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Strongly regular family of boundary-fitted tetrahedral meshes of bounded C^2 domains

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STRONGLY REGULAR FAMILY OF BOUNDARY-FITTED TETRAHEDRAL MESHES OF BOUNDED C^2 DOMAINS.

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Abstract. In this paper we show that for any bounded domain of the class C^2 there exists a strongly regular family of boundary-fitted tetrahedral meshes. We adopt a refinement technique introduced by Křížek and modify it so that a refined mesh is again boundary-fitted. An alternative regularity criterion based on similarity with Sommerville tetrahedron is used and shown to be equivalent to other standard criteria. The sequence of regularities during the refinement process is estimated from below and shown to converge to a positive number by virtue of the convergence of q-Pochhammer symbol. The final result takes the form of an implication with an assumption that can be obviously fulfilled for any bounded C^2 domain.

Key words: boundary fitted mesh, strongly regular family, Sommerville tetrahedron, Sommerville regularity criterion, mesh refinement, tetrahedral mesh.

Subj. AMS Class.: 65N30, 65N50.

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1. Introduction

In numerical schemes, approximating PDE problems, smooth domains Ω are often approximated by polyhedral domains Ω_h that are split into tetrahedral meshes. Each such mesh is characterized by a discretization parameter h, bounding from above the size of elements. For convergence proofs, we need this parameter to decrease to zero, usually by decomposition of every element into several smaller ones. Using this process we create a new, finer mesh. However, during this process we need to control the quality of the mesh, mainly the *shape regularity*, excluding the occurrence of extremely flat or prolonged elements, see [3, Section 14.].

Creating such *strongly regular refinement of the mesh* is elementary in 2D, the technique for 3D case was shown by Křížek in [9]. In our work we will have special requirement of the mesh: The vertices of the mesh that lie on the boundary of the polyhedral domain $\partial\Omega_h$ should lie also on the boundary of the smooth domain $\partial\Omega$. We call such mesh *boundary-fitted*. The proof of existence of such refinement for 2D can be found in [8], for 3D we bring the result in this paper.

The motivation for this work emanates from [5], where the authors define a numerical method to compressible Navier–Stokes equation in a strongly regular family of boundary-fitted meshes.

We start with the following three definitions and state the main result afterwards.

Definition 1. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of the class C^2. We denote by $r_\Omega \in \mathbb{R}^+$ the minimal radius of an osculation sphere of $\partial\Omega$ and set $h_0 := \min\{\frac{r_\Omega}{2}, \frac{\alpha}{2}\}$, where α is a lower bound for the mutual distance of two parts of the boundary $\partial\Omega$.

For the exact definition of α we refer to standard Evans' PDE textbook [4, p. 626].

Definition 2. We say that a couple $(\Omega_h, \mathcal{T}_h)$ is an *approximative domain with a boundary-fitted mesh* of Ω, if $\partial\Omega_h$ consists of triangles, vertices of these triangles belong to $\partial\Omega$ and $\mathcal{T}_h$ is a mesh consisting of closed tetrahedral elements K satisfying the following:

- For any element $K \in \mathcal{T}_h$, any of its faces is either a face of some different element $L \in \mathcal{T}_h$, or a face of the polyhedron Ω_h,
- $\operatorname{diam} K \leq h \leq h_0$ for any $K \in \mathcal{T}_h$,
- $\bigcup_{K\in\mathcal{T}_h} K = \overline{\Omega_h}$.

Further, we denote by $\varrho(K)$ the radius of the largest ball contained in element K.

Definition 3. We say that the infinite sequence $\{\mathcal{T}_h\}_{h\to 0}$ is a *family of boundary-fitted meshes*, if for any $\varepsilon > 0$ there exists $h \in (0, \varepsilon)$ such that $\mathcal{T}_h$ is a boundary-fitted mesh in the sense of Definition 2.

In addition, if there exists $\theta_0 > 0$ independent of h such that for any $\mathcal{T}_h$ and any $K \in \mathcal{T}_h$ it holds that

$$\theta(K) := \frac{\varrho(K)}{\operatorname{diam} K} \geq \theta_0,$$

we say that $\{\mathcal{T}_h\}_{h\to 0}$ is a *strongly regular* family.

There are several equivalent definitions of strong regularity, see [2]. We introduce a different regularity criterion and use it later in this work.

Having introduced the basic definitions, we can state the main theorem.

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^3$ of the class C^2. Let for some $h_1 \leq h_0$ there exists $(\Omega_{h_1}, \mathcal{T}_{h_1})$ an approximative domain with boundary-fitted mesh and let*

$$\theta(K) \geq \frac{4b\sqrt{2}}{r_\Omega}\text{diam } K, \tag{1}$$

for any $K \in \mathcal{T}_{h_1}$, where

$$b > b_0 = \frac{8}{\sqrt{3}}(2+\sqrt{5}). \tag{2}$$

Then there exists a strongly regular family of boundary-fitted meshes $\{\mathcal{T}_h\}_{h\to 0}$.

Moreover, there exists a constant $d_\Omega > 0$ depending solely on the geometric properties of $\partial\Omega$ such that for all $x \in \partial\Omega_h$

$$\text{dist}[x, \partial\Omega] \leq d_\Omega h^2. \tag{3}$$

Remark 1. *Note that (2) is equivalent to*

$$\frac{\left(1+\frac{8}{b\sqrt{3}}\right)^2}{2\left(1-\frac{8}{b\sqrt{3}}\right)} < 1. \tag{4}$$

The rest of the paper is devoted to the proof of Theorem 1.

2. Distance of approximative domain

We start with proving the latter part of Theorem 1 concerning the size of the gap between Ω_h and Ω.

Lemma 1. *Let Ω, r_Ω, h_0 be as in Definition 1. Then for any $h \leq h_0$, and Ω_h an approximative domain from Definition 2, for any $x \in \partial\Omega_h$, the following holds:*

$$\text{dist}[x, \partial\Omega] \leq \frac{(\text{diam } E_h^j)^2}{r_\Omega}, \tag{5}$$

if $x \in E_h^j$, where E_h^j is an edge of $\partial\Omega_h$, and

$$\operatorname{dist}[x, \partial\Omega] \leq 2\frac{(\text{diam } T_h^j)^2}{r_\Omega}, \tag{6}$$

if $x \in T_h^j$, where T_h^j is a boundary triangle of $\partial\Omega_h$.

Proof. From definition of C^2-domain we have $\partial\Omega = \bigcup_{i=1}^M \partial\Omega^i$ where $\partial\Omega^i$ are manifolds that are graphs of C^2 functions from subsets of $\mathbb{R}^2$ to $\mathbb{R}$. Let us denote these functions by $G_i, i = 1, \dots, M$. Then, clearly $r_\Omega = (\max_i ||\nabla^2 G_i||_\infty)^{-1}$.

Take any approximative domain Ω_h. From Definition 2, $\partial\Omega_h = \bigcup_j T_h^j$, where T_h^j are triangles with diameter not exceeding h. Take arbitrary $x \in \partial\Omega_h$. Then there is a triangle $T_h^j : x \in T_h^j$. Without loss of generality $T_h^j \subset G_i^{-1}(\partial\Omega^i)$ for some $i = i(j)$. (Actually, it is true up to a rotation and shift of coordinates.)

If x is a vertex, then $\operatorname{dist}[x, \partial\Omega] = 0$ by assumption and both (5), (6) hold.

Let $x \in T_h^j \backslash \{v_1, v_2, v_3\}$ for some boundary triangle T_h^j, where v_1, v_2, v_3 are its vertices. Define g as a restriction of G_i to the line $v_1 x$, for . Then the Taylor expansion gives

$$g(y) = g'(v_1)(y - v_1) + \frac{1}{2}g''(\widetilde{y})(y - v_1)^2, \tag{7}$$

for any y in the line and some $\widetilde{y} \in T_h^j$. Note that $g(v_r) = 0, r \in 1, 2, 3$, as by assumption $v_r \in \partial\Omega$. Further

$$|g''(\widetilde{y})| \leq ||\nabla^2 G_i||_\infty \leq \frac{1}{r_\Omega}. \tag{8}$$

Let x lie on the edge E_h^j of $T_h^j \subset \partial\Omega_h$. Then we can use (7) twice, for $y = x$ and $y = v_2$, which together with estimate (8) gives

$$|g(x)| \leq |g'(v_1)(x - v_1)| + \frac{(\text{diam } E_h^j)^2}{2r_\Omega}, \quad |g'(v_1)(v_2 - v_1)| \leq \frac{(\text{diam } E_h^j)^2}{2r_\Omega},$$

from which we infer $|g(x)| \leq \frac{1}{r_\Omega}(\text{diam } E_h^j)^2$.

Let $x \in \operatorname{int} T_h^j$. Then we use (7) twice, for $y = x$ and $y = e$, where e is the intersection of the line $v_1 x$ with the edge $v_2 v_3$. With the help of (8) we get

$$|g(x)| \leq |g'(v_1)(x - v_1)| + \frac{1}{2r_\Omega}(\text{diam } T_h^j)^2,$$

$$|g'(v_1)(e - v_1)| \leq |g(e)| + \frac{1}{2r_\Omega}(\text{diam } T_h^j)^2.$$

As we already have $|g(e)| \leq \frac{1}{r_\Omega}(\text{diam } T_h^j)^2$ for an edge point e, we can infer $|g(x)| \leq \frac{2}{r_\Omega}(\text{diam } T_h^j)^2$. The proof is concluded by realizing that $\text{dist}[x, \partial\Omega] \leq \text{dist}[x, g(x)] = |g(x)|$. □

Lemma 1 implies the following corollary.

Corollary 1 (h^2 - property)**.** *Let Ω, r_Ω, h_0 be as in Definition 1. Then there exists $d_\Omega > 0$ depending solely on geometrical properties of Ω such that for any $h \leq h_0, \Omega_h$ from Definition 2, for any $x \in \partial\Omega_h$,*

$$\text{dist}[x, \partial\Omega] \leq d_\Omega h^2.$$

Proof. Set $d_\Omega := \frac{2}{r_\Omega}$ in (5) and (6) and recall that $\text{diam } E_h^j \leq \text{diam } T_h^j \leq \text{diam } K \leq h$. □

Note that in this section we worked only the with the approximative domain, no requirements on the mesh were needed.

3. Preliminaries

To prove the existence of a strongly regular family of boundary-fitted meshes, we will use a decomposition of tetrahedron into eight tetrahedra which inherit the regularity estimate. However, it is not the strong regularity condition introduced in Definition 3 that is being preserved. Therefore, we introduce an alternative criterion of regularity.

Before that, we recall some properties of affine transformations that play a crucial role throughout this paper. Some tetrahedra established by the refinement process need to be modified (boundary vertices should be shifted to the smooth boundary) so that their union satisfies the definition of a boundary-fitted mesh (Definition 2). The shift is performed using affine transformations.

The final part of this section is devoted to so-called q-Pochhammer symbols, which will finally ensure the existence of a lower bound on the regularity ratio θ_0 in (3).

3.1. **Affine transformations and singular values.**

An affine transformation F is a one-to-one mapping of a linear vector space to itself, preserving linearity and the *ratio of division*, see e.g. [1, Proposition 2.8.]. Endowing 3D with Euclidean coordinates, F can be represented by a 3×3 regular matrix Q and a shift vector q,

$$F(x) = Qx + q.$$

In what follows, we will be mainly interested in the effects to geometric properties of the objects undergoing the transformation. As the translation vector q cannot affect the shape change, we focus on properties of the matrix Q.

Lemma 2 (Singular Value Decomposition)**.** *Let $Q \in \mathbb{R}^{3\times 3}$ be a nonsingular matrix. Then there exist matrices U, Σ, V satisfying $Q = U\Sigma V^T$, where $U^T U = I$, $V^T V = I$, and Σ is a diagonal matrix of the so-called* singular values $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3)$*, where all three σ_i are positive.*

Moreover, Q transforms a unit sphere into an ellipsoid with semiaxes of the lengths $\sigma_i, i = 1, 2, 3$.

The proof of above assertion can be found in any linear algebra textbook, see for instance [6, Section 7.3.].

From the above lemma we will use mainly $\sigma_{\min} := \min\{\sigma_1, \sigma_2, \sigma_3\}$ and $\sigma_{\max} := \max\{\sigma_1, \sigma_2, \sigma_3\}$, the maximal shrinking and prolongation factors, respectively. In the sequel we write $\sigma_{\min}(F)$ (and $\sigma_{\max}(F)$), minimal (maximal) singular value of affine transformation F, referring to the minimal (maximal) singular value of its matrix Q.

The following lemma provides a tool for estimating singular values of a composition of affine mappings.

Lemma 3. *Let A and B be affine transformations. Then we have*

$$\sigma_{\min}(A \circ B) \geq \sigma_{\min}(A) \cdot \sigma_{\min}(B)$$

and

$$\sigma_{\max}(A \circ B) \leq \sigma_{\max}(A) \cdot \sigma_{\max}(B).$$

3.2. **Sommerville regularity criterion.** An alternative regularity criterion, introduced in this section, measures the similarity of a general tetrahedron to a reference tetrahedron, which is in our case the *Sommerville tetrahedron*, introduced in 1923 in [10].

Definition 4 (Sommerville tetrahedron)**.** Sommerville tetrahedron is any tetrahedron similar to the unit tetrahedron $\widetilde{K}$, which is defined through Euclidean coordinates of its vertices,

$$\widetilde{A} = \left[\frac{1}{2}, 0, 0\right]^T, \quad \widetilde{B} = \left[-\frac{1}{2}, 0, 0\right]^T, \quad \widetilde{C} = \left[0, \frac{1}{2}, \frac{1}{2}\right]^T, \quad \widetilde{D} = \left[0, -\frac{1}{2}, \frac{1}{2}\right]^T.$$

Unit Sommerville tetrahedron $\widetilde{K}$ (see Figure 1) has two opposite edges of the length 1, the other four of the length $\frac{\sqrt{3}}{2}$ and dihedral angles attain the values 60° and 90°. For further use we will need the following characterization of $\widetilde{K}$,

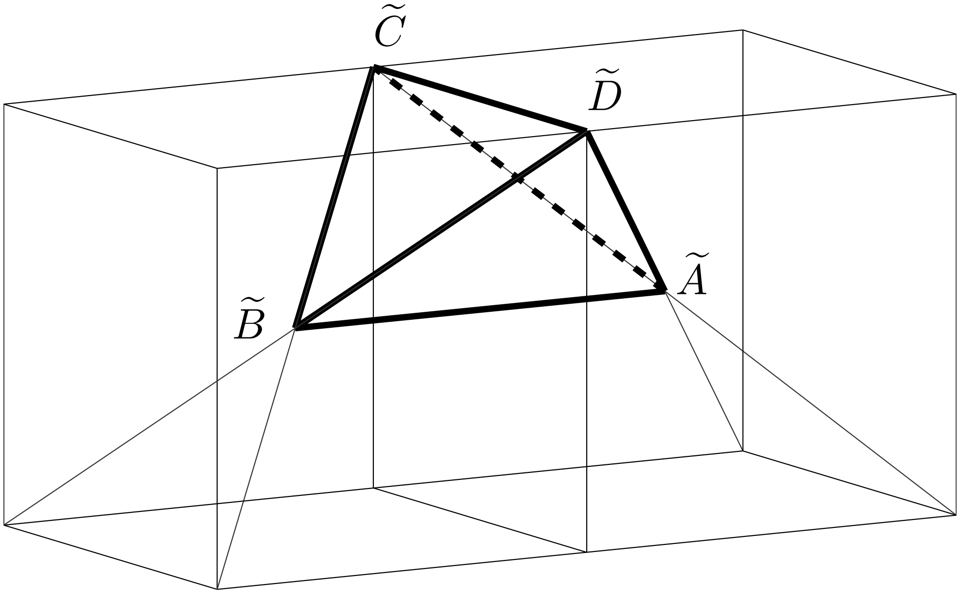

FIGURE 1. Unit Sommerville tetrahedron $\widetilde{K}$ inscribed in two auxilliary cubes. (Axes are omitted for the sake of brevity.)

$$\operatorname{diam}\widetilde{K}=1,\quad e(\widetilde{K})=\frac{\sqrt{3}}{2},\quad \widetilde{\varrho}=\theta(\widetilde{K})=\frac{\sqrt{2}}{8},\quad m(\widetilde{K})=\frac{\sqrt{2}}{2}. \tag{9}$$

where $e(\widetilde{K})$ is the length of the shortest edge, $\widetilde{\varrho}=\varrho(\widetilde{K})$ is the radius of an inscribed sphere and $m(\widetilde{K})$ is the shortest median of a face of Sommerville tetrahedron. For detailed computations, see [7].

Note that for any tetrahedron $K=\operatorname{co}(ABCD)$, there exists unique affine transformation F_K that maps the Sommerville tetrahedron $\widetilde{K}=\operatorname{co}(\widetilde{A}\widetilde{B}\widetilde{C}\widetilde{D})$ onto K, i.e.

$$F_K(\widetilde{x})=Q_K\widetilde{x}+q_K, \tag{10}$$

determined by $F_K(\widetilde{A})=A, F_K(\widetilde{B})=B, F_K(\widetilde{C})=C, F_K(\widetilde{D})=D$. It can be easily shown that $Q_K=[A-B,C-D,C+D-A-B]$ and $q_K=\frac{1}{2}(A+B)$.

However, as we get a different transformation just by relabelling the vertices of tetrahedron K, we must be careful with employing the following alternative regularity criterion.

Definition 5. Let $K=\operatorname{co}(ABCD)$ be a tetrahedron,

$$\mathcal{A}_K := \{F_K; F_K \text{ an affine transformation}, F_K(\widetilde{K}) = K\} \tag{11}$$

be a set of all affine transformations mapping Sommerville tetrahedron $\widetilde{K}$ onto K. Then we define the *Sommerville regularity ratio* of tetrahedron K as

$$\kappa(K) = \max_{F_K \in \mathcal{A}_K} \frac{\sigma_{\min}(F_K)}{\sigma_{\max}(F_K)}, \tag{12}$$

where $\sigma_{\min}(F_K), \sigma_{\max}(F_K)$ are the minimal (and maximal) singular values of F_K.

Note that κ attains its maximum 1 for Sommerville tetrahedron, while the minimal value 0 would be attained for a degenerate tetrahedron. Consequently, κ plays role of a regularity measure.

Remark 2. *Taking regular tetrahedron as the reference one, we could leave out the maximization in (12). However, we prefer Sommerville tetrahedron as its copies tile the 3D space, see* [7], [10]*, while the regular tetrahedron does not.*

We show the equivalence of the Sommerville regularity criterion to other standard criteria in a form of two lemmas that we use directly in the next section.

Lemma 4. *Let $\kappa_0 > 0$ and let there exist a sequence $h_n \to 0$ such that $\{\mathcal{T}_{h_n}\}_{n\in\mathbb{N}}$ is a family of boundary-fitted meshes satisfying*

$$\kappa(K) \geq \kappa_0 > 0,$$

for any $n \in \mathbb{N}$ and any $K \in \mathcal{T}_{h_n}$.

Then $\{\mathcal{T}_{h_n}\}_{n\in\mathbb{N}}$ is a strongly regular family of boundary-fitted meshes.

The proof is strongly based on ideas of Křížek, see [9].

Proof. We take arbitrary $n \in \mathbb{N}$, arbitrary element $K \in \mathcal{T}_{h_n}$ and consider affine function F_K from (11). We denote by $\widetilde{\mathcal{S}}(\widetilde{x_0}, \widetilde{\varrho})$ the inscribed sphere of $\widetilde{K}$, then $F_K(\widetilde{\mathcal{S}}) =: \mathcal{E} \subset K$ is an ellipsoid. Let us label its center with x_0. Taking $r(K)$ as the shortest semi-axis of $\mathcal{E}$, then the sphere $\mathcal{S}(x_0, r(K))$ is contained in K and therefore $\varrho(K) \geq r(K)$.

From the properties of the singular values of an affine transformation we get the estimates $r(K) = \sigma_{\min}(F_K) \cdot \widetilde{\varrho}$ and $\operatorname{diam} K \leq \sigma_{\max}(F_K) \cdot \operatorname{diam} \widetilde{K}$. Hence, we can write

(13)
$$\theta(K) = \frac{\varrho(K)}{\text{diam } K} \geq \frac{r(K)}{\text{diam } K} \geq \frac{\sigma_{\min}(F_K) \cdot \widetilde{\varrho}}{\sigma_{\max}(F_K) \cdot \text{diam } \widetilde{K}} = \kappa(K)\theta(\widetilde{K}),$$

where the last equality holds assuming we take an appropriate F_K that realizes the maximum in (12). By assumption, $\kappa(K) \geq \kappa_0$ and using (13) we can conclude

$$\theta(K) \geq \kappa_0\theta(\widetilde{K}) = \frac{\sqrt{2}}{8}\kappa_0 =: \theta_0$$

for any K in the family of meshes. □

Lemma 5. *Let $s > 0$ and K be a tetrahedron satisfying $\theta(K) \geq s$. Then*

$$\kappa(K) \geq \frac{\sqrt{2}}{8}s.$$

Proof. Setting K into coordinates in such way that its shortest edge belong to the line parallel to the longest edge of the Sommerville tetrahedron, we can write $\varrho(K) \leq \sigma_{\min}(F_K) \cdot \text{diam } \widetilde{K}$. Further, the mapping F_K transforms the inscribed sphere of $\widetilde{K}$ onto an inscribed ellipsoid of K, hence diam $K \geq \sigma_{\max}(F_K)\widetilde{\varrho}$. Therefore,

$$s \leq \theta(K) = \frac{\varrho(K)}{\text{diam } K} \leq \frac{\sigma_{\min}(F_K) \cdot \text{diam } \widetilde{K}}{\sigma_{\max}(F_K) \cdot \varrho(\widetilde{K})} \leq \kappa(K) \cdot \frac{8}{\sqrt{2}}.$$

□

We conclude this part with the following Corollary of Lemma 3.

Corollary 2. *Let K, K' be two tetrahedra, and let S be the an affine transformation that maps K into K'. Then we have*

$$\kappa(K') \geq \kappa(K)\frac{\sigma_{min}(S)}{\sigma_{max}(S)}.$$

3.3. **q-Pochhammer symbol.** Further we prove some properties of the so-called *q-Pochhammer symbol*, which will be the final tool used for showing the existence of a lower bound κ_0.

Definition 6. Let $n \in \mathbb{N}$ and $a, q \in [0, 1]$. The product

$$(a; q)_n := \prod_{j=0}^{n-1}(1 - aq^j),$$

is called the *q-Pochhammer symbol.*

Lemma 6. *Let* $a \in (0,1), q \in (0,1)$. *Then there exists* $P(a,q) > 0$ *such that for any* $n \in \mathbb{N}$

$$(a;q)_n > \lim_{n\to\infty} (a;q)_n = P(a,q).$$

Proof. As $(a;q)_{n+1} = (1 - aq^n) \cdot (a;q)_n$, the sequence is monotonically decreasing. To prove the existence of a positive limit of $(a;q)_n$, it suffices to find its positive lower bound. Consider

$$s_n := \sum_{k=0}^{n-1} \log(1 - aq^k).$$

Clearly $(a;q)_n = \exp s_n$ and using $\log(1 - z) > -\frac{z}{a}$ for $0 \le z < a < 1$, we can estimate

$$s_n > -\sum_{k=0}^{n-1} q^k = -\frac{1 - q^n}{1 - q}. \tag{14}$$

Combining (14) with the monotonicity of both exponential function and of partial sums of geometric series we get

$$(a;q)_n = \exp s_n > \exp\left(-\frac{1 - q^n}{1 - q}\right) > \exp\left(\frac{-1}{1 - q}\right) > 0.$$

□

4. Mesh refinement

In 1982, Křížek proved the following, see [9].

Theorem 2 ([9, Theorem 3.2])**.** *For any polyhedron there exists a strongly regular family of decompositions into tetrahedra.*

For our purpose it is not possible to use this result directly, because the decomposition in [9] creates a mesh that is no longer boundary-fitted, as new vertices on the boundary of polyhedral domain are created and do not lie on $\partial\Omega$, in general. Our idea is to use this decomposition and to modify (i.e. affinely transform) the tetrahedra in the boundary layer to put all boundary vertices to $\partial\Omega$. By virtue of Lemma 1 we show that this change is small in comparison with the diameter of the element and the strong regularity is therefore preserved.

4.1. **Decomposition of a tetrahedron.** We start with the first step, from the proof of Theorem 2 we extract the following lemma.

Lemma 7. *Let $\mathcal{T}_h$ be a mesh of Ω_h. Then for any $K \in \mathcal{T}_h$ there exists its decomposition $\mathcal{D}(K) = \{K_i\}_{i=1}^8$ into eight face-to-face tetrahedra such that vertices of K_i are either vertices of K or midpoints of its edges and for all $i = 1, \ldots, 8$ we have that*

$$\text{diam } K_i \leq \frac{1}{2}\text{diam } K \quad \text{and} \quad \kappa(K_i) \geq \kappa(K). \tag{15}$$

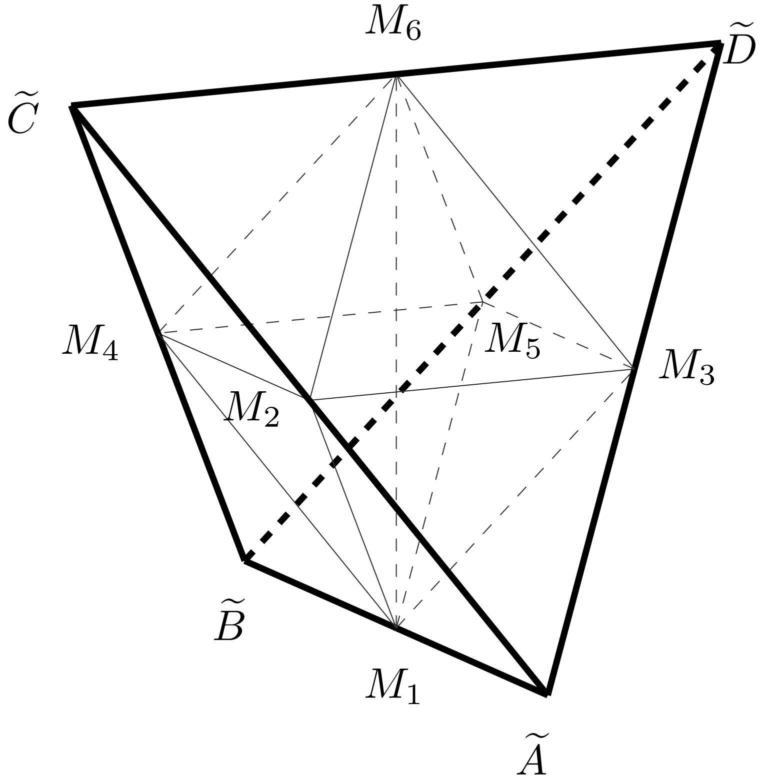

Figure 2. Sketch of Křížek's decomposition of the Sommerville tetrahedron $\widetilde{K}$. Reproduction from [9].

Proof. The Sommerville tetrahedron $\widetilde{K}$ can be decomposed into eight tetrahedra similar to $\widetilde{K}$ - cutting three of four edges in their midpoints creates four tetrahedra and one octahedron which can be decomposed into four identical tetrahedra, see Figure 2 and [9, Proof of Theorem 3.2] or [11, Theorem 4.3]. We denote the decomposition by $\widetilde{\mathcal{D}} = \{\widetilde{K_i}\}_{i=1}^8$

and it follows that diam $\widetilde{K_i} = \frac{1}{2}$. Then we take the affine transformation F_K that realizes $\kappa(K)$. We observe that

$$F_K(\widetilde{\mathcal{D}}) = \{F_K(\widetilde{K_i}), \widetilde{K_i} \in \widetilde{\mathcal{D}}\}_{i=1}^{8}$$

is a decomposition of K.

The key idea is that $\widetilde{K_i}$ are also Sommerville tetrahedra and F_K transforms $\widetilde{K_i}$ into K_i, which implies $\kappa(K_i) \geq \kappa(K)$ for any $K_i \in \mathcal{D}(K)$, since F_K does not have to be the mapping realizing the maximum in $\kappa(K_i)$. The first part of (15) is a consequence of the ratio of division being invariant w.r.t. to affine transformation. □

4.2. **Correction of the decomposition.** The tetrahedra $K_i \in \mathcal{D}(K)$, $K \in \mathcal{T}_h$, do not create a boundary-fitted mesh (according to Definition 2) as new vertices were created on the boundary of polyhedral domain Ω_h that do not belong to the boundary of the smooth domain Ω. To fix that, we apply an affine shift to these vertices. We set the domain of vertices that must be shifted in order to obtain a boundary-fitted mesh,

$$V(\mathcal{T}_h) := \{x \text{ is a vertex of some } K_i \in \mathcal{D}(K), K \in \mathcal{T}_h \text{ and } x \in \partial\Omega_h \backslash \partial\Omega\}.$$

For any $x \in V(\mathcal{T}_h)$ we choose one $y(x) \in \partial\Omega$ such that

$$\operatorname{dist}[x, \partial\Omega] = \operatorname{dist}[x, y(x)]. \tag{16}$$

Then for any $K_i \in \mathcal{D}(K)$ of a given $K \in \mathcal{T}_h$ we consider an affine shift function S_{K_i} defined uniquely by the images of four vertices of the tetrahedron K_i:

$$S_{K_i}(v) = \begin{cases} y(v) & \text{for } v \in V(\mathcal{T}_h), v \text{ vertex of } K_i, \\ v & \text{for } v \notin V(\mathcal{T}_h), v \text{ vertex of } K_i. \end{cases} \tag{17}$$

From Lemma 1, we have an upper bound on the size of this shift. We have to prove that under the assumptions given by Theorem 1, the shift of vertices would not damage the topology of the finer mesh.

Lemma 8. *Let $\Omega, \Omega_h, \mathcal{T}_h$ be as in Definitions 1 and 2. Let v_1, v_2 be distinct vertices of the refined mesh, i.e. $v_i, i = 1, 2$, is either a vertex or a midpoint of an edge of some tetrahedron in $\mathcal{T}_h$. Let*

$$\{tv_1 + (1-t)v_2, t \in (0, t_1)\} \subset K \in \mathcal{T}_h,$$
$$\{tv_1 + (1-t)v_2, t \in (t_2, 1)\} \subset L \in \mathcal{T}_h$$

for some $t_1, t_2 \in (0,1), t_1 \leq t_2$*, and* $K, L \in \mathcal{T}_h$ *not necessarily distinct. Then*

$$\operatorname{dist}[v_1, v_2] \geq \frac{\sqrt{3}}{8}(\sigma_{\min}(F_K) + \sigma_{\min}(F_L)). \tag{18}$$

Proof. Let $K = L$. Then the segment v_1v_2 is either half of an edge, a midsegment of a face triangle, an edge itself, median of a face or a median of a tetrahedron (both v_1, v_2 are midpoints of edges of tetrahedron K). For the first three options, we have clearly $\operatorname{dist}[v_1, v_2] \geq \frac{1}{2}e(K) \geq \frac{1}{2}\frac{\sqrt{3}}{2}\sigma_{\min}(F_K)$. For a median of a triangle we have $\operatorname{dist}[v_1, v_2] \geq m(K) \geq \frac{\sqrt{2}}{2}\sigma_{\min}(F_K)$, as an affine mapping maps median onto median. The same estimates applies for the last option. In both cases we used (9).

Let $K \neq L$. If v_1 is a vertex of K, then we denote by Γ_K the face of K opposite to v_1. Then $\operatorname{dist}[v_1, \Gamma_K] \geq \sigma_{\min}(F_K) \cdot \frac{\sqrt{2}}{2}$, where the last fraction is the (minimal) distance of a vertex from the opposite face in the Sommerville tetrahedron.

In case v_1 is a midpoint of the edge of K, we denote by Γ_K^1, Γ_K^2 the faces of K that do not contain v_1. Then

$$\min_{i=1,2} \operatorname{dist}[v_1, \Gamma_K^i] \geq \sigma_{\min}(F_K)\frac{\sqrt{2}}{4},$$

where the last fraction is the minimal value of such distance in the Sommerville tetrahedron.

Taking minimum over the above listed possibilities, we conclude that (18) holds. □

Lemma 9. *Let for any* $h \leq h_0$*, any* $K \in \mathcal{T}_h$ *satisfies the so-called* minimal regularity condition

$$\kappa(K) \geq b\frac{\operatorname{diam} K}{r_\Omega}, \qquad \text{where } b > b_0 = \frac{8}{\sqrt{3}}(2 + \sqrt{5}). \tag{19}$$

Then for any v_1, v_2 *vertices of* $K_i \in \mathcal{D}(K)$*,* $L_j \in \mathcal{D}(L)$*, respectively, we have that*

$$\operatorname{dist}[v_1, v_2] > \operatorname{dist}[v_1, S_{K_i}(v_1)] + \operatorname{dist}[v_2, S_{L_j}(v_2)],$$

i.e. the shift above does not damage the topological properties of the mesh.

Proof. By construction, if $v_i \in V(\mathcal{T}_h)$, then it is a midpoint of an edge of some boundary triangle T_j^h. By virtue of Lemma 1, in particular from (5), together with (16) and (17) we obtain

$$\frac{1}{r_\Omega}\left((\text{diam } K)^2+(\text{diam } L)^2\right) \geq \text{dist}[v_1, S_{K_i}(v_1)]+\text{dist}[v_2, S_{L_j}(v_2)]. \tag{20}$$

Lemma 8 gives

$$\text{dist}[v_1, v_2] \geq \frac{\sqrt{3}}{8}(\sigma_{\min}(F_K)+\sigma_{\min}(F_L)), \tag{21}$$

where both F_K and F_L realize the maxima in $\kappa(K)$ and $\kappa(L)$, respectively. From the definition of κ and Lemma 2 we have

$$\sigma_{\min}(F_K) = \kappa(K)\sigma_{\max}(F_K) \geq \kappa(K)\text{diam } K\frac{2}{\sqrt{3}} > \kappa(K)\text{diam } K. \tag{22}$$

Using the assumption (19), we can rewrite (22) as

$$\sigma_{min}(F_K)+\sigma_{min}(F_L) \geq b\frac{(\text{diam } K)^2+(\text{diam } L)^2}{r_\Omega}. \tag{23}$$

Substituting (23) into (21), we get

$$\text{dist}[v_1, v_2] > \frac{b\sqrt{3}}{8r_\Omega}\left((\text{diam } K)^2+(\text{diam } L)^2\right), \tag{24}$$

which, combined with (20) concludes the proof, since $\frac{b\sqrt{3}}{8} > 1$. □

Having defined the shift, we focus on the bounds of the singular values of the affine shift, which will be needed in a moment.

Lemma 10. *Let $K \in \mathcal{T}_h$ be a tetrahedron, let $K_i \in \mathcal{D}(K)$ and let the affine shift S_{K_i} be defined with (17). Then for its singular values, we have*

$$\sigma_{\min}(S_{K_i}) \geq 1-\frac{8}{\sqrt{3}r_\Omega}\frac{\text{diam } K}{\kappa(K)}, \tag{25}$$

$$\sigma_{\max}(S_{K_i}) \leq 1+\frac{8}{\sqrt{3}r_\Omega}\frac{\text{diam } K}{\kappa(K)}, \tag{26}$$

and the regularity criterion for the new tetrahedra satisfies the estimate

$$\kappa(S_{K_i}) \geq \frac{1-\frac{8}{\sqrt{3}r_\Omega}\frac{\text{diam } K}{\kappa(K)}}{1+\frac{8}{\sqrt{3}r_\Omega}\frac{\text{diam } K}{\kappa(K)}}\kappa(K) \geq \left(1-\frac{8}{\sqrt{3}r_\Omega}\frac{\text{diam } K}{\kappa(K)}\right)^2\kappa(K). \tag{27}$$

Proof. The maximal singular value of S_{K_i} represents the maximal relative prolongation, which can be achieved at the shortest edge of K_i, i.e. $e(K_i) = \frac{1}{2}e(K)$ by moving the vertices from each other with the maximal radius, i.e.

$$\sigma_{max}(S_{K_i}) \leq \frac{\frac{1}{2}e(K) + \frac{2}{r_\Omega}(\text{diam } K)^2}{\frac{1}{2}e(K)} = 1 + 4\frac{(\text{diam } K)^2}{e(K)r_\Omega}. \tag{28}$$

Using $e(K) \geq e(\widetilde{K})\cdot\sigma_{\min}(F_K)$ and $\text{diam } K \leq \text{diam } \widetilde{K}\cdot\sigma_{\max}(F_K)$, where F_K realizes the maximum in definition of κ, we can deduce that

$$e(K) \geq \kappa(K)\cdot\text{diam } K\frac{e(\widetilde{K})}{\text{diam } \widetilde{K}} = \kappa(K)\cdot\text{diam } K\frac{\sqrt{3}}{2}. \tag{29}$$

Using estimate (29) in (28), we conclude (26). The same steps prove the inequality (25). Then by virtue of Corollary 2 we can estimate

$$\kappa(S_{K_i}(K_i)) \geq \frac{\sigma_{\min}(S_{K_i})}{\sigma_{\max}(S_{K_i})}\kappa(K). \tag{30}$$

The last relation (27) is obtained from (30) using the estimates (25), (26), and the inequality $(1+z)^{-1} \geq (1-z), z \in \mathbb{R}^+$. □

Next we show that with shifting the new vertices to the smooth boundary we do not disturb the uniform decrease of the discretization parameter.

Lemma 11. *Let $h \leq h_0$ and $\mathcal{T}_h$ be a boundary-fitted mesh. Let a tetrahedron $K \in \mathcal{T}_h$ satisfy minimal regularity condition (19) with some admissible b. Then there exists a number $\mu(b) \in (0,1)$, such that for any $K_i \in \mathcal{D}(K)$ it holds that*

$$\text{diam } S_{K_i}(K_i) \leq \mu(b)\cdot\text{diam } K. \tag{31}$$

Proof. From Lemma 7 we recall $\text{diam } K_i \leq \frac{1}{2}\text{diam } K$. From the construction it follows that

$$\text{diam } S_{K_i}(K_i) \leq \frac{\sigma_{\max}(S_{K_i})}{2}\text{diam } K. \tag{32}$$

Substituting from the minimal regularity condition (19) into the upper bound for $\sigma_{\max}(S_{K_i})$ (26) we get the estimate

$$\sigma_{\max}(S_{K_i}) \leq 1 + \frac{8}{b\sqrt{3}}. \tag{33}$$

Then, combining (32) and (33), we conclude that

$$\operatorname{diam} S_{K_i}(K_i) \leq \left(\frac{1}{2} + \frac{4}{b\sqrt{3}}\right) \operatorname{diam} K =: \mu(b) \cdot \operatorname{diam} K.$$

The factor $\mu(b)$ belongs to $(0,1)$ as clearly $b > \frac{8}{\sqrt{3}}$. □

Corollary 3. *Let $h \leq h_0$ and $\mathcal{T}_h$ a boundary-fitted mesh. Let every $K \in \mathcal{T}_h$ satisfy minimal regularity condition (19) with some admissible b. Then*

$$\mathcal{T}_k := \{S_{K_i}(K_i), K_i \in \mathcal{D}(K), K \in \mathcal{T}_h\}$$

is a boundary-fitted mesh in the sense of Definition 2 with

$$k < \left(\frac{1}{2} + \frac{4}{b\sqrt{3}}\right) h. \tag{34}$$

Proof. The construction together with condition (19) ensures that $\mathcal{T}_k$ is a boundary-fitted mesh. Even if every element is transformed by a different affine function, still the common faces (and edges) of two neighbouring elements are transformed identically for both elements, hence the face-to-face property is preserved.

We define k being the maximal diameter of an element in $\mathcal{T}_k$, say L. But clearly this L was created by splitting and shifting of some tetrahedron $M \in \mathcal{T}_h$. Then from Lemma 11 it follows that

$$k = \operatorname{diam} L < \mu(b) \cdot \operatorname{diam} M \leq \mu(b) \cdot h = \left(\frac{1}{2} + \frac{4}{b\sqrt{3}}\right) h.$$

□

Remark 3. *Notice that so far it was sufficient that $b \geq \frac{8}{\sqrt{3}}$. For the next lemma we need the stronger condition (19), indeed.*

Next, we need to show that in the process of refinement, the newly established elements do not violate the minimal regularity condition (19) with given b, which is necessary to allow the repetition of the refinement process.

Lemma 12. *Let K be such that $\kappa(K)$ satisfies condition (19) with some admissible b and let $K_i \in \mathcal{D}(K)$. Then $S_{K_i}(K_i)$ also satisfies (19) with b.*

Proof. We know from (27) that

$$\kappa(S_{K_i}(K_i)) \geq \frac{1 - \frac{8}{\sqrt{3}r_\Omega}\frac{\text{diam } K}{\kappa(K)}}{1 + \frac{8}{\sqrt{3}r_\Omega}\frac{\text{diam } K}{\kappa(K)}}\kappa(K), \tag{35}$$

and from (19)

$$\kappa(K) \geq \frac{b}{r_\Omega}\text{diam } K. \tag{36}$$

Substituting (36) into (35), we get

$$\kappa(S_{K_i}(K_i)) \geq \frac{1 - \frac{8}{b\sqrt{3}}}{1 + \frac{8}{b\sqrt{3}}}\frac{b}{r_\Omega}\text{diam } K. \tag{37}$$

Finally, (34) implies

$$\text{diam } K \geq \frac{2}{1 + \frac{8}{b\sqrt{3}}}\text{diam } S_{K_i}(K_i),$$

which substituted into (37) together with the inequality from Remark 1 recovers (19) with b also for $S_{K_i}(K_i)$. ☐

Theorem 3 (Existence of family.)**.** *Let Ω, h_0 be as in Definition 1 and let for some $h_1 \leq h_0$ there exists a boundary-fitted mesh $\mathcal{T}_{h_1}$ of Ω such that every tetrahedron $K \in \mathcal{T}_{h_1}$ satisfies (19) with some admissible b. Then there exists a family of boundary-fitted meshes $\{\mathcal{T}_{h_n}\}_{n\in\mathbb{N}}$ with $h_n \to 0$.*

Proof. We proceed via mathematical induction. By assumption, for h_1 there exists a boundary-fitted mesh $\mathcal{T}_{h_1}$ with elements satisfying (19) with b.

Corollary 3 gives the following implication. If for h_n there exists boundary-fitted mesh $\mathcal{T}_{h_n}$ with elements satisfying regularity condition (19) with some b, then there exists $h_{n+1} \leq \mu(b)h_n$ such that there exists boundary-fitted mesh $\mathcal{T}_{h_{n+1}}$. By virtue of Lemma 12 all elements of this finer mesh satisfy (19) with b.

The proof is concluded, as we have proven the property for h_1 as well as the induction step. ☐

4.3. **Proof of the Sommerville strong regularity.**

Theorem 4. *Let Ω, h_0 be like in Definition 1. Let for $h_1 \leq h_0$ there exists $\mathcal{T}_{h_1}$ a boundary-fitted mesh to Ω, whose every element satisfies (19) with some admissible b. Then the family $\{\mathcal{T}_{h_n}\}_{n\in\mathbb{N}}$ of boundary-fitted meshes obtained through Theorem 3 is Sommerville strongly regular,* *i.e. there exists* $\kappa_0 > 0$ *such that for any* $n \in \mathbb{N}$*, any* $K \in \mathcal{T}_{h_n}$ *it holds that* $\kappa(K) \geq \kappa_0$*.*

Proof. Consider the family of elements $\{L_n\}_{n\in\mathbb{N}\cup\{0\}}$ such that $L_0 \in \mathcal{T}_{h_1}$ and for any $n \in \mathbb{N}, L_n \in \mathcal{T}_{h_{n+1}}$, and $L_n := S_{K_i}(K_i)$, where $K_i \in \mathcal{D}(L_{n-1})$.

Thanks to Lemma 10 we have

$$\kappa(L_{n+1}) \geq \left(1 - \frac{8}{\sqrt{3}}\frac{\text{diam } L_n}{r_\Omega \kappa(L_n)}\right)^2 \kappa(L_n). \tag{38}$$

Further, we have from Lemma 11 that

$$\text{diam } L_n \leq \frac{1}{2}\left(1 + \frac{8}{b\sqrt{3}}\right) \text{diam } L_{n-1}, \tag{39}$$

and from Lemma 10 combined with (19) also

$$\kappa(L_n) \geq \frac{1 - \frac{8}{b\sqrt{3}}}{1 + \frac{8}{b\sqrt{3}}} \kappa(L_{n-1}). \tag{40}$$

Combining (39) and (40), we get

$$\frac{\text{diam } L_n}{\kappa(L_n)} \leq \frac{1}{2}\frac{\left(1 + \frac{8}{b\sqrt{3}}\right)^2}{1 - \frac{8}{b\sqrt{3}}}\frac{\text{diam } L_{n-1}}{\kappa(L_{n-1})},$$

i.e.

$$\frac{\text{diam } L_n}{\kappa(L_n)} \leq \left(\frac{\left(1 + \frac{8}{b\sqrt{3}}\right)^2}{2\left(1 - \frac{8}{b\sqrt{3}}\right)}\right)^n \frac{\text{diam } L_0}{\kappa(L_0)}.$$

And as the condition (19) holds also for L_0 we have

$$\frac{\text{diam } L_n}{\kappa(L_n)} \leq \left(\frac{\left(1 + \frac{8}{b\sqrt{3}}\right)^2}{2\left(1 - \frac{8}{b\sqrt{3}}\right)}\right)^n \frac{r_\Omega}{b}. \tag{41}$$

Then, substituting (41) to (38), we get

$$\kappa(L_{n+1}) \geq \left(1 - \frac{8}{b\sqrt{3}}\left(\frac{\left(1 + \frac{8}{b\sqrt{3}}\right)^2}{2\left(1 - \frac{8}{b\sqrt{3}}\right)}\right)^n\right) \kappa(L_n).$$

Hence, we can explicitly estimate

$$\kappa(L_{n+1}) \geq \prod_{i=0}^{n} \left(1 - \frac{8}{b\sqrt{3}} \left(\frac{\left(1 + \frac{8}{b\sqrt{3}}\right)^2}{2\left(1 - \frac{8}{b\sqrt{3}}\right)} \right)^i \right) \kappa(L_0). \tag{42}$$

The product on the right-hand side of (42) is a q-Pochhammer symbol with parameters

$$a = \frac{8}{b\sqrt{3}}, \qquad q = \frac{\left(1 + \frac{8}{b\sqrt{3}}\right)^2}{2\left(1 - \frac{8}{b\sqrt{3}}\right)}.$$

Assumption (19) guarantees that $q \in (0,1)$ and also $a \in (0,1)$, see Remark 1. Therefore, we have from Lemma 6 that the right-hand side of (42) has a positive limit $P(a,q) > 0$ for $n \to \infty$ and hence also

$$\kappa(L_n) \geq (a;q)_n \cdot \kappa(L_0) > P(a,q) \cdot \kappa(L_0).$$

We recall that $L_0 \in \mathcal{T}_{h_1}$ and set

$$\kappa_0 := P(a,q) \cdot \min_{L \in \mathcal{T}_{h_1}} \kappa(L),$$

which concludes the proof. ☐

5. Proof of Theorem 1

The final step of the proof is a simple bridging of the main Theorem 1 and Theorem 4.

Proof. As for Lemma 5, the condition (1-2) can be transformed to minimal regularity condition (19). Then we apply Theorem 4 to get the existence of family of boundary-fitted meshes satisfying $\kappa(K) \geq \kappa_0 > 0$ for all tetrahedral elements K in the family of meshes. Then, by virtue of Lemma 4 we conclude the strong regularity of the family.

The estimate (3) is ensured by Corollary 1. ☐

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