INSTITUTE of MATHEMATICS

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Preprint No. 56-2014

PRAHA 2014

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Abstract

A stability result for the compressible Navier-Stokes system with transport equation for entropy s is shown. The proof comes as an outcome of the isentropic case and additional properties of the effective viscous flux. We deal with the pressure term in the form $\rho^\gamma e^s$ with adiabatic index $\gamma > 3/2$; therefore the crucial renormalization method is restricted.

Key words. compressible Navier-Stokes system, entropy transport, effective viscous flux

[1] The research of M. M. was supported by the GA13-00522S grant of the Grant Agency of the Czech Republic.

[2] The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840.

1 Introduction

Our aim is to show a stability result for global solutions of the compressible Navier-Stokes system supplemented by the transport equation for a scalar quantity (Theorem 3.1 and Corollary 3.3). Influence of this quantity on the pressure term is also considered. Systems of this kind are limit models for the Navier-Stokes-Fourier system when the thermal conduction coefficient is taken zero and the heating from viscous dissipation can be neglected. Such models arise e.g. in meteorology, see [Kle04].

The considered system reads

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \tag{1}$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \triangle \mathbf{u} - (\mu + \lambda)\nabla \operatorname{div} \mathbf{u} + \nabla p(\rho, s) = \rho f \tag{2}$$

$$\partial_t s + \nabla s \cdot \mathbf{u} = 0, \tag{3}$$

where ρ, s, are scalar unknown functions on $\Omega \times (0,T)$ and $\mathbf{u} \colon \Omega \times (0,T) \to \mathbb{R}^3$.[3] We suppose $\Omega \subseteq \mathbb{R}^3$ to be a bounded domain with Lipschitz boundary. We also suppose homogeneous Dirichlet condition for $\mathbf{u}$.[4]

We assume that $\mu > 0$ and $\lambda + 2/3\mu > 0$ (which is the widely used assumption) and add the following constitutive relation for the pressure term

$$p(\rho, s) = \rho^\gamma \mathcal{T}(s), \tag{4}$$

where $\mathcal{T}$ is a continuous and possitive function. We also consider initial data ρ_0, $(\rho\mathbf{u})_0$ and s_0.

First result on stability of the system (1), (2) with the transport equation was published by P.-L. Lions under rather non-physical assumption $\gamma > 9/5$, see Chapter 5 and Chapter 8 of [Lio98]. The result (for $\gamma > 9/5$) was then used by Bresch et al. in [BDGL02] where is shown that the low Mach number limit for the considered system is the compressible isentropic Navier-Stokes equation.

Existence of solutions for the compressible Navier-Stokes system with equation for temperature of parabolic type and $\gamma > 3/2$ was provided by Feireisl, see e.g. [Fei04]. For $\gamma < 9/5$ no results have been published if the parabolic equation for temperature is replaced by less regular transport equation for entropy.

We show a kind of stability result for solutions under mild assumtions on the sequence of densities. We apply schemes from [Lio98] and [Fei04]. The lack of space regularity for density in case $\gamma < 9/5$ unables us to renormalize the continuity equation (1) using renormalization techniques including defect measures provided by [Fei01]. The main reason is that in the polytropic case (i. e. with non-constant entropy) the pressure is not a monotone function of density but rather of $\tilde{\rho} = \rho\mathcal{T}(s)^{1/\gamma}$. We use invariance of the transport equation (Lemma 3.2) with respect to renormalization. This gives two consequences - one can work with a more suitable form of the pressure term, namely $\mathcal{T}(s) = 1/s$, and one can combine the continuity equation for density and the transport equation for entropy to conclude thee continuity equation for $\tilde{\rho}$. Then it is possible to use techniques from [Fei04] to show convergence of the pressure term. However, we cannot provide strong convergence of either ρ_n or s_n (only of $\tilde{\rho_n}$). The main

[3] We use the classical terminology for unknown functions - *density function* for ρ, *velocity vector field* for $\mathbf{u}$ and *momentum vector field* for $\rho\mathbf{u}$ and *entropy* for s.

[4] In cases when Ω is the whole space or torus (with periodic boundary conditions on $\mathbf{u}$) we can adapt analogous techniques and obtain the same result.

problem then lies in convergence of $s_n \operatorname{div} u_n$, which can be treated due to a generalized form (Lemma 4.2) of so called effective viscous flux identity.

We specify the difference between this result and the result of Lions. In the case $\gamma > 9/5$ it is possible to improve integrability of the limit density, namely $\rho \in L^2((0,T);L^2(\Omega))$. Under this condition one can renormalize the continuity equation for ρ without any other assumption. If γ is only greater then $3/2$, the structure of the momentum equation is needed[5] to show that the continuity equation for ρ can be renormalized. But as was already mentioned, this structure works for $\tilde{\rho}$ and not for ρ.

1.1 Weak formulation

We call a triplet

$$(\rho, s, \mathbf{u}) \in L^\infty((0,T);L^\gamma(\Omega)) \times \cap_{q\geq 1} L^\infty((0,T);L^q(\Omega)) \times L^2((0,T);W_0^{1,2}(\Omega))$$

a weak solution to (1), (2) and (3) satisfying homogeneous Dirichlet boundary condition and initial conditions ρ_0, $(\rho u)_0$ and s_0 if

- equalities (1) and (3) are satisfied in the sense of distributions, i.e.

$$\int_0^T\int_\Omega \rho\partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \rho\mathbf{u}\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{5}$$

$$\left.\begin{aligned} &\int_0^T\int_\Omega \rho\mathbf{u}\partial_t\eta\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \rho\mathbf{u}\otimes\mathbf{u}\nabla\eta\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega p(\rho,s)\operatorname{div}\eta\,\mathrm{d}x\,\mathrm{d}t \\ -\mu&\int_0^T\int_\Omega \nabla\mathbf{u}\nabla\eta\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega(\lambda+\mu)\operatorname{div}\mathbf{u}\operatorname{div}\eta\,\mathrm{d}x\,\mathrm{d}t = \int_{(0,T)\times\Omega}\rho f\,\mathrm{d}x\,\mathrm{d}t \end{aligned}\right\} \tag{6}$$

$$\int_0^T\int_\Omega s\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega s\mathbf{u}\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega s\operatorname{div}\mathbf{u}\varphi\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{7}$$

  for any $\varphi \in \mathcal{D}((0,T)\times\Omega)$ and $\eta \in \mathcal{D}((0,T)\times\Omega)^3$. Where $\mathcal{D}((0,T)\times\Omega)$ is the space of $\mathcal{C}^\infty$ functions with compact support in $(0,T)\times\Omega$.

- quantities for which are the evolutionary equations prescribed satisfy

$$(\rho, \rho\mathbf{u}, s) \in \mathcal{C}([0,T];L^\gamma_\omega(\Omega)) \times \mathcal{C}([0,T];L^{m_\infty}_\omega(\Omega)) \times \cap_{q\geq 1}\mathcal{C}([0,T];L^q(\Omega)_\omega)$$

  and $\rho(0) = \rho_0$, $(\rho\mathbf{u})(0) = (\rho\mathbf{u})_0$, $s(0) = s_0$.

We note that $\mathcal{C}([0,T];X_\omega)$ is the space of continuous functions from $[0,T]$ to Banach space X endowed with the weak topology.

2 A priori estimates

We assume in this section $(\rho, s, \mathbf{u})$ to be a sufficiently smooth solution to (1), (2) and (3) with smooth initial data. Then entropy is transported along characteristics given by the flow

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t,x) = \mathbf{u}(t,\mathbf{X}(t,x)). \tag{8}$$

[5] at least no result is known if the continuity equation in this case can be renormalized without the momentum equation

As

$$\frac{\mathrm{d}}{\mathrm{d}t}s(t,\mathbf{X}(t,x)) = 0,$$

the entropy stays bounded by the initial condition for all $t \in [0,T]$. By the same method one can derive a priori non-negativity for the density ρ (when ρ_0 is non-negative).

Next, we multiply the momentum equation by $\mathbf{u}$ and integrate both sides over Ω. We obtain (respecting continuity equation for ρ and the boundary condition for u)

$$\partial_t \int_\Omega \frac{1}{2}\rho|\mathbf{u}|^2 + \mu\int_\Omega |\nabla \mathbf{u}|^2 + (\lambda+\mu)\int_\Omega (\operatorname{div}(\mathbf{u}))^2 - \int_\Omega T(s)\rho^\gamma \operatorname{div}(\mathbf{u}) = \int_\Omega \rho\mathbf{u}f. \quad (9)$$

Observe that under homogeneous Dirichlet boundary conditions for $\mathbf{u}$ we have . We multiply (3) by $\rho B'(s)$ and use (1), where B is a smooth function, we obtain "renormalized" version of the equation, particularly

$$\partial_t(\rho B(s)) + \operatorname{div}(\rho B(s)\mathbf{u}) = 0. \quad (10)$$

Put $B(s) = T(s)^{1/\gamma}$ and denote $\tilde\rho = B(s)\rho$. We then derive estimates similar to the isentropic case $p = p(\rho)$ instead we deal with $p = p(\tilde\rho)$. We test (10) by $C'(\tilde\rho)$ and obtain

$$\partial_t(C(\tilde\rho)) + \operatorname{div} C(\tilde\rho)\mathbf{u} + (C'(\tilde\rho)\tilde\rho - C(\tilde\rho))\operatorname{div}\mathbf{u} = 0. \quad (11)$$

We then put $C(\tilde\rho) = \tilde\rho P(\tilde\rho)$ for

$$P(z) = \int_1^z \frac{q^\gamma}{q^2}\,\mathrm{d}q = \frac{1}{\gamma-1}z^{\gamma-1} - 1 \quad (12)$$

and realize that

$$(C'(\tilde\rho)\tilde\rho - C(\tilde\rho))\operatorname{div}\mathbf{u} = \tilde\rho^\gamma \operatorname{div}\mathbf{u}.$$

Applying this equality to (9) we end with energy equality in form

$$\partial_t \int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \tilde\rho P(\tilde\rho)\right) + \mu\int_\Omega |\nabla\mathbf{u}|^2 + (\mu+\lambda)\int_\Omega(\operatorname{div}(\mathbf{u}))^2 = \int_\Omega \rho\mathbf{u}f \quad (13)$$

from which can be deduced the following global in time estimates.

Claim 2.1. *Let* $(\rho, s, \mathbf{u})$ *be a smooth solution to* (1)-(3) *then*

- s *is bounded in* $L^\infty((0,T)\times\Omega)$,
- $\tilde\rho$ *and* ρ *are bounded in* $L^\infty((0,T); L^\gamma(\Omega))$ *and nonegative,*
- $\mathbf{u}$ *is bounded in* $L^2((0,T); W_0^{1,2}(\Omega))$,
- $\rho\mathbf{u}$ *is bounded in* $L^\infty((0,T); L^{m_\infty}(\Omega))$,
- $\rho\mathbf{u}$ *is bounded in* $L^2((0,T); L^{m_2}(\Omega))$,

where exponents m_2 *and* m_∞ *are given through*

$$m_\infty = \frac{2\gamma}{\gamma+1},$$
$$m_2 = \frac{6\gamma}{6+\gamma}.$$

3 Weak sequential stability and global existence

We state the main result on the stability of weak solutions. First observe that if s is a solution of the transport equation and B a differentiable function then (at least formally) $B(s)$ is a solution of the same equation with initial condition $B(s_0)$. This invariance with respect to renormalization gives us flexibility in the form of the pressure term. We set $\zeta = \left(T^{-1}(s)\right)^{1/\gamma}$ and observe that

$$p = \left(\frac{\rho}{\zeta}\right)^{\gamma}, \quad \tilde{\rho} = \frac{\rho}{\zeta}. \tag{14}$$

As T is positive, ζ has values in $(1/C, C)$ for some $C > 0$ if and only if s is bounded. As we will see later, the quantity ρ/ζ has more suitable form when passing to limit than $\rho\zeta$.

Theorem 3.1. *Let $(\rho_n, \mathbf{u}_n, \zeta_n)$ be a sequence of weak solutions to (1) - (3) with initial data*

$$(\rho_{n,0}, (\rho\mathbf{u})_{n,0}, \zeta_{n,0}) \to (\rho_0, (\rho\mathbf{u})_0, \zeta_0) \quad \textit{strongly in } L^{\gamma} \times L^{m_\infty} \times L^{\infty}$$

satisfying energy inequality

$$\left[\int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \tilde{\rho}P(\tilde{\rho})\right)\right]_0^T + \mu\int_\Omega |\nabla\mathbf{u}|^2 + (\mu+\lambda)\int_\Omega (\mathrm{div}(\mathbf{u}))^2 \le \int_0^T\int_\Omega \rho\mathbf{u}f \tag{15}$$

for P given by (12) and

$$1/C \le \operatorname{ess\,inf} \zeta_n \le \operatorname{ess\,sup} \zeta_n \le C.$$

Let the initial data converge strongly in corresponding norms and $\rho_n \in L^2(0,T;L^2)$. Then there exists a subsequence $(\rho_{n_k}, \mathbf{u}_{n_k}, \zeta_{n_k})$ convergent weakly to a solution to (1)-(3) with initial data $(\rho_0, (\rho\mathbf{u})_0, \zeta_0)$ and p given by (14).

Remark. We emphasize that we do not suppose ρ_n to be equibounded in $L^2((0,T)\times(\Omega))$ because this bound is not given a priori (unless $\gamma \ge 2$).[6] This assumption provide renormalization of the continuity equation. We denote that the mostly used approximative scheme (see [FNP01]) provides such regularity for ρ_n in the final approximative step.

Proof. (Theorem 3.1). Step 1 - strong convergence of the makeshift density.

We put $\tilde{\rho_n} = \rho_n/\zeta_n$ and observe that $p(\rho,\zeta) = \tilde{\rho}^{\gamma}$. The function $\tilde{\rho}$ also satisfies the continuity equation (see Lemma 4.1 - recall also that $(\rho_n, \mathbf{u}_n)$ can be extended from Ω to the whole space by zero). Hence we use the well-known results for the isentropic case (see [Fei04]) and obtain

$$\tilde{\rho_n} \to \tilde{\rho} \quad \text{a. e. and also in } \mathcal{C}([0,T];L^{\gamma}(\Omega)). \tag{16}$$

Step 2 - passing to the limit in the transport equation. From (16) we derive a weak convergence of

$$\rho_n = \tilde{\rho_n}\zeta_n \rightharpoonup \tilde{\rho}\zeta,$$

therefore $\rho/\zeta = \tilde{\rho}$ and $\rho^{\gamma}/\zeta^{\gamma} = \tilde{\rho}^{\gamma}$. Hence we satisfied the momentum equation.

[6] In the case $\gamma > 9/5$ we can improve the a priori regularity using appropriate test function to obtain $L^2(L^2)$ bound.

The pair $(\zeta_n, \mathbf{u}_n)$ solves the transport equation in the weak sense, so

$$\int_0^T \int_\Omega \zeta_n \partial_t \phi + \int_0^T \int_\Omega \zeta_n \mathbf{u}_n \cdot \nabla \phi - \zeta_n \operatorname{div} \mathbf{u}_n \phi = 0 \tag{17}$$

for any $\phi \in \mathcal{D}(\Omega)$. Passing to the limit in (17) we conclude that

$$\int_0^T \int_\Omega \zeta \partial_t \phi + \int_0^T \int_\Omega \zeta \mathbf{u} \cdot \nabla \phi - \overline{\zeta \operatorname{div} \mathbf{u}} \phi = 0.$$

Next we use properties of the effective viscous flux (Lemma 4.2) and realize that for any $\phi \in \mathcal{D}([0,T])$ and $\eta \in \mathcal{D}(\Omega)$:

$$\begin{aligned} \lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^3} \phi\eta \left(\tilde{\rho}_n^\gamma - (2\mu+\lambda) \operatorname{div} \mathbf{u}_n\right) \zeta_n \,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T \int_{\mathbb{R}^3} \phi\eta \left(\tilde{\rho}^\gamma - (2\mu+\lambda) \operatorname{div} \mathbf{u}\right) \zeta \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{18}$$

As $\tilde{\rho}_n$ converges strongly, one realizes that

$$\overline{\zeta \operatorname{div} \mathbf{u}} = \zeta \operatorname{div} \mathbf{u}$$

and so $(\zeta, \mathbf{u})$ solves the transport equation in the weak sense. Weak continuity in time of ρ, $\rho\mathbf{u}$, ζ and satisfaction of the initial conditions are standard for evolutionary equations. □

Remark. The proof did not provide strong (or pointwise) convergence of ζ_n or ρ_n. We sketch the main obstructions which we cannot avoid. For any continuous B we can renormalize equations for ζ_n and ζ. Then due to Lemma (4.2) we deduce that

$$\partial_t(\overline{B(\zeta)} - B(\zeta))) + \mathbf{u} \cdot \nabla(\overline{B(\zeta)} - B(\zeta))).$$

in the weak sense. Therefore

$$\begin{aligned} &\left[\int_\Omega \overline{B(\zeta)}(s,x) - B(\zeta)(s,x) \,\mathrm{d}x\right]_0^t \\ &\qquad = -\int_0^t \int_\Omega \operatorname{div} \mathbf{u}(s,x) \left(\overline{B(\zeta)}(s,x) - B(\zeta)(s,x)\right) \mathrm{d}x\,\mathrm{d}s \end{aligned}$$

but we cannot utilise Gronwall's lemma, unless $\operatorname{div} \mathbf{u} \in L^\infty((0,T)\times\Omega)$.[7] One may also try to derive almost everywhere convergence of density. However, for $\gamma < 9/5$ it is more complex to renormalize the equation of continuity. Approach using defect measures developed in [Fei01] demands compatible structure of the pressure term and is straightforwardly applicable only in slight perturbations of the isentropic case $p = p(\rho)$.

The following claim is a corollary of renormalization techniques - based on smoothing of equations and Friedrich's commutator lemma. For proof see e.g. Chapter 4 of [Fei04].

[7] It is well known that the boundedness of divergence of the velocity field is one of the most important open problems in the case of compressible models.

Lemma 3.2. *Let* $(\zeta, \mathbf{u}) \in (L^\infty((0,T)\times\Omega) \cap \mathcal{C}([0,T]; L^q_\omega) \times L^2((0,T); W^{1,2}(\Omega))$ *be a weak solution to* (3) *with* $\zeta(0) = \zeta_0 \in L^\infty$. *Then for every* $B \in \mathcal{C}(\mathbb{R})$ *is* $(B(\zeta), \mathbf{u})$ *a weak solution to* (3) *with* $B(\zeta) \in \mathcal{C}([0,T]; L^q(\Omega))$ *and* $(B(\zeta))(0) = B(\zeta_0)$.

This invariance result for the week solution enlarges the class of possible forms of the pressure term. The next theorem is a straigtforward corollary of Theorem 3.1 and Lemma 3.2.

Corollary 3.3. *Let* $T \in \mathcal{C}(\mathbb{R})$ *be a positive invertible function. Let* $(\rho_n, s_n, \mathbf{u}_n)$ *be a sequence of weak solutions to* (1) - (3) *with initial data*

$$(\rho_{n,0}, (\rho\mathbf{u})_{n,0}, s_{n,0}) \to (\rho_0, (\rho\mathbf{u})_0, s_0) \quad \textit{strongly in } L^\gamma \times L^{m_\infty} \times L^\infty$$

and $p = \rho^\gamma T(s)$ *satisfying inequality*

$$\left[\int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \tilde{\rho}P(\tilde{\rho})\right)\right]_0^T + \mu\int_\Omega |\nabla\mathbf{u}|^2 + (\mu+\lambda)\int_\Omega (\operatorname{div}(\mathbf{u}))^2 \le \int_0^T\int_\Omega \rho\mathbf{u}f \tag{19}$$

for P *given by* (12), $\tilde{\rho} = \rho T^{1/\gamma}(s)$ *and* s_n *uniformly bounded in* $L^\infty((0,T)\times\Omega)$. *Let the correspondent initial data converge strongly in corresponding norms and* $\rho_n \in L^2(0,T;L^2)$. *Then there exists a weak solution to* (1) - (3) *with the limit initial data and* $p = \rho^\gamma T(s)$.

4 Auxiliary lemmas

In this section we summarize additional claims which were used during the main proof. The firts one is based on renormalization techniques famously presented in [DL89].

Lemma 4.1. *Let*

$$(\rho, \mathbf{u}) \in L^2((0,T); L^2(\mathbb{R}^d)) \times L^2((0,T); W^{1,2}(\mathbb{R}^d))$$

be a weak solution to the continuity equation and

$$(\zeta, \mathbf{u}) \in L^\infty((0,T)\times\mathbb{R}^d) \times L^2((0,T); W^{1,2}(\mathbb{R}^d))$$

a weak solution to a transport equation. Then $(\rho\zeta, \mathbf{u})$ *is a weak solution to the continuity equation.*

Proof. Let $\eta \in \mathcal{D}(\mathbb{R}^d)$ be a non-negative function with $\|\eta\|_{L^1(\mathbb{R}^d)} = 1$ and denote $\eta_\varepsilon = 1/\varepsilon^n\eta(\cdot/\varepsilon)$. We mollify both equations with respect to space variables by testing the weak formulation for any $y \in \mathbb{R}^d$ by functions $\eta_\varepsilon(\cdot - y)$. We obtain equations

$$\partial_t[\rho]_\varepsilon + \operatorname{div}([\rho]_\varepsilon\mathbf{u}) = \operatorname{div}([\rho]_\varepsilon\mathbf{u}) - \operatorname{div}([\rho\mathbf{u}]_\varepsilon), \tag{20}$$

$$\partial_t[\zeta]_\varepsilon + \mathbf{u}\cdot\nabla[\zeta]_\varepsilon = \mathbf{u}\cdot\nabla[\zeta]_\varepsilon - [\mathbf{u}\cdot\nabla\zeta]_\varepsilon \tag{21}$$

where $[g]_\varepsilon = g * \eta_\varepsilon$. We then multiply (20) by $[\zeta]_\varepsilon$ and with respect to (21) we get

$$\begin{aligned} &\partial_t\left([\rho]_\varepsilon[\zeta]_\varepsilon\right) + \operatorname{div}([\rho]_\varepsilon[\zeta]_\varepsilon\mathbf{u}) \\ &\quad = \left(\operatorname{div}([\rho]_\varepsilon\mathbf{u}) - \operatorname{div}([\rho\mathbf{u}]_\varepsilon)\right)[\zeta]_\varepsilon + \left(\mathbf{u}\cdot\nabla[\zeta]_\varepsilon - [\mathbf{u}\cdot\nabla\zeta]_\varepsilon\right)[\rho]_\varepsilon. \end{aligned} \tag{22}$$

The right hand side converges to zero in $L^1((0,T)\times\mathbb{R}^d)$ due to the well-known Friedrich's commutator lemma. The weak convergence of derivatives on the left-hand side is assured by the strong convergence of the mollified functions. □

We recall the celebrated effective viscous flux identity, which can be postulated in a slightly generalized form.

Lemma 4.2. *Let* $(\rho_n, \mathbf{u}_n, s_n)$ *be weak solutions to* (1), (2) *and* (3) *uniformly bounded by a priori estimates and weakly convergent to* $(\rho, \mathbf{u}, s)$. *Let*

- p_n *be uniformly bounded in* $L^r((0,T)\times\Omega)$ *for some* $r>1$ *and weakly convergent to* p,
- $\sigma_n \rightharpoonup^* \sigma$ *in* $L^\infty((0,T)\times\Omega)$ *with* $\partial_t\sigma_n + \operatorname{div}(\sigma_n u_n) = \kappa_n$ *for* κ_n *boudned in* $L^2((0,T);L^2(\Omega))$.

Then after passing to a subsequence, if needed, we obtain

$$\lim_{n\to\infty}\int_0^T\int_{\mathbb{R}^3}\phi\eta\,(p_n-(2\mu+\lambda)\operatorname{div}\mathbf{u}_n)\,\sigma_n\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{R}^3}\phi\eta\,(p-(2\mu+\lambda)\operatorname{div}\mathbf{u})\,\sigma\,\mathrm{d}x\,\mathrm{d}t \tag{23}$$

for any $\eta\in\mathcal{D}(\Omega)$ *and* $\phi\in\mathcal{D}((0,T))$.

Remark. Broadly speaking, the sequence $\{p_n-(2\mu+\lambda)\operatorname{div}\mathbf{u}_n\}$ behaves as L^1 strongly convergent if tested by a bounded solutions of (nonhomogeneous) continuity equation with streamlines induced by $\mathbf{u}_n$.

Remark. The proof of Lemma 4.2 follows from the proof for the known special case $\sigma_n = B(\rho_n)$ and

$$\partial_t(\rho_n)+\operatorname{div}(B(\rho_n)\mathbf{u}_n) = (B(\rho)-B'(\rho)\rho)\operatorname{div}u$$

for a B bounded $\mathcal{C}^1$ function with compactly supported $B'(t)$.

The only difference is the presence of κ_n. However,

$$\int_{\mathbb{R}^3}\phi\eta\rho_n\mathbf{u}_n\nabla\triangle^{-1}\kappa_n \to \int_{\mathbb{R}^3}\phi\eta\rho\mathbf{u}\nabla\triangle^{-1}\kappa$$

as $\rho_n\mathbf{u}_n$ converges in $L^\infty([0,T];L^{2\gamma/(\gamma+1)}_\omega)\hookrightarrow L^2([0,T];W^{-1,2})$ and

$$\nabla\triangle^{-1}\kappa_n\to\nabla\triangle^{-1}\kappa \quad \text{in } L^2((0,T);W^{1,2}_0)$$

because of linearity and degree of the operator $\nabla\triangle^{-1}$. For more details see [Lio98] or [Fei04]. A version of this theorem can be also found in [PS12].

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