangians under the additional condition that they depend on spacetime connections (up to finite order k).

Theorem 4.1. ([**7**]) *All* 1*st order (with respect to sections of the quantum bundle) natural quantum Lagrangians induced by the gravitational and quantum structure of spacetime are of the form*

$$\mathcal{L}(\Psi) = a\,\mathcal{L}_{can}(\Psi) - b\frac{\hbar}{2m} R dt \wedge h(\Psi, \Psi)\,,$$

where $\mathcal{L}_{can}(\Psi)$ *is the canonical quantum Lagrangian,* R *is the scalar curvature of the vertical metric connection* $\varkappa$ *and* a, b *are real numbers.* $\square$

The *Schrödinger operator* associated with a natural quantum Lagrangian $\mathcal{L}$ is then the sheaf morphism

$$\mathcal{O}_{Sch}(\mathcal{L}) = \langle \bar{\upsilon}; {}^\sharp\mathcal{E}(\mathcal{L}) \rangle : \mathcal{S}(Q) \to \mathcal{S}(\mathbb{T}^* \otimes Q)\,,$$

where ${}^\sharp\mathcal{E}(\mathcal{L}) : J_2 Q \to \mathbb{L}^3 \otimes Q \otimes \Lambda^4 T^* E$ is the *Euler-Lagrange morphism*, i.e., for any section $\Psi \in \mathcal{S}(Q)$, we have the *Euler-Lagrange operator* ${}^\sharp\mathcal{E}(\mathcal{L})(\Psi) = {}^\sharp\mathcal{E}(\mathcal{L}) \circ j^2\Psi$ associated with $\mathcal{L}$.

Let us consider a section (an observer) $o : E \to JE \hookrightarrow \mathbb{T}^* \otimes TE$ and let us define the divergence of o as a $\mathbb{T}^*$-valued function given by

$$L_o \eta = \operatorname{div}(o)\eta : E \to \mathbb{T}^* \otimes \mathbb{L}^3 \otimes \Lambda^3 V^* E$$

which, in coordinates adapted to o, has a coordinate expression $\operatorname{div}(o) = \frac{\partial_0\sqrt{g}}{\sqrt{g}}$. Further we have the (observed) Laplacian

$$\overset{o}{\Delta}(\Psi) = \langle \bar{g}; \nabla[o^*Ч \otimes K]\nabla[o^*Ч]\Psi\rangle : E \to Q\,,$$

where the second order covariant differential is considered with respect to the tensor product linear connection $o^*Ч\otimes K$. Then the Schödinger operator associated with the natural quantum Lagrangian of Theorem 4.1 can be expressed as

$$\mathcal{O}_{Sch}(\mathcal{L}(\Psi)) = u^0 \otimes \left(a\big(i\,(\nabla_0 + \frac{1}{2}\operatorname{div}(o)) + \frac{\hbar_0}{2m}\overset{o}{\Delta}\big) - b\frac{\hbar_0}{2m}R \right)(\Psi)\,. \tag{4.1}$$

Let us note that even if the operators ∇_0, $\operatorname{div}(o)$ and $\overset{o}{\Delta}$ depend on an observer o, the Schrödinger operator (4.1) is observer independent.

Now we shall classify all natural operators $\mathcal{O}_{Sch} : \mathcal{S}(Q) \to \mathcal{S}(\mathbb{T}^* \otimes Q)$ of the Schrödinger type, i.e., we shall classify all second order operators depending on the vertical metric field, the spacetime connection and its derivatives of finite order k and the quantum connection and its first order derivatives.

Theorem 4.2. ([**7**]) *All 2nd order natural operators of Schrödinger type are of the form*

$$\mathcal{O}_{Sch}(\Psi) = u^0 \otimes \left(a\big(i\,(\nabla_0 + \frac{1}{2}\operatorname{div}(o)) + \frac{\hbar_0}{2m}\overset{o}{\Delta}\big) - b\frac{\hbar_0}{2m}R \right)(\Psi),$$

where R is the scalar curvature of the vertical metric connection, o is an observer and a, b are complex numbers. □

In Theorem 4.1 we have classified all 1st order natural quantum Lagrangians. Naturally, there is a question if higher order natural quantum Lagrangians exist. The answer is positive, at least in the second order. If we consider the Schrödinger operator $\mathcal{O}_{Sch}(\mathcal{L}_{can})$ associated with the canonical quantum Lagrangian $\mathcal{L}_{can}$, then it is easy to see that

$$\mathcal{L}_{Sch}(\Psi) = \frac{1}{2}\big(dt \wedge h(\Psi, \mathcal{O}_{Sch}(\mathcal{L}_{can}(\Psi))) + dt \wedge h(\mathcal{O}_{Sch}(\mathcal{L}_{can}(\Psi)), \Psi)\big)$$

is the 2nd order natural quantum Lagrangian. Moreover, we can classify all 2nd order Lagrangians and we get.

Theorem 4.3. ([**7**]) *All 2nd order (with respect to sections of the quantum bundle) natural quantum Lagrangians induced by the gravitational and quantum structure of spacetime are of the form*

$$\mathcal{L}(\Psi) = a\,\mathcal{L}_{can}(\Psi) - b\frac{\hbar}{2m}R dt \wedge h(\Psi, \Psi) + c\,\mathcal{L}_{Sch}(\Psi)$$

where a, b, c are real numbers. □

Remark 4.4. Let us note that the Schrödinger operator associated with the 2nd order natural quantum Lagrangians from Theorem 4.3 is

$$\mathcal{O}_{Sch}(\mathcal{L}(\Psi)) = (a+c)\mathcal{O}_{Sch}(\mathcal{L}_{can}(\Psi)) - b\frac{\hbar}{2m}R(\Psi)\,,$$

so the second order part of the above quantum Lagrangian does not give new physical information. □

4.2. **Reduction theorems for general linear connections.** In Section 4.1 we have studied second order natural quantum Lagrangians and second order Schrödinger operators. In both situations such operators are factorized through the covariant differentials of sections of the quantum bundle where the first order covariant differentials are given by the quantum connection but the second order covariant differentials are given by both quantum and spacetime connections. This fact was a motivation how to generalize the reduction theorems for general linear connections on vector bundles.

Let $E \to M$ be a vector bundle with a m-dimensional base and n-dimensional fibres. Local linear fiber coordinate charts on E will be denoted by (x^λ, y^i).

We define a *linear connection* on E to be a linear splitting $K : E \to J^1E$. Considering the contact morphism $J^1E \to T^*M \otimes TE$ over the identity of TM, a linear connection can be regarded as a TE-valued 1-form $K : E \to T^*M \otimes TE$ projecting into the identity of TM. The coordinate expression of a linear connection K is of the type

$$K = d^\lambda \otimes \big(\partial_\lambda + K_j{}^i{}_\lambda\, y^j\, \partial_i\big)\,, \qquad \text{with} \qquad K_j{}^i{}_\lambda \in C^\infty(M,\mathbb{R})\,.$$

Linear connections can be regarded as sections of a (1,1)-order $G = Gl(n,\mathbb{R})$-gauge-natural bundle $\operatorname{Lin} E \to M$ described in Example 3.7.2.

The curvature of a linear connection K on E turns out to be the vertical valued 2–form $R[K] = -[K,K] : E \to VE \otimes \Lambda^2T^*M$, where $[,]$ is the Froelicher-Nijenhuis bracket. If we consider the identification $VE = E \underset{M}{\times} E$ and linearity of $R[K]$, the curvature $R[K]$ can be considered as the curvature tensor field $R[K] : M \to E^* \otimes E \otimes \Lambda^2T^*M$ and

$$R[K] : C^\infty(\operatorname{Lin} E) \to C^\infty(E^* \otimes E \otimes \Lambda^2T^*M)$$

is a natural operator which is of order one.

Let us set $E^{p,r}_{q,s} = \otimes^p E \otimes \otimes^q E^* \otimes \otimes^r TM \otimes \otimes^s T^*M$. Then a classical connection Λ on M and a linear connection K on E induce the linear tensor product connection $K^p_q \otimes \Lambda^r_s = \otimes^p K \otimes \otimes^q K^* \otimes \otimes^r \Lambda \otimes \otimes^s \Lambda^*$ on $E^{p,r}_{q,s}$

$$K^p_q \otimes \Gamma^r_s : E^{p,r}_{q,s} \to T^*M \underset{M}{\otimes} TE^{p,r}_{q,s}$$

which can be considered as a linear splitting $K^p_q \otimes \Lambda^r_s : E^{p,r}_{q,s} \to J^1 E^{p,r}_{q,s}$. Let $\Phi \in C^\infty(E^{p,r}_{q,s})$. We define the *covariant differential of* Φ *with respect to the pair of connections* (K,Λ) as a section of $E^{p,r}_{q,s} \otimes T^*M$ defined by

$$\nabla^{(K,\Lambda)}\Phi = j^1\Phi - (K^p_q \otimes \Lambda^r_s) \circ \Phi \,.$$

The iterated rth order covariant differential applied on the curvature tensor of a linear connection is a natural operator which is of order $(r-1)$ with respect to classical connection and of order $(r+1)$ with respect to linear connections. Let us denote by $C^r_L E$ the image of this operator and by $C^{(s)}_C M \times_M C^{(r)}_L E$ the (s,r)*-order curvature bundle of classical and linear connections* given as the image of the pair of the operators $(\nabla^{(s+1)}R, \nabla^{(r+1)}R)$, $s \geq r-2$, $\nabla^{(s)} = (\mathrm{id}, \nabla, \dots, \nabla^s)$, defined on $\mathrm{Cla}_\tau M \times \mathrm{Lin}\, E$. Let us assume a $(1,0)$-order $Gl(n,\mathbb{R})$-gauge natural bundle FE, then the *first reduction theorem for linear and classical connections* can be formulated as follows.

Theorem 4.5. ([**8**]) *Let* $s \geq r-2$*,* $r \geq 0$*. All natural differential operators*

$$f : C^\infty(\mathrm{Cla}_\tau M \underset{M}{\times} \mathrm{Lin}\, E) \to C^\infty(FE)$$

which are of order s *with respect to classical connections and of order* r *with respect to linear connections are of the form*

$$f(j^s\Lambda, j^r K) = g(\nabla^{(s-1)}R[\Lambda], \nabla^{(r-1)}R[K])$$

where g *is a zero order natural operator*

$$g : C^\infty(C^{(s-1)}_C M \underset{M}{\times} C^{(r-1)}_L E) \to C^\infty(FE)\,. \qquad \square$$

Let us assume the kth order covariant differential of sections of $E^{p_1,p_2}_{q_1,q_2}$. It is a natural operator of order k with respect to sections of $E^{p_1,p_2}_{q_1,q_2}$ and of order $(k-1)$ with respect to classical and linear connections. Let us define the k-th order *Ricci bundle* $Z^{(k)}E$ as the image of the triplet of the operators $(\nabla^{(k-2)}R, \nabla^{(k-2)}R, \nabla^{(k)})$ defined on $\mathrm{Cla}_\tau M \times \mathrm{Lin}\, E \times E^{p_1,p_2}_{q_1,q_2}$. Then the *second reduction theorem for linear and classical connections* can be formulated as follows.

Theorem 4.6. ([**8**]) *Let* $s, r \geq k-1$*,* $s \geq r-2$*. All natural differential operators*

$$f : C^\infty(\mathrm{Cla}_\tau M \underset{M}{\times} \mathrm{Lin}\, E \underset{M}{\times} E^{p_1,p_2}_{q_1,q_2}) \to C^\infty(FE)$$

which are of order s *with respect to classical connections, of order* r *with respect to linear connections and of order* k *with respect to sections of* $E^{p_1,p_2}_{q_1,q_2}$ *are of the form*

$$f(j^s\Lambda, j^r K, j^k\Phi) = g(\nabla^{(s-1)}R[\Lambda], \nabla^{(r-1)}R[K], \nabla^{(k)}\Phi)$$

where g is a zero order natural operator

$$g : C^\infty((C_C^{(s-1)} M \underset{M}{\times} C_L^{(r-1)} E) \underset{C_C^{(k-2)} M \underset{M}{\times} C_L^{(k-2)} E}{\times} Z^{(k)} E) \to C^\infty(FE)\,. \quad \square$$

4.3. **Higher order valued reduction theorems for general linear connections.** In Section 2.5 we have generalized the classical RTs for higher order valued natural operators. The similar generalization can be done also for reduction Theorems 4.5 and 4.6 of general linear connections. Let us denote by $C_C^{(k,s)} M \underset{M}{\times} C_L^{(k,r)} E$ the (k,s,r)-order *curvature bundle of classical and linear connections* given as the image of the pair of the operators $(\nabla^{(k,s)} R, \nabla^{(k,r)} R)$, $s \geq r-2$, defined on $\mathrm{Cla}_\tau M \times \mathrm{Lin}\, E$. Then the *first k-th order valued reduction theorem for linear and classical connections* can be formulated as follows.

Theorem 4.7. ([**9**]) *Let $s \geq r-2$, $r+1, s+2 \geq k \geq 1$. Let F be a $Gl(n,\mathbb{R})$-gauge-natural bundle of order k. All natural differential operators*

$$f : C^\infty(\mathrm{Cla}_\tau M \underset{M}{\times} \mathrm{Lin}\, E) \to C^\infty(FE)$$

which are of order s with respect to classical connections and of order r with respect to linear connections are of the form

$$f(j^s\Lambda, j^r K) = g(j^{k-2}\Lambda, j^{k-1}K, \nabla^{(k-2,s-1)} R[\Lambda], \nabla^{(k-1,r-1)} R[K])$$

where g is a unique natural operator

$$g : J^{k-2}\,\mathrm{Cla}_\tau M \underset{M}{\times} J^{k-1}\,\mathrm{Lin}\, E \underset{M}{\times} C_C^{(k-2,s-1)} M \underset{M}{\times} C_L^{(k-1,r-1)} E \to FE\,. \quad \square$$

Let us denote by $Z^{(k,r)}E$ the (k,r)-order *Ricci bundle* defined as the image of the triplet of the operators $(\nabla^{(k-2,r-2)} R, \nabla^{(k-2,r-2)} R, \nabla^{(k,r)})$ defined on $\mathrm{Cla}_\tau M \times \mathrm{Lin}\, E \times E^{p_1,p_2}_{q_1,q_2}$. Then the *second k-th order valued reduction theorem for linear and classical connections* can be formulated as follows.

Theorem 4.8. ([**9**]) *Let F be a $Gl(n,\mathbb{R})$-gauge-natural bundle of order $k \geq 1$ and let $r+1 \geq k$. All natural differential operators*

$$f : C^\infty(\mathrm{Cla}_\tau M \underset{M}{\times} \mathrm{Lin}\, E \underset{M}{\times} E^{p_1,p_2}_{q_1,q_2}) \to C^\infty(FE)$$

of order r with respect sections of $E^{p_1,p_2}_{q_1,q_2}$ are of the form

$$\begin{aligned} f(j^{r-1}\Lambda, j^{r-1}K, j^r\Phi) = g(&j^{k-2}\Lambda, j^{k-2}K, j^{k-1}\Phi,\\ &\nabla^{(k-2,r-2)} R[\Lambda], \nabla^{(k-2,r-2)} R[K], \nabla^{(k,r)}\Phi) \end{aligned}$$

where g is a unique natural operator

$$g : J^{k-2}\,\mathrm{Cla}_\tau M \underset{M}{\times} J^{k-2}\,\mathrm{Lin}\, E \underset{M}{\times} J^{k-1} E^{p_1,p_2}_{q_1,q_2} \underset{M}{\times} Z^{(k,r)} E \to FE\,. \quad \square$$

Remark 4.9. The order $(r-1)$ of the operator of Theorem 4.8 with respect to linear and classical connections is the minimal order we have to use. The second reduction theorem can be easily generalized for any operators of orders s_1 or s_2 with respect to connections Λ or K, respectively, where $s_1 \geq s_2 - 2$, $s_1, s_2 \geq r - 1$. Then

$$f(j^{s_1}\Lambda, j^{s_2}K, j^r\Phi) = g(j^{k-2}\Lambda, j^{k-2}K, j^{k-1}\Phi,$$
$$\nabla^{(k-2,s_1-1)}R[\Lambda], \nabla^{(k-2,s_2-1)}R[K], \nabla^{(k,r)}\Phi)\,. \quad \square$$

Remark 4.10. The above higher order valued valued reduction theorems deal with symmetric classical connections on the base manifolds. If Λ is a non-symmetric classical connection, then there is its unique splitting $\Lambda = \widetilde{\Lambda} + T$, where $\widetilde{\Lambda}$ is the symmetric classical connection obtained from Λ by symmetrization, i.e., $\widetilde{\Lambda}_\mu{}^\lambda{}_\nu = \frac{1}{2}\,(\Lambda_\mu{}^\lambda{}_\nu + \Lambda_\nu{}^\lambda{}_\mu)$, and T is the torsion $(1,2)$-tensor, i.e., $T_\mu{}^\lambda{}_\nu = \frac{1}{2}\,(\Lambda_\mu{}^\lambda{}_\nu - \Lambda_\nu{}^\lambda{}_\mu)$. Then any finite order natural operator for Λ and K is of the form, $s \geq r-2$,

$$f(j^s\Lambda, j^rK) = f(j^s\widetilde{\Lambda}, j^rK, j^sT) =$$
$$= g(j^{k-2}\widetilde{\Lambda}, j^{k-2}K, j^{k-1}T, \widetilde{\nabla}^{(k-2,s-1)}R[\widetilde{\Lambda}], \widetilde{\nabla}^{(k-2,r-1)}R[K], \widetilde{\nabla}^{(k,s)}T)\,,$$

where $\widetilde{}$ refers to $\widetilde{\Lambda}$. $\square$

As applications of higher order valued reduction theorems we shall classify all classical connections on the total space of a vector bundle and all connections on the 1st jet prolongation of a vector bundle given naturally by a general linear connection K and a classical connection Λ on the base, [30]. We have an induced natural classical connection $D(\Lambda, K)$ on E given by, [49, 60],

Proposition 4.11. *There exists a unique classical connection $D = D(\Lambda, K)$ on the total space E with the following properties*

$$\nabla^D_{h^K(X)} h^K(Y) = h^K(\nabla^\Lambda_X Y), \quad \nabla^D_{h^K(X)} s^V = (\nabla^K_X s)^V,$$
$$\nabla^D_{s^V} h^K(X) = 0, \quad \nabla^D_{s^V}\sigma^V = 0\,,$$

for all vector fields X, Y on M and all sections s, σ of E, where h^K is the horizontal lift with respect to K, $\nabla^K, \nabla^\Lambda, \nabla^D$ are covariant differentials with respect to K, Λ, D, respectively, and s^V, σ^V denote the vertical lifts of the sections s, σ, respectively. $\square$

Remark 4.12. The gauge–natural bundle $\mathrm{Cla}\, E$ is a $Gl(n,\mathbb{R})$-gauge-natural bundle of order (2,2) and $D(\Lambda, K)$ defines the natural operator D from $C^\infty(\mathrm{Cla}\, M \underset{M}{\times} \mathrm{Lin}\, E)$ to $C^\infty(\mathrm{Cla}\, E)$ which is of order zero with respect to Λ and of order one with respect to K. $\square$

The difference of any two classical connections on E is a tensor field on E of the type $(1,2)$. So, having the connection $D(\Lambda, K)$, all classical connections on E naturally given by K and Λ are of the type $D(\Lambda, K) + \Phi(\Lambda, K)$, where $\Phi(\Lambda, K)$ is a natural (1,2)-tensor field on E. Hence, the problem of classification of natural classical connections on E is reduced to the problem of classification of natural tensor fields on E. any tensor field on E is a section of a $Gl(n,\mathbb{R})$-gauge-natural bundle of order $(1,1)$. Then, by Theorem 4.8 and Remark 4.10, we get

Corollary 4.13. ([30]) *Let Φ be a tensor field on E naturally given by a classical connection Λ on M (in order s) and by a general linear connection K on E (in order r, $s \geq r-2$). Then*

$$\Phi(u, j^s\Lambda, j^r K) = \Psi(u, \widetilde{\nabla}^{(s)}T, \widetilde{\nabla}^{(s-1)}R[\widetilde{\Lambda}], K, \widetilde{\nabla}^{(r-1)}R[K]) \,,$$

where $u \in E$ and $\widetilde{}$ refers to the classical symmetrized connection $\widetilde{\Lambda}$. □

Now we can use the above Corollary 4.13 to classify (1,2)-tensor fields on E. We have the following families of natural operators given by Λ and K.

A) Λ gives 3-parameter family of (1,2)-tensor fields on M, [60], given by

$$S(\Lambda) = a_1\, T + a_2 I_{TM} \otimes \hat{T} + a_3\, \hat{T} \otimes I_{TM} \,,$$

where T is the torsion tensor of Λ, $\hat{T}$ is its contraction and $I_{TM} : M \to TM \otimes T^*M$ is the identity tensor.

B) Λ and K define naturally the following 9-parameter family of (0,2) tensor fields on M, [60], given by

$$\begin{aligned} G(\Lambda, K) = {} & b_1\, C^{12}_{13}(T \otimes T) + b_2\, C^{12}_{31}(T \otimes T) + b_3\, C^{12}_{12}(T \otimes T) \\ & + c_1\, C^1_1 \widetilde{\nabla} T + c_2\, \overline{C^1_1 \widetilde{\nabla} T} + c_3\, C^1_3 \widetilde{\nabla} T \\ & + d_1\, C^1_1 R[\widetilde{\Lambda}] + d_2\, C^1_2 R[\widetilde{\Lambda}] + e_1\, C^1_1 R[K] \,, \end{aligned}$$

where C^{ij}_{kl} is the contraction with respect to indicated indices and $\overline{C^1_1 \widetilde{\nabla} T}$ denotes the conjugated tensor obtained by the exchange of subindices.

C) The value of the curvature tensor $R[K]$ applied on the Liouville vector field L is in $T^*M \otimes VE \otimes T^*M$.

D) Finally, if we consider ν_K as the vertical valued 1-form $\nu_K : E \to T^*E \otimes VE$ with coordinate expression

$$\nu_K = (d^i - K_j{}^i{}_\lambda\, y^j\, d^\lambda) \otimes \partial_i \,,$$

we have 2-parameter family of operator obtained by applying the morphism $\iota_{T^*M} \otimes \iota_{VE} \otimes \mathrm{id}_{T^*E}$ on

$$H(\Lambda, K) = h_1\, \nu_K \otimes \hat{T} + h_2\, \hat{T} \otimes \nu_K \,.$$

Summarizing the above constructions we get

Theorem 4.14. ([30]) *All classical connections on E naturally given by Λ (in order s) and by K (in order r, $s \geq r-2$) are of the maximal order one and are of the form*

$$\widetilde{D}(\Lambda, K) = D(\Lambda, K) + h^K\big(S(\Lambda)\big) + L \otimes G(\Lambda, K) + R[K](L) + H(\Lambda, K)\,,$$

i.e. form a 15-parameter family of connections. □

Remark 4.15. In [50, 60] the same result was obtained by direct calculations without using the reduction theorems. Our result coincides with the result of [50, 60] but our base of the 15-parameter family of operators differ from the base used in [50, 60]. □

In Section 2.2 (see [**5**]), we have described a natural operator χ transforming a classical connection on the total space of a fibered manifold and a classical connection on the base manifold into a connection on the 1st jet prolongation of the fibered manifold. Applying this operator on a classical connection $D(\Lambda, K)$ on the total space of a vector bundle $E \to M$ we get

Theorem 4.16. *A general linear connection K on E and a classical connection Λ on M give naturally the connection $\Gamma(\Lambda, K) = \chi(D(\Lambda, K))$ on J^1E.* □

Any natural connection on J^1E is then of the form $\widetilde{\Gamma}(\Lambda, K) = \Gamma(\Lambda, K) + \Psi(\Lambda, K)$, where $\Psi(\Lambda, K)$ is a natural section of $T^*E \otimes T^*M \otimes VE$. Then we have

Theorem 4.17. ([30]) *All connections on J^1E naturally given by Λ (in order s) and by K (in order r, $s \geq r-2$) are of the maximal order one and are of the form*

$$\widetilde{\Gamma}(\Lambda, K) = \Gamma(\Lambda, K) + \theta \circ h^K\big(S(\Lambda)\big) + L \otimes G(\Lambda, K) + R[K](L) + \nu_K \otimes \hat{T}\,,$$

i.e. form a 14-parameter family of connections. □

Remark 4.18. Let us note that $\widetilde{\Gamma}(\Lambda, K) = \chi\big(\widetilde{D}(\Lambda, K)\big)$. □

4.4. **Higher order Utiyama's theorem.** One of the most famous results in gauge invariant theories is the *Utiyama's theorem*, [88], that classifies those Lagrangians for gauge fields (principal connections on principal bundles) which are (locally) gauge invariant. In his original paper, [88], Utiyama considered his theorem only locally with specific gauge transformations. Later the Utiyama's theorem was reproved by many authors also globally, see for instance [36, 42, 69]. The Utiyama's theorem can be formulated globally as follows: given a principal connection Γ then any gauge invariant first order Lagrangian is given by a gauge invariant Lagrangian of the curvature tensor $R[\Gamma]$, i.e. $\Omega(j^1\Gamma) = \widetilde{\Omega}(R[\Gamma])$. The Utiyama's theorem can be very simply generalized for operators with values in a gauge-natural bundle of order $(1,0)$. In this case we shall use the term *Utiyama-like theorem* instead of the Utiyama's theorem. The Utiyama-like theorem was proved (in order 1) in [60].

Higher order local version of the Utiyama-like theorem was studied in [52] where the author generalized the replacement theorem for gauge fields. The results obtained in [52] are local and not complete since only concomitants obtained from the covariant differentials of the curvature tensor of the gauge field are assumed, while concomitants obtained from classical connections on the base are not considered. By using the methods of gauge–natural bundles we obtain complete and global coordinate free description of higher order Utiyama-like theorem.

Let G be an n-dimensional Lie group, $P \in \operatorname{Ob}\mathcal{PB}_m(G)$, Γ a principal connection on P and $\operatorname{ad} P$ the adjoint vector bundle associated with the principal bundle P. Then we have the induced *adjoint linear connection* $\operatorname{ad}(\Gamma)$ on $\operatorname{ad} P$.

The curvature tensor of a principal connection is a 1-order natural operator from $\operatorname{Pri} P$ into $\operatorname{ad} P \otimes \bigwedge^2 T^*M$. The covariant differential of the curvature tensor $R[\Gamma]$ with respect to Γ and a classical connection Λ on the base M is then defined as the covariant differential with respect to $\operatorname{ad}(\Gamma)$ and Λ, see Section 4.2. Then the iterated rth order covariant differential $\nabla^r R[\Gamma]$ is a natural operator on $\operatorname{Cla} M \times \operatorname{Pri} P$ which is of order $(r-1)$ with respect to classical connections and of order $(r+1)$ with respect to principal connections. Let us denote by $C^{(s)}_C M \times C^{(r)}_P P$, $s \geq r-2$, (s,r)-order *curvature bundle for classical and principal connections* obtained as the image of the pair of the operators $(\nabla^{(s)} R, \nabla^{(r)} R)$ defined on $\operatorname{Cla}_\tau \times \operatorname{Pri} P$. Then *higher order Utiyama-like theorem for principal and classical connections* can be formulated as follows.

Theorem 4.19. ([**11**]) *Let $s \geq r-2$, $r \geq 0$, and let F be a (1,0)-order G-gauge-natural bundle functor. All natural differential operators*

$$f : C^\infty(\operatorname{Cla}_\tau M \underset{M}{\times} \operatorname{Pri} P) \to C^\infty(FP)$$

which are of order s with respect to classical connections and of order r with respect to principal connections are of the form

$$f(j^s\Lambda, j^r\Gamma) = g(\nabla^{(s-1)} R[\Lambda], \nabla^{(r-1)} R[\Gamma])$$

where g is a zero order natural operator

$$g : C^\infty(C^{(s-1)}_C M \underset{M}{\times} C^{(r-1)}_P P) \to C^\infty(FP)\,. \qquad \square$$

Remark 4.20. The curvature bundle of classical and principal connections is given by identities depending on the structure constants of the group G. So all natural operators defined on the curvature bundle depend also on the structure constants, i.e.,

$$f(j^s\Lambda, j^r\Gamma) = g(c, \nabla^{(s-1)}R[\Lambda], \nabla^{(r-1)}R[\Gamma])\,.$$

For instance $c^b_{ab}\,\nabla_{\rho_r}\dots\nabla_{\rho_1}R^a{}_{\lambda\mu}$ is an example of a natural tensor field of the type $(0, r{+}2)$ on M given by Λ (in order $(r-1)$) and Γ (in order $(r+1)$). In the case of (general) linear connections the structure constants are given by the Kronecker deltas and they contract with the curvature tensor fields, i.e., they are not "visible". $\square$

The DSc. dissertation consists of the set of 11 research papers [**1**] – [**11**]. [12] – [34] are other papers of the author concerning natural operators. The others references [35] – [92] have been used in the text.

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