INSTITUTE of MATHEMATICS

ACADEMY of SCIENCES of the CZECH REPUBLIC

A convergent numerical method for the full Navier-Stokes-Fourier system in smooth physical domains

Eduard Feireisl
Radim Hošek
Martin Michálek

Preprint No. 3-2015

PRAHA 2015

A convergent numerical method for the full Navier-Stokes-Fourier system in smooth physical domains

Eduard Feireisl [*] Radim Hošek [†] Martin Michálek

Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, CZ-115 67 Praha 1, Czech Republic

Abstract

We propose a mixed finite volume - finite element numerical method for solving the full Navier-Stokes-Fourier system describing the motion of a compressible, viscous, and heat conducting fluid. The physical domain occupied by the fluid has smooth boundary and it is approximated by a family of polyhedral numerical domains. Convergence and stability of the numerical scheme is studied. The numerical solutions are shown to converge, up to a subsequence, to a weak solution of the problem posed on the limit domain.

Key words: Navier-Stokes-Fourier system, finite element method, finite volume method, stability, general domain

Contents

[*]The research of E.F. leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840.

[†]The research of R.H. leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840.

1 Introduction

Numerical methods based on finite elements approximation use a mesh over the physical domain Ω. If the boundary of the latter is curved, meshes built up by means of *polygonal* elements can only approximate the kinematic boundary $\partial\Omega$. On the other hand, rigorous *error* estimates of the numerical methods usually require the exact solution of the problem to be smooth. Smooth solutions, however, can exist only on *regular* physical domains. It is therefore of interest to study the convergence of a numerical scheme in the situation when a family of numerical polyhedral domains Ω_h approaches, in a certain sense, the limit physical domain Ω. To avoid technicalities and since we are primarily interested in *smooth* solutions of the problem, only bounded domains with a sufficiently smooth boundary $\partial\Omega \in C^1$ will be considered although the principal results of this paper can be easily extended to less regular geometries, say $\partial\Omega$ Lipschitz.

1.1 Navier-Stokes-Fourier system

The motion of a compressible, viscous, and heat conducting fluid in the framework of continuum mechanics is characterized by three basic macroscopic (observable) quantities: The mass density $\varrho = \varrho(t,x)$, the absolute temperature $\vartheta = \vartheta(t,x)$, and the velocity field $\mathbf{u} = \mathbf{u}(t,x)$, depending on the time $t \in (0,T)$ and the reference (Eulerian) spatial position $x \in \Omega$. The time evolution of the fluid is governed by the *Navier-Stokes-Fourier system* of equations:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.2}$$

$$c_v \left[\partial_t(\varrho \vartheta) + \mathrm{div}_x(\varrho \vartheta \mathbf{u}) \right] + \mathrm{div}_x \mathbf{q}(\vartheta, \nabla_x \vartheta) = \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \vartheta \frac{\partial p(\varrho, \vartheta)}{\partial \vartheta} \mathrm{div}_x \mathbf{u}, \tag{1.3}$$

where $p = p(\varrho, \vartheta)$ denotes the pressure, $\mathbb{S}(\nabla_x \mathbf{u})$ is the viscous stress tensor, c_v the specific heat at constant volume, and $\mathbf{q}(\vartheta, \nabla_x \vartheta)$ the heat flux, see Gallavotti [11]. For the sake of simplicity, the effect of external mechanical and heat sources is omitted in (1.2) and (1.3), respectively.

Furthermore, we suppose the fluid is Newtonian (linearly viscous), with the tensor $\mathbb{S}$ determined by *Newton's law*

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0, \tag{1.4}$$

with constant viscosity coefficients μ and η. Similarly, the heat flux is given by *Fourier's law*

$$\mathbf{q} = -\kappa(\vartheta) \nabla_x \vartheta = -\nabla_x K(\vartheta), \ K(\vartheta) = \int_0^\vartheta \kappa(z) \, \mathrm{d}z, \tag{1.5}$$

where, in contrast with (1.4), the heat conductivity coefficient κ is a continuously differentiable function of the temperature,

$$\kappa = \kappa(\vartheta), \ \underline{\kappa}(1 + \vartheta^2) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^2), \ \underline{\kappa} > 0. \tag{1.6}$$

Finally, we assume that the specific heat at constant volume $c_v > 0$ is constant, and the pressure $p = p(\varrho, \vartheta)$ satisfies

$$p(\varrho,\vartheta) = a\varrho^\gamma + b\varrho + \varrho\vartheta,\ a,b > 0,\ \gamma > 3. \tag{1.7}$$

Remark 1.1 *The specific form of the constitutive relations (1.5 – 1.7) is inspired by similar assumptions introduced in [8]. These hypotheses may be seen as the simplest possible that are still tractable by the available analytical methods. In particular, the problem (1.1 – 1.7), supplemented with suitable boundary conditions, admits a global-in-time weak solution for any finite energy initial data, see [8, Chapter 7, Theorem 7.1]. In the context of the existence theory developed in [8], the assumption* $\gamma > 3$ *is optimal.*

The system of equations (1.1 – 1.3) is supplemented with the *no-slip boundary condition*

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{1.8}$$

and the *no-flux boundary condition*

$$\mathbf{q}(\vartheta, \nabla_x\vartheta)\cdot\mathbf{n}|_{\partial\Omega} = -\kappa(\vartheta)\nabla_x\vartheta\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{1.9}$$

where the symbol $\mathbf{n}$ stands for the outer normal vector to $\partial\Omega$.

The initial state of the fluid is given by

$$\varrho(0,\cdot) = \varrho_0,\ \vartheta(0,\cdot) = \vartheta_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0, \tag{1.10}$$

where, in accordance with their physical interpretation, the functions ϱ_0, ϑ_0 are strictly positive in Ω.

1.2 Numerical analysis

We propose a modification of the numerical method for the Navier-Stokes-Fourier system developed in [10] adapted to the physical domain with a smooth boundary, where the target domain Ω is approximated by a family of polyhedral (numerical) domains $\{\Omega_h\}_{h>0}$. A similar problem has been treated in [9] in the context of *barotropic fluids*, where the original numerical method of Karlsen and Karper [14], [15] has been adapted to the smooth domain setting. In contrast with [9], the presence of the heat equation (1.3), together with the Neumann type boundary condition (1.9), create new difficulties addressed in the present paper.

Motivated by Karper [15], we use a mixed finite-element finite-volume method, where the convective terms are approximated by the standard upwind operator, while the diffusive term in the momentum equation is handled by means of the discontinuous Galerkin method based on the non-conformal finite elements of Cruzeix-Raviart type. Accordingly, we consider an *unfitted tetrahedral mesh* generating a family of numerical domains $\{\Omega_h\}_{h>0}$ such that

$$\Omega \subset \overline{\Omega} \subset \Omega_h \subset \mathcal{U}_h[\Omega] \equiv \left\{ x \in R^3 \ \middle| \ \mathrm{dist}[x, \Omega] < h \right\}, \tag{1.11}$$

see Section 2.2.1 for details.

Since the diffusion coefficient in the heat equation (1.3) is nonlinear, it seems more convenient to use the finite-volume scheme for the discretization of the heat flux as well. In order to prove stability and, more importantly, *consistency* of the resulting numerical method, the underlying mesh should be shape regular in the sense of Eymard et al. [5] and satisfy (1.11) at the same time. Examples of tetrahedral meshes complying with this stipulation were constructed in [13].

The paper is organized as follows. In Section 2, we introduce the the concept of *weak solution* to the Navier-Stokes-Fourier system, together with the necessary numerical framework including the basic notation and properties of the underlying function spaces. In Section 3, we define the numerical method and state our main result concerning convergence towards a weak solution of the Navier-Stokes-Fourier system. Having exhausted the preliminary material, we report certain relations and estimates already obtained in [10]. Section 4 deals with numerical analogues of the renormalized version of the continuity and thermal energy balance as well as discrete version of the total energy balance. Section 5 addresses the issue of stability of the scheme, recalling the uniform bounds necessary for the limit passage. The material in these two sections is presented without proofs, with the references to the relevant parts of [10]. Section 6 is devoted to the problem of consistency and convergence of the scheme mimicking certain steps of the existence theory developed in [8, Chapter 7]. We conclude the paper in Section 7 by showing *unconditional convergence* of the scheme on condition that the numerical solutions remain bounded independently of the step parameter h.

2 Preliminaries, weak solutions, numerical framework

In this section, we collect the preliminary material concerning solvability of the Navier-Stokes-Fourier system and the apparatus of numerical analysis used in the paper.

2.1 Weak solutions

We use the concept of *weak formulation* of the problem (1.1 – 1.10) introduced in [8, Chapter 4]:

Definition 2.1 *A triple of functions* $[\varrho, \vartheta, \mathbf{u}]$ *is a* weak solution *to the problem (1.1 – 1.10) in the space-time cylinder* $(0,T) \times \Omega$ *if:*

$$\varrho \in L^\infty(0,T; L^\gamma(\Omega)), \ \vartheta \in L^2(0,T; L^6(\Omega)), \ \mathbf{u} \in L^2(0,T; W_0^{1,2}(\Omega; R^3)), \tag{2.1}$$

$$\varrho\mathbf{u} \in L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^3)), \ \ \varrho\vartheta \in L^\infty(0,T; L^1(\Omega)); \tag{2.2}$$

$$\varrho \geq 0, \ \ \vartheta > 0 \ \ a.a. \ \ in \ (0,T)\times\Omega; \tag{2.3}$$

$$\int_0^T \int_\Omega \Big[\varrho \partial_t \varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi \Big] \ \mathrm{d}x \ \mathrm{d}t = - \int_\Omega \varrho_0 \varphi(0,\cdot) \ \mathrm{d}x \tag{2.4}$$

for any $\varphi \in C^\infty_c([0,T)\times\overline{\Omega})$;

$$\int_0^T \int_\Omega \Big[\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p(\varrho,\vartheta)\mathrm{div}_x\varphi \Big] \ \mathrm{d}x \ \mathrm{d}t \tag{2.5}$$

$$= \int_0^T \int_\Omega \Big[\mu\nabla_x\mathbf{u} : \nabla_x\varphi + \lambda\mathrm{div}_x\mathbf{u} \ \mathrm{div}_x\varphi \Big] \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega \varrho_0\mathbf{u}_0\cdot\varphi(0,\cdot) \ \mathrm{d}x, \ \ \lambda = \frac{1}{3}\mu + \eta > 0,$$

for any $\varphi \in C^\infty_c([0,T)\times\Omega; R^3)$;

$$\int_0^T \int_\Omega \Big[c_v\Big(\varrho\vartheta\partial_t\varphi + \varrho\vartheta\mathbf{u}\cdot\nabla_x\varphi\Big) - \overline{K(\vartheta)}\Delta\varphi \Big] \ \mathrm{d}x \ \mathrm{d}t \tag{2.6}$$

$$+ \int_0^T \int_\Omega \Big[\mu|\nabla_x\mathbf{u}|^2 + \lambda|\mathrm{div}_x\mathbf{u}|^2 \Big]\varphi \ \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_\Omega \varrho\vartheta\mathrm{div}_x\mathbf{u}\varphi \ \mathrm{d}x \ \mathrm{d}t \leq \int_\Omega c_v\varrho_0\vartheta_0\varphi(0,\cdot) \ \mathrm{d}x$$

for any $\varphi \in C^\infty_c([0,T)\times\overline{\Omega})$, $\varphi \geq 0$, $\nabla_x\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0$, *where*

$$\varrho\overline{K(\vartheta)} = \varrho K(\vartheta); \tag{2.7}$$

the energy inequality

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + c_v\varrho\vartheta + \frac{a}{\gamma-1}\varrho^\gamma + b\varrho\log(\varrho) \right](\tau,\cdot) \ \mathrm{d}x \tag{2.8}$$

$$\leq \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + c_v\varrho_0\vartheta_0 + \frac{a}{\gamma-1}\varrho_0^\gamma + b\varrho_0\log(\varrho_0) \right] \ \mathrm{d}x$$

holds for a.a. $\tau \in (0,T)$.

The *existence* of weak solutions to the Navier-Stokes-Fourier system on an arbitrary time interval $(0,T)$ was proved in [8, Chapter 7, Theorem 7.1]. The interested reader may consult [8] for a thorough discussion concerning the inequalities in (2.6), (2.8) as well as the interpretation of (2.7). Further properties of weak solutions, and, in particular, the problem of weak-strong uniqueness and conditional regularity are discussed in Section 7.

2.2 Mesh, finite elements

In what follows, we make systematically use of the following notation:

$$a \lesssim b \text{ if } a \leq cb,\ c > 0 \text{ a constant},\ a \approx b \text{ if } a \lesssim b \text{ and } b \lesssim a.$$

Here, "constant" means a generic quantity independent of the size of the mesh and the time step used in the numerical scheme as well as other parameters as the case may be.

2.2.1 Mesh

Our numerical scheme is constructed over a family of *polyhedral* domains Ω_h approximating Ω in the sense specified in (1.11). Furthermore, we suppose that each Ω_h admits a *tetrahedral mesh* consisting of a set of compact elements $E \in E_h$, a set of faces $\Gamma \in \Gamma_h$, along with the associated normals $\mathbf{n}$, and a family of control points $x_E \in \mathrm{int}[E]$, enjoying the following properties, cf. Eymard et al. [5, Chapter 3]:

1. The intersection $E \cap F$ of two elements $E, F \in E_h$, $E \neq F$ is either empty or their common face, edge, or vertex.

2. For any $E \in E_h$, $\mathrm{diam}[E] \approx h$, $r[E] \approx h$, where r denotes the radius of the largest sphere contained in E.

3. If E and F are two neighboring elements sharing a common face Γ, then the segment $[x_E, x_F]$ is perpendicular to Γ. We denote

$$d_\Gamma = |x_E - x_F| > 0.$$

Remark 2.1 *If the mesh is well-centered (cf. VanderZee et al. [20], [21]), the point x_E can be taken the center of the circumsphere of the element E. A well centered mesh satisfying (1.11) for a given domain Ω was constructed in [13].*

Remark 2.2 *The hypothesis $x_E \in \mathrm{int}[E]$ can be relaxed by assuming the so-called strict Delaunay condition, see Eymard et al. [5].*

Remark 2.3 *Since our method is based on finite elements of first order, the expected rate of convergence should be the same even if the polygonal approximation of the physical domain is replaced by more sophisticated "curved" elements, cf. Lenoir [16].*

Each face $\Gamma \in \Gamma_h$ is associated with a normal vector $\mathbf{n}$. We shall write Γ_E whenever a face $\Gamma_E \subset \partial E$ is considered as a part of the boundary of the element E. In such a case, the normal vector to Γ_E is always the *outer* normal vector with respect to E. Moreover, for any function g continuous on each element E, we set

$$g^{\rm out}|_\Gamma = \lim_{\delta \to 0+} g(\cdot + \delta \mathbf{n}),\ g^{\rm in}|_\Gamma = \lim_{\delta \to 0+} g(\cdot - \delta \mathbf{n}),\ [[g]]_\Gamma = g^{\rm out} - g^{\rm in}, \{g\}_\Gamma = \frac{1}{2}\left(g^{\rm out} + g^{\rm in} \right). \tag{2.9}$$

For $\Gamma_E \subset \partial E$ we simply write g for $g^{\rm in}$. We also omit the subscript Γ if no confusion arises.

Finally, we distinguish two families of faces,

$$\Gamma_{h,\rm ext} = \left\{ \Gamma \in \Gamma_h \ \Big| \ \Gamma \subset \partial \Omega_h \right\},\ \Gamma_{h,\rm int} = \Gamma_h \setminus \Gamma_{h,\rm ext}.$$

2.2.2 Piecewise linear finite elements

We start by introducing the space of piecewise constant functions

$$Q_h(\Omega_h) = \left\{ v \in L^2(\Omega_h) \ \Big| \ v|_E = a_E \in R \text{ for any } E \in E_h \right\},$$

along with the associated projection

$$\Pi_h^Q : L^1(\Omega_h) \to Q_h(\Omega_h),\ \Pi_h^Q[v] \equiv \hat{v},\ \Pi_h^Q[v]|_E = \frac{1}{|E|}\int_E v \ {\rm d}x \text{ for any } E \in E_h.$$

Recalling Poincaré's inequality

$$\int_E \left| v - \frac{1}{|E|}\int_E v \ {\rm d}x \right|^q \ {\rm d}x \lesssim h^q \int_E |\nabla_x v|^q \ {\rm d}x,$$

we get

$$\left\| v - \Pi_h^Q[v] \right\|_{L^q(\Omega_h)} \lesssim h \|\nabla_x v\|_{L^q(\Omega_h;R^3)},\ \text{for any } v \in W^{1,q}(\Omega_h),\ 1 \le q \le \infty. \tag{2.10}$$

Remark 2.4 *There are other forms of Poincaré's inequality that come handy in what follows:*

$$\int_E \left| v - \frac{1}{|\Gamma_E|}\int_{\Gamma_E} v \ {\rm dS}_x \right|^q \ {\rm d}x \lesssim h^q \int_E |\nabla_x v|^q \ {\rm d}x,\ \textit{for any } \Gamma_E \subset \partial E \tag{2.11}$$

$$\int_{\Gamma_E} \left| v - \frac{1}{|\Gamma_E|}\int_{\Gamma_E} v \ {\rm dS}_x \right|^q \ {\rm dS}_x \lesssim h^{q-1} \int_E |\nabla_x v|^q \ {\rm d}x,\ \textit{for any } \Gamma_E \subset \partial E, \tag{2.12}$$

$1 \le q < \infty$.

In order to establish the consistency of the numerical approximation of the heat flux term in (1.3), we shall need another projection

$$\Pi_h^B : C(\overline{\Omega}_h) \to Q_h(\Omega_h),\ \Pi_v^B[v]|_E = v(x_E),\ E \in E_h.$$

Obviously,

$$\|v - \Pi_h^B[v]\|_{L^\infty(\Omega_h)} \lesssim h \|\nabla_x v\|_{L^\infty(\Omega_h;R^3)} \text{ for any Lipschitz } v. \tag{2.13}$$

Next, we introduce the *Crouzeix-Raviart finite element spaces* (see for instance Brezzi and Fortin [1]):

$$V_h(\Omega_h) = \left\{ v \in L^2(\Omega_h) \ \Big|\ v|_E = \text{affine function},\ E \in E_h,\ \int_\Gamma [[v]] \ \mathrm{dS}_x = 0 \text{ for any } \Gamma \in \Gamma_{h,\mathrm{int}} \right\}, \tag{2.14}$$

$$V_{h,0}(\Omega_h) = \left\{ v \in V_h \ \Big|\ \int_\Gamma v \ \mathrm{dS}_x = 0 \text{ for any } \Gamma \in \Gamma_{h,\mathrm{ext}} \right\}, \tag{2.15}$$

and the projection

$$\Pi_h^V : W^{1,q}(\Omega_h) \to V_h(\Omega_h),\ \int_\Gamma \Pi_h^V[v] \ \mathrm{dS}_x = \int_\Gamma v \ \mathrm{dS}_x \text{ for any } \Gamma \in \Gamma_h.$$

For a differential operator D, we denote

$$D_h v|_E = D(v|_E) \text{ for any } v \text{ differentiable on each element } E \in E_h.$$

It is easy to check that

$$\int_{\Omega_h} \mathrm{div}_h \Pi_h^V[\mathbf{u}] \ w \ \mathrm{d}x = \int_{\Omega_h} \mathrm{div}_h \mathbf{u} \ w \ \mathrm{d}x \text{ for any } w \in Q_h(\Omega_h), \tag{2.16}$$

and

$$\int_\Omega \nabla_h v \cdot \nabla_h \Pi_h^V[\varphi] \ \mathrm{d}x = \int_\Omega \nabla_h v \cdot \nabla_x \varphi \ \mathrm{d}x \text{ for all } v \in V_{h,0}(\Omega),\ \varphi \in W_0^{1,2}(\Omega), \tag{2.17}$$

see Karper [15, Lemma 2.11]. Moreover, as a direct consequence of the shape regularity of the mesh, we record the error estimates

$$\left\| v - \Pi_h^V[v] \right\|_{L^q(\Omega_h)} + h \left\| \nabla_h \left(v - \Pi_h^V[v] \right) \right\|_{L^q(\Omega_h;R^3)} \lesssim h^m \left\| \nabla^m v \right\|_{L^q(\Omega_h;R^{3^m})},\ m = 1,2,\ 1 < q < \infty, \tag{2.18}$$

for any $v \in W^{m,q}(\Omega_h)$, see Crouzeix and Raviart [4], Karper [15, Lemma 2.7].

2.2.3 Upwind

We introduce the standard *upwind* operator $\mathrm{Up}[r,\mathbf{u}]$ defined on a face Γ as

$$\mathrm{Up}[r,\mathbf{u}] = r^{\mathrm{in}}[\tilde{\mathbf{u}}\cdot\mathbf{n}]^{+} + r^{\mathrm{out}}[\tilde{\mathbf{u}}\cdot\mathbf{n}]^{-}, \tag{2.19}$$

where we have denoted

$$[c]^{+} = \max\{c,0\},\ [c]^{-} = \min\{c,0\},\ \tilde{v} = \frac{1}{|\Gamma|}\int_{\Gamma} v \ \mathrm{dS}_x.$$

Such a definition makes sense as soon as $r\in Q_h(\Omega_h)$, $\mathbf{u}\in V_h(\Omega_h;R^3)$ and $\Gamma\in\Gamma_{h,\mathrm{int}}$.

After a bit tedious but straightforward manipulation carried over in full detail in [10, Section 2.4, formula (2.17)], we deduce the formula

$$\int_{\Omega_h} r\mathbf{u}\cdot\nabla_x\phi \ \mathrm{d}x = \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_{\Gamma}\mathrm{Up}[r,\mathbf{u}][[F]] \ \mathrm{dS}_x \tag{2.20}$$

$$+\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}(F-\phi)\ [[r]]\ [\tilde{\mathbf{u}}\cdot\mathbf{n}]^{-} \ \mathrm{dS}_x + \sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\phi r(\mathbf{u}-\tilde{\mathbf{u}})\cdot\mathbf{n} \ \mathrm{dS}_x$$

$$+\int_{\Omega_h}(F-\phi)r\mathrm{div}_h\mathbf{u} \ \mathrm{d}x$$

for any $r,F\in Q_h(\Omega_h)$, $\mathbf{u}\in V_{h,0}(\Omega_h;R^3)$, $\phi\in C^1(\overline{\Omega}_h)$.

Finally, we recall Jensen's inequality in the form

$$\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}|\tilde{v}|^q \ \mathrm{dS}_x \lesssim \sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}|v|^q \ \mathrm{dS}_x,\ 1\le q<\infty \tag{2.21}$$

for any $v\in C(\overline{E})$, $E\in E_h$, together with

$$\sum_{\Gamma\in\Gamma_h}\int_{\Gamma}|v-\tilde{v}|^2 \ \mathrm{dS}_x \lesssim h\int_{\Omega_h}|\nabla_h v|^2 \ \mathrm{d}x \text{ for any } v\in V_{h,0}(\Omega_h;R^3) \tag{2.22}$$

that follows directly from Poincarè's inequality (2.11).

2.2.4 $L^p - L^q$ estimates and traces

Since the mesh is shape regular, we can derive the following estimates by a scaling argument. First, we have

$$\|v\|^q_{L^q(\partial E)} \lesssim \frac{1}{h}\left(\|v\|^q_{L^q(E)} + h^q\|\nabla_x v\|^q_{L^q(E;R^3)}\right),\ 1\le q<\infty \text{ for any } v\in C^1(E); \tag{2.23}$$

whence

$$\|w\|^q_{L^q(\partial E)} \lesssim \frac{1}{h}\|w\|^q_{L^q(E)} \text{ for any } 1 \le q < \infty,\ w \in P_m, \tag{2.24}$$

where P_m denotes the space of polynomials of order m.

Similarly, we obtain

$$\|w\|_{L^p(E)} \lesssim h^{3\left(\frac{1}{p}-\frac{1}{q}\right)}\|w\|_{L^q(E)}\ 1 \le q < p \le \infty,\ w \in P_m, \tag{2.25}$$

and, by virtue of the inequality,

$$\left(\sum a_i^p\right)^{1/p} \le \left(\sum a_i^q\right)^{1/q} \text{ whenever } p \ge q,$$

we deduce

$$\|w\|_{L^p(\Omega_h)} \le ch^{3\left(\frac{1}{p}-\frac{1}{q}\right)}\|w\|_{L^q(\Omega_h)}\ 1 \le q < p \le \infty, \text{ for any } w|_E \in P_m(E),\ E \in E_h. \tag{2.26}$$

There is an analogue of (2.25) and (2.26) for piecewise smooth functions of the time variable $t \in (0,T)$ for the discretization of order Δt. Specifically, we have

$$\|w\|_{L^p(\Delta t)} \lesssim (\Delta t)^{\left(\frac{1}{p}-\frac{1}{q}\right)}\|w\|_{L^q(\Delta t)}\ 1 \le q < p \le \infty, \tag{2.27}$$

and, therefore

$$\|w\|_{L^p(0,T)} \lesssim (\Delta t)^{\left(\frac{1}{p}-\frac{1}{q}\right)}\|w\|_{L^q(0,T)}\ 1 \le q < p \le \infty \tag{2.28}$$

for any w that is constant on any time segment $[j\Delta t, (j+1)\Delta t]$ contained in $[0,T]$.

2.2.5 Discrete Sobolev spaces

For $v \in Q_h(\Omega_h)$, let

$$\|v\|^2_{H^1_{Q_h}(\Omega_h)} = \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{[[v]]^2}{h}\ \mathrm{dS}_x$$

be a discrete analogue of the Sobolev gradient norm. Similarly, we introduce

$$\|v\|^2_{H^1_{V_h}(\Omega_h)} = \int_\Omega \left(|\nabla_h v|^2\right)\ \mathrm{d}x \text{ for } v \in V_h(\Omega_h).$$

Recall that

$$\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[v]]^2\ \mathrm{dS}_x \lesssim h\|v\|^2_{H^1_{V_h}(\Omega_h)} \text{ for any } v \in V_h(\Omega_h), \tag{2.29}$$

see Gallouet et al. [12, Lemma 2.2].

We report a discrete analogue of the standard Sobolev embedding relations:

$$\|v\|_{L^6(\Omega_h)} \stackrel{<}{\sim} \left(\|v\|_{H^1_{Q_h}(\Omega_h)} + \|v\|_{L^2(\Omega_h)} \right), \ v \in Q_h(\Omega_h), \tag{2.30}$$

see Chenais-Hillairet, Droniou [2, Lemma 6.1], and

$$\|v\|_{L^6(\Omega_h)} \stackrel{<}{\sim} \|v\|_{H^1_{V_h}(\Omega_h)}, \ v \in V_{h,0}(\Omega_h), \tag{2.31}$$

see Gallouet et al. [12, Lemma 3.2].

Finally, let $[v]_\delta = v * \omega_\delta$ denote the spatial regularization by a convolution with a family of smooth kernels, specifically,

$$\omega_\delta(y) = \frac{1}{\delta^3}\omega\left(\frac{y}{\delta}\right), \ \omega \in C^\infty_c(\{x \in R^3 \mid |x| < 1\}), \ \omega \geq 0, \ \omega(y) = \omega(|y|), \ \int_{R^3} \omega(y) \ \mathrm{d}y = 1.$$

We have

$$\int_{\{x \in \Omega_h \ \mid \ \mathrm{dist}[x, \partial\Omega_h] > \delta\}} |\nabla_x [v]_\delta|^2 \ \mathrm{d}x \stackrel{<}{\sim} \frac{h}{\delta} \|v\|^2_{H^1_{Q_h}(\Omega_h)} \text{ for any } v \in Q_h(\Omega_h), \tag{2.32}$$

and

$$\int_{\Omega_h} |\nabla_x [v]_\delta|^2 \ \mathrm{d}x \stackrel{<}{\sim} \frac{h}{\delta} \|v\|^2_{H^1_{V_h}(\Omega_h)} \text{ for any } v \in V_{h,0}(\Omega_h) \tag{2.33}$$

provided $0 < \delta \leq h$, see Christiansen et al. [3, Proposition 5.67]. Note that the functions from $V_{h,0}$ can be extended to be zero outside Ω_h so that the regularization is well defined.

3 Numerical scheme, main result

The numerical scheme is formally the same as in [10], the only difference is that the numerical domains Ω_h depend on the discretization step h. For this reason, it is convenient the initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ to be defined on the whole space R^3, $\mathbf{u}_0$ vanishing outside Ω.

We set

$$\varrho^0_h = \Pi^Q_h[\varrho_0] \in Q_h(\Omega_h), \ \vartheta^0_h = \Pi^Q_h[\vartheta_0] \in Q_h(\Omega_h), \ \mathbf{u}^0_h = \Pi^Q_h[\mathbf{u}_0] \in Q_h(\Omega_h; R^3). \tag{3.1}$$

We fix the time step $\Delta t \approx h$ and introduce the discrete time derivative

$$D_t b^k_h = \frac{b^k_h - b^{k-1}_h}{\Delta t}.$$

The numerical solutions $[\varrho^k_h, \vartheta^k_h, \mathbf{u}^k_h]_{h>0}$, $k = 1, 2, \ldots$,

$$\varrho^k_h, \ \vartheta^k_h \in Q_h(\Omega_h), \ \mathbf{u}^k_h \in V_{h,0}(\Omega_h; R^3)$$

are defined successively by means of the numerical method:

Continuity method

$$\int_{\Omega_h} D_t \varrho_h^k \phi \ \mathrm{d}x - \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k, \mathbf{u}_h^k] \ [[\phi]] \ \mathrm{dS}_x + h^\alpha \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h^k]][[\phi]] \ \mathrm{dS}_x = 0 \tag{3.2}$$

for all $\phi \in Q_h(\Omega_h)$, with a parameter $0 < \alpha < 1$;

Momentum method

$$\begin{aligned} &\int_{\Omega_h} D_t (\varrho_h^k \widehat{\mathbf{u}}_h^k) \cdot \phi \ \mathrm{d}x - \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k \widehat{\mathbf{u}}_h^k, \mathbf{u}_h^k] \cdot [[\widehat{\phi}]] \ \mathrm{dS}_x \\ &+ \int_{\Omega_h} \Big[\mu \nabla_h \mathbf{u}_h^k : \nabla_h \phi + \lambda \mathrm{div}_h \mathbf{u}_h^k \mathrm{div}_h \phi \Big] \ \mathrm{d}x - \int_{\Omega_h} p(\varrho_h^k, \vartheta_h^k) \mathrm{div}_h \phi \ \mathrm{d}x \\ &+ h^\alpha \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h^k]] \left\{ \widehat{u_h^k} \right\} \cdot [[\widehat{\phi}]] \ \mathrm{dS}_x = 0 \end{aligned} \tag{3.3}$$

for any $\phi \in V_{h,0}(\Omega_h; R^3)$;

Thermal energy method

$$\begin{aligned} c_v \int_{\Omega_h} D_t (\varrho_h^k \vartheta_h^k) \phi \ \mathrm{d}x - c_v \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k \vartheta_h^k, \mathbf{u}_h^k] \ [[\phi]] \ \mathrm{dS}_x + \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma} [[K(\vartheta_h^k)]] \ [[\phi]] \ \mathrm{dS}_x \\ = \int_{\Omega_h} \Big[\mu |\nabla_h \mathbf{u}_h^k|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h^k|^2 \Big] \phi \ \mathrm{d}x - \int_{\Omega_h} \varrho_h^k \vartheta_h^k \mathrm{div}_h \mathbf{u}_h^k \phi \ \mathrm{d}x \end{aligned} \tag{3.4}$$

for any $\phi \in Q_h(\Omega_h)$.

Remark 3.1 *The terms proportional to h^α are needed for technical reasons explained in detail in [10, Section 7.3]. They represent numerical counterparts of the artificial viscosity regularization used in [8, Chapter 7] and were introduced by Eymard et al. [6] to prove convergence of the momentum method. Note that they can be "eliminated" from the scheme by using the "dissipative" upwind operator introduced in [9].*

Before stating our main result, it is convenient to extend the numerical solution to be defined for *any* $t \in R$. To this end, we set

$$\varrho_h(t,\cdot) = \varrho_h^0,\ \vartheta_h(t,\cdot) = \vartheta_h^0,\ \mathbf{u}_h(t,\cdot) = \mathbf{u}_h^0 \text{ for } t \le 0,$$

$$\varrho_h(t,\cdot) = \varrho_h^k,\ \vartheta_h(t,\cdot) = \vartheta_h^k,\ \mathbf{u}_h(t,\cdot) = \mathbf{u}_h^k \text{ for } t \in [k\Delta t, (k+1)\Delta t),\ k = 1, 2, \dots,$$

and, accordingly, the discrete time derivative of a quantity v_h is

$$D_t v_h(t,\cdot) = \frac{v_h(t) - v_h(t - \Delta t)}{\Delta t},\ t > 0.$$

The main result of the present paper reads as follows:

Theorem 3.1 *Let $\Omega \subset R^3$ be a bounded domain of class C^1 approximated by a family of polyhedral domains $\{\Omega_h\}_{h>0}$ in the sense specified in (1.11), where each Ω_h admits a tetrahedral mesh satisfying the hypotheses introduced in Section 2.2.1. Suppose that $\mu > 0$, $\lambda = \mu/3 + \eta > 0$, and that the pressure $p = p(\varrho, \vartheta)$ and the heat conductivity coefficient $\kappa = \kappa(\vartheta)$ comply with (1.6), (1.7). Let $[\varrho_h, \vartheta_h, \mathbf{u}_h]_{h>0}$ be a family of numerical solutions resulting from the scheme (3.1 - 3.4), with*

$$\Delta t \approx h$$

such that

$$\varrho_h > 0,\ \vartheta_h > 0 \text{ for all } h > 0.$$

Then, at least for a suitable subsequence,

$$\varrho_h \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\Omega)) \text{ and strongly in } L^1((0,T)\times\Omega),$$

$$\vartheta_h \to \vartheta \text{ weakly in } L^2(0,T;L^6(\Omega)),$$

$$\mathbf{u}_h \to \mathbf{u} \text{ weakly in } L^2(0,T;L^6(\Omega;R^3)),\ \nabla_h \mathbf{u}_h \to \nabla_x \mathbf{u} \text{ weakly in } L^2((0,T)\times\Omega;R^{3\times3}),$$

where $[\varrho, \vartheta, \mathbf{u}]$ is a weak solution of the Navier-Stokes-Fourier system (1.1 – 1.10) in $(0,T)\times\Omega$ in the sense of Definition 2.1.

The rest of the paper is basically devoted to the proof of Theorem 3.1. As many steps are essentially the same as in [10] we omit technicalities and focus only on the necessary modifications to accommodate the variable numerical domains.

4 Renormalization

The proof of convergence of the numerical method (3.1 – 3.4) mimicks the principal steps of the existence theory developed in [8] based, among other things, on suitable *renormalization* of both the equation of continuity (1.1) and the heat equation (1.3). At the level of numerical solutions, we can deduce the following (see [10, Sections 4.1, 4.2]):

1. **Renormalized continuity method.**

$$\int_{\Omega_h} D_t b(\varrho_h^k)\phi \ \mathrm{d}x - \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[b(\varrho_h^k), \mathbf{u}_h^k] \ [[\phi]] \ \mathrm{dS}_x + \int_{\Omega_h} \phi \ \left(b'(\varrho_h^k)\varrho_h^k - b(\varrho_h^k) \right) \mathrm{div}_h \mathbf{u}_h^k \ \mathrm{d}x \tag{4.1}$$

$$= - \int_{\Omega_h} \frac{\Delta t}{2} b''(\xi_{\varrho,h}^k) \left(\frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \right)^2 \phi \ \mathrm{d}x - h^\alpha \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \phi \ b''(\overline{\eta}_{\varrho,h}^k)[[\varrho_h^k]]^2 \ \mathrm{dS}_x$$

$$- \frac{1}{2} \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \phi \ b''(\eta_{\varrho,h}^k)[[\varrho_h^k]]^2 |\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}| \ \mathrm{dS}_x$$

for any $\phi \in Q_h(\Omega_h)$, $b \in C^2(0,\infty)$, where

$\xi_{\varrho,h}^k \in \mathrm{co}\{\varrho_h^{k-1}, \varrho_h^k\}$ on each element $E \in E_h$, $\eta_{\varrho,h}^k, \overline{\eta}_{\varrho,h}^k \in \mathrm{co}\{\varrho_h^k, (\varrho_h^k)^{\mathrm{out}}\}$ on each face $\Gamma \in \Gamma_{h,\mathrm{int}}$.

2. **Renormalized thermal energy method.**

$$c_v \int_{\Omega_h} D_t \left(\varrho_h^k \chi(\vartheta_h^k) \right) \phi \ \mathrm{d}x - c_v \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}(\varrho_h^k \chi(\vartheta_h^k), \mathbf{u}_h^k) \ [[\phi]] \ \mathrm{dS}_x \tag{4.2}$$

$$+ \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma} [[K(\vartheta_h^k)]] \ [[\chi'(\vartheta_h^k)\phi]] \ \mathrm{dS}_x$$

$$= \int_{\Omega_h} \left(\mu |\nabla_h \mathbf{u}_h^k|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h^k|^2 \right) \chi'(\vartheta_h^k)\phi \ \mathrm{d}x - \int_{\Omega_h} \chi'(\vartheta_h^k) \varrho_h^k \vartheta_h^k \mathrm{div}_h \mathbf{u}_h^k \phi \ \mathrm{d}x$$

$$- c_v \frac{\Delta t}{2} \int_{\Omega_h} \chi''(\xi_{\vartheta,h}^k) \varrho_h^{k-1} \left(\frac{\vartheta_h^k - \vartheta_h^{k-1}}{\Delta t} \right)^2 \phi \ \mathrm{d}x$$

$$+ \frac{c_v}{2} \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \phi \chi''(\eta_{\vartheta,h}^k)[[\vartheta_h^k]]^2 (\varrho_h^k)^{\mathrm{out}} [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ \mathrm{dS}_x$$

$$- h^\alpha c_v \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h^k]] \ [[\left(\chi(\vartheta_h^k) - \chi'(\vartheta_h^k)\vartheta_k^k \right) \phi]] \ \mathrm{dS}_x$$

for any $\phi \in Q_h(\Omega_h)$, $\chi \in C^2(0,\infty)$, with

$$\xi^k_{\vartheta,h} \in \mathrm{co}\{\vartheta_h^{k-1}, \vartheta_h^k\},\ \ \eta^k_{\vartheta,h} \in \mathrm{co}\{\vartheta_h^k, (\vartheta_h^k)^{\rm out}\}.$$

Finally, exactly as in [10, Section 4.3] we may use (4.1), (4.2) and the momentum method (3.3) to deduce:

- **Total energy balance.**

$$\begin{aligned} D_t \int_{\Omega_h} \left[\frac{1}{2}\varrho_h^k |\widehat{\mathbf{u}}_h^k|^2 + c_v \varrho_h^k \vartheta_h^k + \frac{a}{\gamma-1}\left(\varrho_h^k\right)^\gamma + b\varrho_h^k \log(\varrho_h^k)\right] \,\mathrm{d}x \\ + \frac{\Delta t}{2}\int_{\Omega_h} \left(A\left|\frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t}\right|^2 + \varrho_h^{k-1}\left|\frac{\widehat{\mathbf{u}}_h^k - \widehat{\mathbf{u}}_h^{k-1}}{\Delta t}\right|^2\right) \,\mathrm{d}x \\ - \sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E} (\varrho_h^k)^{\rm out}[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^- \frac{\left|\widehat{\mathbf{u}}_h^k - \left(\widehat{\mathbf{u}}_h^k\right)^{\rm out}\right|^2}{2} \,\mathrm{dS}_x \\ + \frac{A}{2}\sum_{\Gamma\in\Gamma_{h,\rm int}}\int_\Gamma \left(h^\alpha + |\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}|\right)[[\varrho_h^k]]^2 \,\mathrm{dS}_x \le 0 \\ \text{with } A = \min_{\varrho>0}\left\{a\gamma\varrho^{\gamma-2} + \frac{b}{\varrho}\right\} > 0. \end{aligned} \tag{4.3}$$

5 Stability

Similarly to [10, Section 5] we derive uniform bounds on the family of numerical solutions independent of the step h.

5.1 Mass conservation and energy bounds

Taking $\phi \equiv 1$ in the continuity method (3.2) we obtain

$$\int_{\Omega_h} \varrho_h(t,\cdot)\,\mathrm{d}x = \int_{\Omega_h} \varrho_h^0 \,\mathrm{d}x \approx \int_\Omega \varrho_0 \,\mathrm{d}x \text{ for any } h>0, \tag{5.1}$$

meaning the total mass is conserved by the scheme.

The total energy balance (4.3) gives rise to

$$\int_{\Omega_h} \left[\frac{1}{2} \varrho_h |\widehat{\mathbf{u}}_h|^2 + c_v \varrho_h \vartheta_h + \frac{a}{\gamma - 1} (\varrho_h)^\gamma + b \varrho_h \log(\varrho_h) \right] (\tau, \cdot) \, \mathrm{d}x \tag{5.2}$$

$$\leq \int_{\Omega_h} \left[\frac{1}{2} \varrho_h |\widehat{\mathbf{u}}_h|^2 + c_v \varrho_h \vartheta_h + \frac{a}{\gamma - 1} (\varrho_h)^\gamma + b \varrho_h \log(\varrho_h) \right] (\tau, \cdot) \, \mathrm{d}x$$

$$\leq \int_{\Omega_h} \left[\frac{1}{2} \varrho_h^0 |\widehat{\mathbf{u}}_h^0|^2 + c_v \varrho_h^0 \vartheta_h^0 + \frac{a}{\gamma - 1} \left(\varrho_h^0\right)^\gamma + b \varrho_h^0 \log(\varrho_h^0) \right] \mathrm{d}x \equiv E_{0,h}, \ E_{0,h} \lesssim 1.$$

In particular, we deduce the uniform bounds:

$$\sup_{\tau \in (0,T)} \| \sqrt{\varrho_h} \widehat{\mathbf{u}}_h (\tau, \cdot) \|_{L^2(\Omega_h)} \lesssim 1, \tag{5.3}$$

$$\sup_{\tau \in (0,T)} \| \varrho_h \vartheta_h (\tau, \cdot) \|_{L^1(\Omega_h)} \lesssim 1, \tag{5.4}$$

$$\sup_{\tau \in (0,T)} \| \varrho_h [\log \vartheta_h]^+ (\tau, \cdot) \|_{L^1(\Omega_h)} \lesssim 1, \tag{5.5}$$

and

$$\sup_{\tau \in (0,T)} \| \varrho_h (\tau, \cdot) \|_{L^\gamma(\Omega_h)} \lesssim 1, \tag{5.6}$$

independent of $h \to 0$.

We also record the bounds on the numerical dissipation:

$$\sum_{k \geq 0} \int_{\Omega_h} \left[\left| \varrho_h^k - \varrho_h^{k-1} \right|^2 + \varrho_h^{k-1} \left| \widehat{\mathbf{u}}_h^k - \widehat{\mathbf{u}}_h^{k-1} \right|^2 \right] \mathrm{d}x \lesssim 1, \tag{5.7}$$

$$- \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_0^T \int_{\Gamma_E} (\varrho_h)^{\mathrm{out}} [\tilde{\mathbf{u}}_h \cdot \mathbf{n}]^- \left| \widehat{\mathbf{u}}_h - (\widehat{\mathbf{u}}_h)^{\mathrm{out}} \right|^2 \mathrm{dS}_x \, \mathrm{d}t \lesssim 1 \tag{5.8}$$

and

$$\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_0^T \int_\Gamma \left(|\tilde{\mathbf{u}}_h \cdot \mathbf{n}| + h^\alpha \right) [[\varrho_h]]^2 \, \mathrm{dS}_x \, \mathrm{d}t \lesssim 1. \tag{5.9}$$

5.2 Entropy bounds

The bounds resulting from the dissipation mechanism encoded in (3.3), (3.4) are obtained by taking $\chi = \log$, $\phi = 1$ in the renormalized thermal energy method (4.2). Using the fact that

$$\int_{\Omega_h} \varrho_h^k \mathrm{div}_h \mathbf{u}_h^k \, \mathrm{d}x \leq - \int_{\Omega_h} D_t \Big(\varrho_h^k \log(\varrho_h^k) \Big) \, \mathrm{d}x \tag{5.10}$$

(cf. (4.1)), we arrive at

$$c_v \int_{\Omega_h} D_t \left(\varrho_h^k \log(\vartheta_h^k) \right) \ \mathrm{d}x - \int_{\Omega_h} D_t \left(\varrho_h^k \log(\varrho_h^k) \right) \ \mathrm{d}x \geq \tag{5.11}$$

$$- \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma} [[K(\vartheta_h^k)]] \ [[(\vartheta_h^k)^{-1}]] \ \mathrm{dS}_x + \int_{\Omega_h} \left(\mu |\nabla_h \mathbf{u}_h^k|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h^k|^2 \right) \frac{1}{\vartheta_h^k} \ \mathrm{d}x$$

$$+ \frac{\Delta t}{2} c_v \int_{\Omega_h} (\xi_{\vartheta,h}^k)^{-2} \varrho_h^{k-1} \left(\frac{\vartheta_h^k - \vartheta_h^{k-1}}{\Delta t} \right)^2 \ \mathrm{d}x$$

$$- \frac{1}{2} c_v \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} (\eta_{\vartheta,h}^k)^{-2} \left(\vartheta_h^k - \left(\vartheta_h^k \right)^{\mathrm{out}} \right)^2 \left(\varrho_h^k \right)^{\mathrm{out}} [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ \mathrm{dS}_x$$

$$- h^\alpha c_v \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h^k]] \ [[\log(\vartheta_h^k)]] \ \mathrm{dS}_x,$$

where the parameters appearing in the numerical dissipation are the same as in (4.1), (4.2).

Now, exactly as in [10, Section 5], inequality (5.11), together with the bounds already established, gives rise to the following estimates:

$$\sup_{\tau \in (0,T)} \| \varrho_h \log(\vartheta_h)(\tau, \cdot) \|_{L^1(\Omega_h)} \lesssim 1, \tag{5.12}$$

$$\int_0^T \int_{\Omega_h} \frac{1}{\vartheta_h} |\nabla_h \mathbf{u}_h|^2 \ \mathrm{d}x \ \mathrm{d}t \lesssim 1, \tag{5.13}$$

$$\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_0^T \int_\Gamma \frac{[[\vartheta_h^\beta]]^2}{d_\Gamma} \mathrm{dS}_x \ \mathrm{d}t \lesssim 1, \ 0 \leq \beta \leq 1, \ \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_0^T \int_\Gamma \frac{[[\log(\vartheta_h)]]^2}{d_\Gamma} \mathrm{dS}_x \ \mathrm{d}t \lesssim 1, \tag{5.14}$$

and

$$\| \vartheta_h \|_{L^2(0,T;L^6(\Omega_h))} + \| \log(\vartheta_h) \|_{L^2(0,T;L^6(\Omega_h))} \lesssim 1. \tag{5.15}$$

We have also bounds on the numerical dissipation:

$$\sum_{k \geq 0} \int_{\Omega_h} (\xi_{\vartheta,h}^k)^{-2} \varrho_h^{k-1} \left(\vartheta_h^k - \vartheta_h^{k-1} \right)^2 \ \mathrm{d}x \lesssim 1, \ \xi_{\vartheta,h}^k \in \mathrm{co}\{\vartheta_h^{k-1}, \vartheta_h^k\}, \tag{5.16}$$

$$- \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_0^T \int_{\Gamma_E} (\eta_{\vartheta,h})^{-2} [[\vartheta_h]]^2 \left(\varrho_h \right)^{\mathrm{out}} [\tilde{\mathbf{u}}_h \cdot \mathbf{n}]^- \ \mathrm{dS}_x \ \mathrm{d}t \lesssim 1, \ \eta_{\vartheta,h} \in \mathrm{co}\{\vartheta_h, \vartheta_h^{\mathrm{out}}\}. \tag{5.17}$$

5.3 Temperature estimates

Revisiting the thermal energy balance (4.2) for $\chi(\vartheta_h^k) = (\vartheta_h^k)^\beta$, $0 < \beta < 1$, and with the test function $\phi = 1$, we obtain

$$
\begin{aligned}
-\beta \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma} [[K(\vartheta_h)]]\ [[(\vartheta_h)^{\beta-1}]]\ \mathrm{dS}_x + \beta\mu \int_{\Omega_h} \vartheta_h^{\beta-1} |\nabla_h \mathbf{u}_h|^2\ \mathrm{d}x\ \mathrm{d}t \qquad (5.18)\\
+c_v \beta(1-\beta) \frac{\Delta t}{2} \sum_{k=1} \int_{\Omega_h} (\xi_{\vartheta,h}^k)^{\beta-2} \varrho_h^{k-1} \left(\frac{\vartheta_h^k - \vartheta_h^{k-1}}{\Delta t} \right)^2\ \mathrm{d}x \\
+\frac{c_v}{2} \beta(1-\beta) \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_\Gamma (\eta_{\vartheta,h}^k)^{\beta-2} [[\vartheta_h^k]]^2 \left(\varrho_h^k \right)^{\mathrm{out}} [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^-\ \mathrm{dS}_x \\
\lesssim c_v \int_{\Omega_h} D_t(\varrho_h^k (\vartheta_h^k)^\beta)\ \mathrm{d}x + \beta \int_{\Omega_h} \varrho_h^k (\vartheta_h^k)^\beta \mathrm{div}_h \mathbf{u}_h^k\ \mathrm{d}x + h^\alpha c_v (1-\beta) \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h^k]]\ [[(\vartheta_h^k)^\beta]]\ \mathrm{dS}_x.
\end{aligned}
$$

Arguing as in [10, Section 5.3] we deduce from (5.18) the following estimates:

$$
- \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \int_0^T \frac{1}{d_\Gamma} [[K(\vartheta_h)]]\ [[(\vartheta_h)^{\beta-1}]]\ \mathrm{dS}_x \lesssim 1 \text{ for all } 0 < \beta < 1, \tag{5.19}
$$

$$
\sum_{\Gamma \in \Gamma_h} \int_0^T \int_\Gamma \frac{[[\vartheta_h^{1+\frac{\beta}{2}}]]^2}{h}\ \mathrm{dS}_x \lesssim 1 \text{ for all } 0 \le \beta < 1; \tag{5.20}
$$

whence, in accordance with (2.30),

$$
\|\vartheta_h\|_{L^p(0,T;L^q(\Omega_h))} \lesssim 1 \text{ for any } 1 \le p < 3,\ 1 \le q < 9. \tag{5.21}
$$

Finally, returning to the thermal energy method (3.4) with $\phi = 1$, we may use the previous estimates to conclude

$$
\int_0^T \int_{\Omega_h} |\nabla_h \mathbf{u}_h|^2\ \mathrm{d}x\ \mathrm{d}t \lesssim 1, \tag{5.22}
$$

and, in accordance with (2.31),

$$
\|\mathbf{u}_h\|^2_{L^2(0,T;L^6(\Omega_h;R^3))} \lesssim 1. \tag{5.23}
$$

6 Consistency and convergence

Our goal is to check that **(i)** the numerical method is *consistent* with the original weak formulation, **(ii)** the numerical solutions converge, modulo a suitable subsequence, to a weak solution of the problem as stated in Theorem 3.1.

6.1 Consistency

To begin, we claim that the proof of consistency for the continuity method (3.2) and the momentum method (3.3) is exactly the same as in [10, Sections 6.1, 6.2], where the upwind terms may be handled by means of formula (2.20).

6.1.1 Continuity and momentum method

Taking $\Pi_h^Q[\phi]$, $\phi \in C_c^\infty(R^3)$ as a test function in the continuity method (3.2) gives rise to

$$\int_{R^3} \left[D_t \varrho_h - \varrho_h \mathbf{u}_h \cdot \nabla_x \phi \right] \, \mathrm{d}x = \int_{R^3} R_h^1(t, \cdot) \cdot \nabla_x \phi \, \mathrm{d}x \tag{6.1}$$

for any $\phi \in C_c^\infty(R^3)$ provided ϱ_h, $\mathbf{u}_h$ were extended to be zero outside Ω_h. The remainder satisfies

$$\left\| R_h^1 \right\|_{L^2(0,T;L^{\frac{6\gamma}{5\gamma-6}(R^3;R^3)}(R^3;R^3))} \lesssim h^\beta \text{ for some } \beta > 0, \tag{6.2}$$

see [9, Section 6.1].

The choice $\Pi_h^V[\phi]$, $\phi \in C_c^\infty(\Omega; R^3)$ as a test function in the momentum balance (3.3) gives rise to

$$\int_\Omega D_t(\varrho_h \widehat{\mathbf{u}}_h) \cdot \phi \, \mathrm{d}x - \int_\Omega (\varrho_h \widehat{\mathbf{u}}_h \otimes \mathbf{u}_h) : \nabla_x \phi \, \mathrm{d}x \tag{6.3}$$

$$+ \int_\Omega \left[\mu \nabla_h \mathbf{u}_h : \nabla_x \phi + \lambda \mathrm{div}_h \mathbf{u}_h \mathrm{div}_x \phi \right] \, \mathrm{d}x - \int_\Omega p(\varrho_h, \vartheta_h) \mathrm{div}_x \phi \, \mathrm{d}x = \int_\Omega \mathbb{R}_h^2 : \nabla_x \phi \, \mathrm{d}x$$

for any $\phi \in C_c^\infty(\Omega; R^3)$, where the remainder satisfies

$$\left\| \mathbb{R}_h^2 \right\|_{L^1(0,T;L^{\frac{\gamma}{\gamma-1}}(\Omega;R^{3\times 3}))} \lesssim h^\beta \text{ for some } \beta > 0, \tag{6.4}$$

see [9, Section 6.2].

Remark 6.1 *As $\Omega \subset \Omega_h$ for any h and ϕ has compact support in Ω, all terms in (6.3) are well defined.*

6.1.2 Consistency for the thermal energy balance

Instead of working directly with the thermal energy method (3.4), we consider its renormalized variant (4.2). Motivated by [10, Section 6.3], we take the nonlinearities χ belonging to the class

$$\chi \in W^{2,\infty}[0, \infty), \ \chi'(\vartheta) \geq 0, \ \chi''(\vartheta) \leq 0, \quad \chi(\vartheta) = \text{const for all } \vartheta > \vartheta_\chi. \tag{6.5}$$

We start by rewriting

$$\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \frac{1}{d_\Gamma}[[K(\vartheta_h^k)]]\ [[\chi'(\vartheta_h^k)\phi]]\ \mathrm{dS}_x \tag{6.6}$$

$$= \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \frac{1}{d_\Gamma}\left\{\phi\right\}[[K(\vartheta_h^k)]]\ [[\chi'(\vartheta_h^k)]]\mathrm{dS}_x + \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \frac{1}{d_\Gamma}\left\{\chi'\left(\vartheta_h^k\right)\right\}[[K(\vartheta_h^k)]]\ [[\phi]]\mathrm{dS}_x$$

for any $\phi\in Q_h(\Omega_h)$.

Next, take $\phi\in C^2(R^3)$ such that

$$\nabla_x\phi\cdot\mathbf{n}=0 \text{ on } \partial\Omega$$

and use $\Pi_h^B[\phi]$ as a test function in the renormalized thermal energy method (4.2). In view of (6.6), we obtain

$$c_v\int_{\Omega_h} D_t\left(\varrho_h^k\chi(\vartheta_h^k)\right)\Pi_h^B[\phi]\ \mathrm{d}x - c_v\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \mathrm{Up}(\varrho_h^k\chi(\vartheta_h^k),\mathbf{u}_h^k)\ [[\Pi_h^B[\phi]\ ]]\ \mathrm{dS}_x \tag{6.7}$$

$$+\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \frac{1}{d_\Gamma}\left\{\chi'\left(\vartheta_h^k\right)\right\}[[K(\vartheta_h^k)]]\ [[\Pi_h^B[\phi]\ ]]\mathrm{dS}_x$$

$$=\int_{\Omega_h}\left(\mu|\nabla_h\mathbf{u}_h^k|^2+\lambda|\mathrm{div}_h\mathbf{u}_h^k|^2\right)\chi'(\vartheta_h^k)\Pi_h^B[\phi]\ \mathrm{d}x - \int_{\Omega_h}\chi'(\vartheta_h^k)\vartheta_h^k\varrho_h^k\mathrm{div}_h\mathbf{u}_h^k\Pi_h^B[\phi]\ \mathrm{d}x$$

$$-h^\alpha c_v\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma[[\varrho_h^k]]\ [[\left(\chi(\vartheta_h^k)-\chi'(\vartheta_h^k)\vartheta_k^k\right)\Pi_h^B[\phi]\ ]]\ \mathrm{dS}_x + \left\langle D_h,\phi\right\rangle,$$

where

$$\left\langle D_h(t),\phi\right\rangle = -\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\frac{1}{d_\Gamma}\left\{\Pi_h^B[\phi]\right\}[[K(\vartheta_h^k)]]\ [[\chi'(\vartheta_h^k)]]\mathrm{dS}_x$$

$$-c_v\frac{\Delta t}{2}\int_{\Omega_h}\chi''(\xi_{\vartheta,h}^k)\varrho_h^{k-1}\left(\frac{\vartheta_h^k-\vartheta_h^{k-1}}{\Delta t}\right)^2\Pi_h^B[\phi]\ \mathrm{d}x$$

$$+\frac{c_v}{2}\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\Pi_h^B[\phi]\chi''(\eta_{\vartheta,h}^k)[[\vartheta_h^k]]^2\left(\varrho_h^k\right)^{\mathrm{out}}[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\ \mathrm{dS}_x.$$

As χ satisfies (6.5), it is easy to check that

$$\left\langle D_h(t),\phi\right\rangle\geq 0 \text{ whenever } \phi\geq 0.$$

Moreover, applying (6.7) with $\phi=1$ we get

$$0\leq\left\langle D_h(t),1\right\rangle\leq h^\alpha c_v\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma[[\varrho_h^k]]\ [[\left(\chi(\vartheta_h^k)-\chi'(\vartheta_h^k)\vartheta_k^k\right)]]\ \mathrm{dS}_x$$

$$+\int_{\Omega_h} \chi'(\vartheta_h^k)\vartheta_h^k\varrho_h^k \mathrm{div}_h \mathbf{u}_h^k \ \mathrm{d}x + c_v \int_{\Omega_h} D_t \left(\varrho_h^k \chi(\vartheta_h^k) \right) \ \mathrm{d}x$$

where the three integrals on the right-hand side are controlled by the estimates (5.4), (5.6), (5.9), (5.14), and (5.22). We may therefore conclude that

$$0 \leq \langle D_h(t), \phi \rangle \stackrel{<}{\sim} R_h^3(t) \|\phi\|_{L^\infty(\Omega_h)}, \ \|R_h^3\|_{L^1(0,T)} \stackrel{<}{\sim} 1 \text{ whenever } \phi \geq 0. \tag{6.8}$$

Remark 6.2 *Note that (6.8) as well as other estimates derived in this section depend on the structural properties of the function χ postulated in (6.5).*

Now, the discrete time derivative can be written as

$$\int_{\Omega_h} D_t \left(\varrho_h^k \chi(\vartheta_h^k) \right) \Pi_h^B[\phi] \ \mathrm{d}x = \int_{\Omega_h} D_t \left(\varrho_h^k \chi(\vartheta_h^k) \right) \phi \ \mathrm{d}x + \int_{\Omega_h} D_t \left(\varrho_h^k \chi(\vartheta_h^k) \right) \left(\Pi_h^B[\phi] - \phi \right) \ \mathrm{d}x,$$

where

$$\int_{\Omega_h} D_t \left(\varrho_h^k \chi(\vartheta_h^k) \right) \left(\Pi_h^B[\phi] - \phi \right) \ \mathrm{d}x =$$

$$\int_{\Omega_h} \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \chi(\vartheta_h^k) \left(\Pi_h^B[\phi] - \phi \right) \ \mathrm{d}x + \int_{\Omega_h} \varrho_h^{k-1} \frac{\chi(\vartheta_h^k) - \chi(\vartheta_h^{k-1})}{\Delta t} \left(\Pi_h^B[\phi] - \phi \right) \ \mathrm{d}x.$$

As χ is bounded and $\Delta t \approx h$, we may use (2.13) to deduce

$$\left| \int_{\Omega_h} \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \chi(\vartheta_h^k) \left(\Pi_h^B[\phi] - \phi \right) \ \mathrm{d}x \right|$$

$$\stackrel{<}{\sim} \left(\Delta t \int_{\Omega_h} \left(\frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \right)^2 \ \mathrm{d}x \right)^{1/2} \sqrt{h} \|\nabla_x \phi\|_{L^\infty(\Omega_h;R^3)},$$

where the right-hand side is controlled by (5.7).

Similarly,

$$\left| \int_{\Omega_h} \varrho_h^{k-1} \frac{\chi(\vartheta_h^k) - \chi(\vartheta_h^{k-1})}{\Delta t} \left(\Pi_h^B[\phi] - \phi \right) \ \mathrm{d}x \right|$$

$$\left| \int_{\Omega_h} \sqrt{\varrho_h^{k-1}} \sqrt{\varrho_h^{k-1}} \frac{\chi(\vartheta_h^k) - \chi(\vartheta_h^{k-1})}{\Delta t} \left(\Pi_h^B[\phi] - \phi \right) \ \mathrm{d}x \right|$$

$$\stackrel{<}{\sim} \left(\Delta t \int_{\Omega_h} \varrho_h^{k-1} \left(\frac{\chi(\vartheta_h^k) - \chi(\vartheta_h^{k-1})}{\Delta t} \right)^2 \ \mathrm{d}x \right)^{1/2} \sqrt{h} \|\nabla_x \phi\|_{L^\infty(\Omega_h;R^3)} \|\varrho_h^{k-1}\|_{L^\gamma(\Omega_h)}^{1/2},$$

which can be bounded by means of (5.16). Indeed it is enough to check that

$$\chi(A) - \chi(B) \lesssim \frac{A-B}{A} \text{ whenever } A>B \geq 0$$

as long as χ belongs to the class (6.5).

Summing up the previous estimates, we may infer that

$$\left| \int_{\Omega_h} D_t \left(\varrho_h \chi(\vartheta_h) \right) \left(\Pi_h^B[\phi] - \phi \right) \ \mathrm{d}x \right| \lesssim \sqrt{h} \ R_h^4(t) \|\nabla_x \phi\|_{L^\infty(\Omega_h;R^3)}, \ \|R_h^4\|_{L^2(0,T)} \lesssim 1. \tag{6.9}$$

To handle the upwind term, we use formula (2.20) yielding

$$\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k \chi(\vartheta_h^k), \mathbf{u}_h^k] \ [[\Pi_h^B[\phi] \ ]] \ \mathrm{dS}_x \tag{6.10}$$

$$= \int_{\Omega_h} \varrho_h^k \chi(\vartheta_h^k) \mathbf{u}_h^k \cdot \nabla_x \phi \ \mathrm{d}x - \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left(\Pi_h^B[\phi] - \phi \right) [[\varrho_h^k \chi(\vartheta_h^k)]] [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ \mathrm{dS}_x$$

$$+ \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \varrho_h^k \chi(\vartheta_h^k) \phi (\tilde{\mathbf{u}} - \mathbf{u}) \cdot \mathbf{n} \ \mathrm{dS}_x + \sum_{E \in E_h} \int_{E_h} \varrho_h^k \chi(\vartheta_h^k) \mathrm{div}_h \mathbf{u}_h^k \left(\phi - \Pi_h^B \phi \right) \ \mathrm{d}x.$$

We write

$$\sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left(\Pi_h^B[\phi] - \phi \right) [[\varrho_h^k \chi(\vartheta_h^k)]] [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ \mathrm{dS}_x =$$

$$\sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left(\Pi_h^B[\phi] - \phi \right) \varrho_h^k [[\chi(\vartheta_h^k)]] [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ \mathrm{dS}_x$$

$$+ \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left(\Pi_h^B[\phi] - \phi \right) [[\varrho_h^k]] \chi((\vartheta_h^k)^{\mathrm{out}}) [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ \mathrm{dS}_x,$$

where, by means of Hölder's and Jensen's inequalities, the error estimates (2.13), and the trace estimates (2.24),

$$\left| \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left(\Pi_h^B[\phi] - \phi \right) (\varrho_h^k) [[\chi(\vartheta_h^k)]] [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ \mathrm{dS}_x \right|$$

$$\lesssim h^{3/2} \|\nabla_x \phi\|_{L^\infty(\Omega_h;R^3)} \left(\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{[[\chi(\vartheta_h^k)]]^2}{h} \ \mathrm{dS}_x \right)^{1/2} \left(\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma |\varrho_h^k|^2 |\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}|^2 \ \mathrm{dS}_x \right)^{1/2}$$

$$\lesssim h^{3/2} \|\nabla_x \phi\|_{L^\infty(\Omega_h;R^3)} \left(\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{[[\chi(\vartheta_h^k)]]^2}{h} \ \mathrm{dS}_x \right)^{1/2} \left(\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma |\varrho_h^k|^2 |\mathbf{u}_h^k|^2 \ \mathrm{dS}_x \right)^{1/2}$$

$$\lesssim h\|\nabla_x\phi\|_{L^\infty(\Omega_h;R^3)}\left(\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\frac{[[\chi(\vartheta_h^k)]]^2}{h}\ \mathrm{dS}_x\right)^{1/2}\left(\sum_{E\in E_h}\int_E|\varrho_h^k|^2|\mathbf{u}_h^k|^2\mathrm{d}x\right)^{1/2}.$$

Now, the relations (5.6), (5.14), and (5.23) may be used to control both integrals on the right-hand side in $L^2(0,T)$.

Furthermore, as χ is bounded, the integral

$$\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left(\Pi_h^B[\phi]-\phi\right)[[\varrho_h^k]]\chi((\vartheta_h^k)^{\mathrm{out}})[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\ \mathrm{dS}_x$$

can be handled with the help of the energy estimate (5.9), (5.23) combined and with the error estimate (2.13).

Finally, we observe that the remaining two integrals on the right-hand side of (6.10) can be estimated by means of (2.11) and the available energy bounds (5.6), (5.22). Thus we conclude that

$$\left|\sum_{\Gamma\in\Gamma_h}\int_\Gamma \mathrm{Up}[\varrho_h^k\chi(\vartheta_h^k),\mathbf{u}_h^k]\ [[\Pi_h^B[\phi]\ ]]\ \mathrm{dS}_x-\int_{\Omega_h}\varrho_h^k\chi(\vartheta_h^k)\mathbf{u}_h^k\cdot\nabla_x\phi\ \mathrm{d}x\right| \tag{6.11}$$

$$\lesssim h^{\frac{\gamma-2}{\gamma}}R_h^5(t)\|\nabla_x\phi\|_{L^\infty(\Omega_h;R^3)},\ \|R_h^5\|_{L^1(0,T)}\lesssim 1.$$

The most delicate part of the proof of consistency of the thermal energy method is the heat-flux term. We need the following auxiliary result.

Lemma 6.1 *Let $\phi\in C^2(R^3)$ such that $\nabla_x\phi\cdot\mathbf{n}|_{\partial\Omega}=0$.*
Then

$$\left|\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_{\Gamma_h}\frac{1}{d_\Gamma}[[v]]\ [[\ \Pi_h^B[\phi]\ ]]\mathrm{dS}_x+\int_\Omega v\Delta\phi\ \mathrm{d}x\right|\lesssim h^{1/2}\left(\|v\|_{H^1_{Q_h}(\Omega_h)}+\|v\|_{L^\infty(\Omega_h)}\right)\|\phi\|_{C^2(R^3)}$$

for any $v\in Q_h(\Omega_h)$.

Proof:
First, by Gauss-Green theorem,

$$\int_{\Omega_h}v\Delta\phi\ \mathrm{d}x=\sum_{E\in E_h}\int_E v\Delta\phi\ \mathrm{dx}=\sum_{E\in E_h}\int_{\partial E}v\nabla_x\phi\cdot\mathbf{n}\ \mathrm{dS}_x$$

$$=-\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma[[v]]\nabla_x\phi\cdot\mathbf{n}\ \mathrm{dS}_x+\int_{\partial\Omega_h}v\nabla_x\phi\cdot\mathbf{n}\ \mathrm{dS}_x,$$

where, furthermore,

$$\left| \int_{\Omega_h} v \Delta \phi \ \mathrm{d}x - \int_{\Omega} v \Delta \phi \ \mathrm{d}x \right| \leq \left| \int_{\Omega_h \setminus \Omega} |v| |\Delta \phi| \ \mathrm{d}x \right| \stackrel{<}{\sim} h \|v\|_{L^\infty(\Omega_h)} \|\phi\|_{C^2(R^3)}. \tag{6.12}$$

Next, going back to the definition of the projection Π_h^B, we get

$$\left| \nabla_x \phi \cdot \mathbf{n} - \frac{[[\Pi_h^B \phi]]}{d_\Gamma} \right| \stackrel{<}{\sim} h \|\phi\|_{C^2(\overline{\Omega})} \text{ on any face } \Gamma,$$

and, by Hölder's inequality,

$$\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma | \ [[v]] \ | \, \mathrm{dS}_x \leq \left(\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{[[v]]^2}{d_\Gamma} \ \mathrm{dS}_x \right)^{1/2} \left(\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma d_\Gamma \mathrm{dS}_x \right)^{1/2} \stackrel{<}{\sim} \|v\|_{H^1_{Q_h}(\Omega)} |\Omega|^{1/2}. \tag{6.13}$$

Thus it remains to control the integral

$$\int_{\partial \Omega_h} v \nabla_x \phi \cdot \mathbf{n} \ \mathrm{dS}_x.$$

To this end, write

$$\int_{\Omega_h \setminus \Omega} v \Delta \phi \ \mathrm{d}x = \sum_{E \in E_h, E \not\subset \overline{\Omega}} \int_{E \setminus \Omega} v \Delta \phi \ \mathrm{d}x,$$

where the left-hand side is small in view of (6.12). Moreover, by Gauss-Green theorem,

$$\sum_{E \in E_h, E \not\subset \overline{\Omega}} \int_{E \setminus \Omega} v \Delta \phi \ \mathrm{d}x = \int_{\partial \Omega_h} v \nabla_x \phi \cdot \mathbf{n} \ \mathrm{dS}_x + \sum_{E \in E_h, E \not\subset \overline{\Omega}} \int_{\partial (E \setminus \Omega) \setminus \partial \Omega_h} v \nabla_x \phi \cdot \mathbf{n} \ \mathrm{dS}_x.$$

Seeing that $\nabla_x \phi \cdot \mathbf{n}|_{\partial \Omega} = 0$ we may infer that

$$\sum_{E \in E_h, E \not\subset \overline{\Omega}} \int_{\partial (E \setminus \Omega) \setminus \partial \Omega_h} v \nabla_x \phi \cdot \mathbf{n} \ \mathrm{dS}_x = - \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}, \ \Gamma \subset \partial E, \ E \not\subset \overline{\Omega}} \int_{\Gamma \setminus \Omega} [[v]] \nabla_x \phi \cdot \mathbf{n} \ \mathrm{dS}_x,$$

where, similarly to (6.13),

$$\left| \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}, \ \Gamma \subset \partial E, \ E \not\subset \overline{\Omega}} \int_{\Gamma \setminus \Omega} [[v]] \nabla_x \phi \cdot \mathbf{n} \ \mathrm{dS}_x \right|$$

$$\stackrel{<}{\sim} \|\phi\|_{C^1(R^3)} \left(\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{[[v]]^2}{d_\Gamma} \ \mathrm{dS}_x \right)^{1/2} \left(\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}, \ \Gamma \subset \partial E, \ E \not\subset \overline{\Omega}} \int_\Gamma d_\Gamma \ \mathrm{dS}_x \right)^{1/2}$$

$$\lesssim \|\phi\|_{C^1(R^3)} \|v\|_{H^1_{Q_h}(\Omega_h)} \left| \left\{ x \in R^3 \ \Big| \ \mathrm{dist}[x, \partial\Omega_h] < 2h \right\} \right|^{1/2} \approx h^{1/2} \|\phi\|_{C^1(R^3)} \|v\|_{H^1_{Q_h}(\Omega_h)}.$$

Q.E.D.

Now, we are ready to deal with the diffusion term

$$\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma} \left\{ \chi' \left(\vartheta_h^k \right) \right\} [[K(\vartheta_h^k)]] \ [[\Pi_h^B[\phi]]] \mathrm{dS}_x.$$

Introducing a new function K_χ,

$$K_\chi'(\vartheta) = \chi'(\vartheta) K'(\vartheta),$$

we rewrite the diffusive term with the help of the mean-value theorem as

$$\left\{ \chi' \left(\vartheta_h^k \right) \right\} [[K(\vartheta_h^k)]] = [[K_\chi(\vartheta_h^k)]] + c_h^k(x) [[\vartheta_n^k]]^2,$$

where c_h^k is uniformly bounded. Consequently, we get

$$\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma} \left\{ \chi' \left(\vartheta_h^k \right) \right\} [[K(\vartheta_h^k)]] \ [[\Pi_h^B[\phi]]] \mathrm{dS}_x$$

$$= \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma} [[K_\chi(\vartheta_h^k)]] \ [[\Pi_h^B[\phi]]] \mathrm{dS}_x + \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma c_h^k \frac{[[\vartheta_h^k]]^2}{d_\Gamma} [[\Pi_h^B[\phi]]] \mathrm{dS}_x.$$

Seeing that

$$|[[\Pi_h^B \phi]]| \leq h \|\nabla_x \phi\|_{L^\infty(\Omega_h; R^3)}$$

we can estimate the last integral using the entropy bounds (5.14), while the first integral can be "replaced" by $\int_\Omega K_\chi(\vartheta_h^k) \Delta\phi \ \mathrm{d}x$ in view of Lemma 6.1.

Finally, observing that the remaining terms in (6.7) can be treated in a similar way, we sum up the previous estimates to obtain

$$\int_{\Omega_h} D_t \left(\varrho_h^k \chi(\vartheta_h^k) \right) \phi \ \mathrm{d}x - \int_{\Omega_h} \varrho_h^k \chi(\vartheta_h^k) \mathbf{u}_h^k \cdot \nabla_x \phi \ \mathrm{d}x - \int_\Omega K_\chi(\vartheta_h^k) \Delta\phi \ \mathrm{d}x \tag{6.14}$$

$$= \int_{\Omega_h} \left(\mu |\nabla_h \mathbf{u}_h^k|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h^k|^2 \right) \chi'(\vartheta_h^k) \phi \ \mathrm{d}x - \int_{\Omega_h} \chi'(\vartheta_h^k) \vartheta_h^k \varrho_h^k \mathrm{div}_h \mathbf{u}_h^k \phi \ \mathrm{d}x + \langle D_h, \phi \rangle + h^\beta \left\langle R_h^6, \phi \right\rangle,$$

for a certain $\beta > 0$, where

$$\left| \left\langle R_h^6(t), \phi \right\rangle \right| \lesssim R_h^7(t) \|\phi\|_{C^2(R^3)}, \ \|R_h^7\|_{L^1(0,T)} \lesssim 1. \tag{6.15}$$

Relation (6.14) holds for any test function $\phi \in C^2(R^2)$ such that $\nabla_x \phi \cdot \mathbf{n}|_{\partial\Omega} = 0$, and any χ enjoying the properties stated in (6.5). The quantity D_h is a bounded measure satisfying (6.8).

We conclude by a simple observation that (6.14) gives rise to

$$\int_{\Omega_h} D_t \left(\varrho_h^k \chi(\vartheta_h^k) \right) \phi \ \mathrm{d}x - \int_{\Omega} \varrho_h^k \chi(\vartheta_h^k) \mathbf{u}_h^k \cdot \nabla_x \phi \ \mathrm{d}x - \int_{\Omega} K_\chi(\vartheta_h^k) \Delta \phi \ \mathrm{d}x \tag{6.16}$$

$$= \int_{\Omega} \left(\mu |\nabla_h \mathbf{u}_h^k|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h^k|^2 \right) \chi'(\vartheta_h^k) \phi \ \mathrm{d}x - \int_{\Omega} \chi'(\vartheta_h^k) \vartheta_h^k \varrho_h^k \mathrm{div}_h \mathbf{u}_h^k \phi \ \mathrm{d}x + \langle D_h, \phi \rangle + h^\beta \left\langle R_h^6, \phi \right\rangle,$$

where the integrals over the complements $\Omega_h \setminus \Omega$ were incorporated in D_h and R_h^6. As for the discrete time derivative, we claim that

$$\int_0^T \psi(t) \int_{\Omega_h} D_t \left(\varrho_h \chi(\vartheta_h) \right) \phi \ \mathrm{d}x \ \mathrm{d}t \tag{6.17}$$

$$= \psi(0) \int_{\Omega_h} \varrho_h^0 \chi(\vartheta_h^0) \phi \ \mathrm{d}x - \int_0^T \int_{\Omega_h} \left(\frac{\psi(t + \Delta t) - \psi(t)}{\Delta t} \right) \varrho_h \chi(\vartheta_h) \phi \ \mathrm{d}x$$

for any $\psi \in C_c^\infty[0, T)$, where, by the mean-value theorem,

$$\left| \left(\frac{\psi(t + \Delta t) - \psi(t)}{\Delta t} \right) - \partial_t \psi \right| \lesssim \Delta t \sup_{s \in [0,T]} |\psi''(s)|.$$

Thus, with (6.17) in mind, we observe that (6.16) coincides with its analogue proved in [10, Section 6.3, formula (6.25)].

6.2 Convergence

As observed above, the consistency formulation (6.3), (6.4), (6.16), and (6.17) is the same as in [10]; whence the proof of convergence can be carried over by means of the arguments specified in [10, Section 7]. We have proved Theorem 3.1.

7 Unconditional convergence

If the initial data $[\varrho_0, \vartheta_0, \mathbf{u}_0]$ are regular and the physical domain has sufficiently smooth boundary, the Navier-Stokes-Fourier system is known to admit strong solutions, at least on a possibly short time interval. If, for instance,

$$\varrho_0, \ \vartheta_0 \in W^{3,2}(\Omega), \ \varrho_0 > 0, \ \vartheta_0 > 0, \ \mathbf{u}_0 \in W^{3,2}(\Omega; R^3) \tag{7.1}$$

are the initial data satisfying the relevant *compatibility conditions*, and if Ω is of class $C^{2+\nu}$, then the problem (1.1 – 1.10) admits a (classical) solution

$$\varrho, \vartheta \in C([0, T_{\max}); W^{3,2}(\Omega)), \ \mathbf{u}_0 \in C([0, T_{\max}); W^{3,2}(\Omega; R^3)) \tag{7.2}$$

on a maximal time interval $[0, T_{\max})$, see Valli [17], [18] , Valli and Zajaczkowski [19].

On the other hand, as shown in [8, Chapter 7], the problem (1.1 – 1.10) endowed with the regular initial data (7.1) possesses a global in time weak solution in the sense of Definition 2.1. Weak and strong solutions emanating from the same initial data *should* coincide on their common existence time interval. As a matter of fact, the answer is not completely straightforward, however, the following result holds, see [7, Lemma 3.2].

Proposition 7.1 *In addition to the hypotheses of Theorem 3.1, suppose that $\Omega \subset R^3$ is a bounded domain of class $C^{2+\nu}$, and that the initial data satisfy (7.1). Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Navier-Stokes-Fourier system (1.1 – 1.10) enjoying extra regularity*

$$\varrho, \ \vartheta, \ \mathrm{div}_x \mathbf{u} \in L^\infty((0, T) \times \Omega), \ \mathbf{u} \in L^\infty((0, T) \times \Omega; R^3).$$

Then $[\varrho, \vartheta, \mathbf{u}]$ coincides with the strong solution of the same problem as long as the latter exists.

It turns out that the weak solutions possessing the regularity claimed in Proposition 7.1 are in fact strong. More specifically, we report the following assertion, see [7, Theorem 2.2]:

Proposition 7.2 *Under the hypotheses of Proposition 7.1, let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Navier-Stokes-Fourier system, emanating from regular initial data satisfying (7.1), and enjoying the extra regularity*

$$\varrho, \ \vartheta, \ \mathrm{div}_x \mathbf{u} \in L^\infty((0, T) \times \Omega), \ \mathbf{u} \in L^\infty((0, T) \times \Omega; R^3).$$

Then $[\varrho, \vartheta, \mathbf{u}]$ is a strong (classical) solution of the problem in $(0, T) \times \Omega$.

Combining the previous results with Theorem 3.1, we obtain the following statement concerning *unconditional convergence* of the numerical scheme (3.1 – 3.4).

Theorem 7.1 *In addition to the hypotheses of Theorem 3.1, suppose that $\Omega \subset R^3$ is a bounded domain of class $C^{2+\nu}$, and the initial data satisfy (7.1). Let $[\varrho_h, \vartheta_h, \mathbf{u}_h]_{h>0}$ be a family of numerical solutions constructed by means of the scheme (3.1 – 3.4) such that*

$$\varrho_h > 0, \ \vartheta_h > 0,$$

and, in addition,

$$\varrho_h, \ \vartheta_h, \ |\mathbf{u}_h|, \ |\mathrm{div}_h \mathbf{u}_h| \leq M$$

a.a. in $(0,T) \times \Omega$ *for a certain constant* M *independent of* h.

Then

$$\varrho_h \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\Omega)) \text{ and strongly in } L^1((0,T)\times\Omega),$$

$$\vartheta_h \to \vartheta \text{ weakly in } L^2(0,T;L^6(\Omega)),$$

$$\mathbf{u}_h \to \mathbf{u} \text{ weakly in } L^2(0,T;L^6(\Omega;R^3)),\ \nabla_h \mathbf{u}_h \to \nabla_x \mathbf{u} \text{ weakly in } L^2((0,T)\times\Omega;R^{3\times 3}),$$

where $[\varrho, \vartheta, \mathbf{u}]$ *is the (unique) strong solution of the Navier-Stokes-Fourier system (1.1 – 1.10) in* $(0,T) \times \Omega$ *emanating from the initial data (7.1).*

References

[1] F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods*, volume 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 1991.

[2] C. Chainais-Hillairet and J. Droniou. Finite-volume schemes for noncoercive elliptic problems with Neumann boundary conditions. *IMA J. Numer. Anal.*, **31**:61–85, 2011.

[3] S. H. Christiansen, H. Z. Munthe-Kaas, and B. Owren. Topics in structure-preserving discretization. *Acta Numer.*, **20**:1–119, 2011.

[4] M. Crouzeix and P.A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge*, **7**:33–75, 1973.

[5] R. Eymard, T. Gallouët, and R. Herbin. Finite volume methods. In *Handbook of numerical analysis, Vol. VII*, Handb. Numer. Anal., VII, pages 713–1020. North-Holland, Amsterdam, 2000.

[6] R. Eymard, T. Gallouët, R. Herbin, and J. C. Latché. A convergent finite element-finite volume scheme for the compressible Stokes problem. II. The isentropic case. *Math. Comp.*, **79**(270):649–675, 2010.

[7] Feireisl and Y. Sun. Conditional regularity of very weak solutions to the Navier-Stokes-Fourier system. 2014. Preprint IM Prague.

[8] E. Feireisl. *Dynamics of viscous compressible fluids.* Oxford University Press, Oxford, 2004.

[9] E. Feireisl, T. Karper, and M. Michálek. Convergence of a numerical method for the compressible Navier-Stokes system on general domains. *Numer. Math.*, 2014. Submitted.

[10] E. Feireisl, T. Karper, and A. Novotný. On a convergent numerical scheme for the full Navier-Stokes-Fourier system. *IMA J. Numer. Math.*, 2014. Submitted.

[11] G. Gallavotti. *Foundations of fluid dynamics*. Springer-Verlag, New York, 2002.

[12] T. Gallouet, E. Herbin, and J.-C. Latché. A convergent finite element-finite colume scheme for the compressible Stokes problem. I. The isothermal case. *Math. Comput.*, **78**(267):1333–1352, 2009.

[13] R. Hošek. Tetrahedrization of the 3D space with identical well-centered elements. 2015. Preprint.

[14] K. H. Karlsen and T. K. Karper. A convergent mixed method for the Stokes approximation of viscous compressible flow. *IMA J. Numer. Anal.*, 32(**3**):725–764, 2012.

[15] T. K. Karper. A convergent FEM-DG method for the compressible Navier-Stokes equations. *Numer. Math.*, **125**(3):441–510, 2013.

[16] M. Lenoir. Optimal isoparametric finite elements and error estimates for domains involving curved boundaries. *SIAM J. Numer. Anal.*, **23**(3):562–580, 1986.

[17] A. Valli. A correction to the paper: "An existence theorem for compressible viscous fluids". *Ann. Mat. Pura Appl. (4)*, **132**:399–400 (1983), 1982.

[18] A. Valli. An existence theorem for compressible viscous fluids. *Ann. Mat. Pura Appl. (4)*, **130**:197–213, 1982.

[19] A. Valli and M. Zajaczkowski. Navier-Stokes equations for compressible fluids: Global existence and qualitative properties of the solutions in the general case. *Commun. Math. Phys.*, **103**:259–296, 1986.

[20] E. Vanderzee, A. N. Hirani, D. Guoy, and E. A. Ramos. Well-centered triangulation. *SIAM J. Sci. Comput.*, **31**(6):4497–4523, 2009/10.

[21] E. VanderZee, A. N. Hirani, D. Guoy, V. Zharnitsky, and E. A. Ramos. Geometric and combinatorial properties of well-centered triangulations in three and higher dimensions. *Comput. Geom.*, **46**(6):700–724, 2013.