On certain models of liquid crystals

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Basic fields in liquid crystal modeling

Bulk velocity

$$\mathbf{v} = \mathbf{v}(t, x), \ \operatorname{div}_x \mathbf{v} = 0$$

Director field description - liquid crystal orientation

$$\mathbf{d} = \mathbf{d}(t, x), \ |\mathbf{d}| = 1$$

Q-tensor desription

$$\mathbb{Q} = \mathbb{Q}(t, x), \ \mathbb{Q} = \mathbb{Q}^T, \ \mathrm{trace}[\mathbb{Q}] = 0$$

Q-tensor system

Field equations (parabolic model)

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = \nu \Delta \mathbf{v} + \mathrm{div}_x \Sigma[\mathbb{Q}]$$

$$\partial_t \mathbb{Q} + \mathbf{v} \cdot \nabla_x \mathbb{Q} - \mathbb{S}[\nabla_x \mathbf{v}, \mathbb{Q}] = \partial \mathcal{G}(\mathbb{Q})$$

General constitutive relations

Constitutive relations

$$\mathbb{S}[\nabla_x \mathbf{v}, \mathbb{Q}] = (\xi\varepsilon(\mathbf{v}) + \omega(\mathbf{v}))\left(\mathbb{Q} + \frac{1}{3}\mathbb{I}\right) + \left(\mathbb{Q} + \frac{1}{3}\mathbb{I}\right)(\xi\varepsilon(\mathbf{v}) - \omega(\mathbf{v}))$$

$$-2\xi\left(\mathbb{Q} + \frac{1}{3}\mathbb{I}\right)\mathbb{Q} : \nabla_x \mathbf{v}$$

$$\Sigma[\mathbb{Q}] = 2\xi\mathbb{H} : \mathbb{Q}\left(\mathbb{Q} + \frac{1}{3}\mathbb{I}\right) - \xi\left[\mathbb{H}\left(\mathbb{Q} + \frac{1}{3}\mathbb{I}\right) - \left(\mathbb{Q} + \frac{1}{3}\mathbb{I}\right)\mathbb{H}\right]$$

$$-(\mathbb{Q}\mathbb{H} - \mathbb{H}\mathbb{Q}) - \nabla_x \mathbb{Q} \odot \nabla_x \mathbb{Q}$$

$$\mathbb{H} = \Delta\mathbb{Q} - \partial\mathcal{G}(\mathbb{Q}), \ \varepsilon(\mathbf{v}) = \nabla_x \mathbf{v} + \nabla_x^t \mathbf{v}, \ \omega(\mathbf{v}) = \nabla_x \mathbf{v} - \nabla_x^t \mathbf{v}$$

Toy models

Model proposed by F.Lin and C.Liu with director field description

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x vcv + \nabla_x \Pi = \nu \Delta \mathbf{v} - \mathrm{div}_x (\nabla_x \mathbf{d} \odot \nabla_x \mathbf{d})$$

$$\partial_t \mathbf{d} + \mathbf{v} \cdot \nabla_x \mathbf{d} - \boxed{\mathbf{d} \cdot \nabla_x \mathbf{v}} = \Delta \mathbf{d} + \partial \mathcal{G}(\mathbf{d})$$

Well-posedness results

F.Lin, C.Liu [weak solutions], **J.Ball** [new approach via penalizing potential], **S.Shkoller** [local existence with stretching term], **M.Paicu, A.Zarnescu** [Q-tensor model], **M.Hieber, M. Nesensohn J.Pruess, K.Schade** [system with temperature, smooth local solutions via maximal regularity], and many others

Toy models revisited

Incompressibility - equation of continuity

$$\mathrm{div}_x \mathbf{v} = 0$$

Momentum equation - "Euler" or "Navier-Stokes" system

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = \nu \Delta \mathbf{v} - \mathrm{div}_x \left(\nabla_x \mathbb{Q} \odot \nabla_x \mathbb{Q} \right), \ \nu \geq 0$$

Q-tensor field equation - parabolic type

$$D_t \mathbb{Q} \equiv \boxed{\partial_t \mathbb{Q} + \mathbf{v} \cdot \nabla_x \mathbb{Q}} = \Delta \mathbb{Q} + \mathcal{F}(\mathbb{Q}) - \lambda \mathbb{Q}$$

Q-tensor field equation - hyperbolic type

$$D_t^2 \mathbb{Q} = \Delta \mathbb{Q} + \mathcal{F}(\mathbb{Q}) - \lambda \mathbb{Q}$$

Basic system of equations revisited

Incompressibility - equation of continuity

$$\mathrm{div}_x \mathbf{v} = 0$$

Momentum equation - "Navier–Stokes" system

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = \nu \Delta \mathbf{v} - \mathrm{div}_x \left(\nabla_x \mathbb{Q} \odot \nabla_x \mathbb{Q} \right)$$

Q-tensor field equation - hyperbolic

$$\partial_t \mathbb{Q} + \mathbf{v} \cdot \nabla_x \mathbb{Q} = \mathbb{P}$$

$$\partial_t \mathbb{P} + \mathbf{v} \cdot \nabla_x \mathbb{P} = \Delta \mathbb{Q} + \mathcal{F}(\mathbb{Q}) - \lambda \mathbb{Q}$$

Local existence of strong solutions

Periodic boundary conditions

$$\Omega = \left([-\pi, \pi]|_{\{-\pi,\pi\}} \right)^N, \ N = 2, 3$$

Sobolev framework

$$W^{s,2}(\Omega)$$

Local existence

$$[\mathbf{v}_0, \mathbb{P}_0, \mathbb{Q}] \in W^{s,2} \times W^{s,2} \times W^{s+1,2}, \ s \geq 4$$

The problem admits a local continuous solution up to a maximal time $T_{\max}$.

Energy and relative energy

Energy - energy balance

$$E(\mathbf{v}, \mathbb{P}, \mathbb{Q}) = \frac{1}{2} \int_\Omega |\mathbf{v}|^2 + |\mathbb{P}|^2 + |\nabla_x \mathbb{Q}|^2 + 2\mathcal{G}(\mathbb{Q}) \ \mathrm{d}x, \ \ \partial G = \lambda \mathbb{I} - \mathcal{F}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\mathbf{v}, \mathbb{P}, \mathbb{Q}) + \nu \int_\Omega |\nabla_x \mathbf{v}|^2 \ \mathrm{d}x \boxed{\leq} 0$$

Relative energy

$$\mathcal{E}\left(\mathbf{v}, \mathbb{P}, \mathbb{Q} \,\middle|\, \tilde{\mathbf{v}}, \tilde{\mathbb{P}}, \tilde{\mathbb{Q}}\right)$$

$$= \frac{1}{2} \int_\Omega \left[|\mathbf{v} - \tilde{\mathbf{v}}|^2 + |\mathbb{P} - \tilde{\mathbb{P}}|^2 + |\nabla_x \mathbb{Q} - \nabla_x \tilde{\mathbb{Q}}|^2 \right] \ \mathrm{d}x$$

$$+ \int_\Omega \left[\mathcal{G}(\mathbb{Q}) - \partial G(\tilde{\mathbb{Q}})(\mathbb{Q} - \tilde{\mathbb{Q}}) - \mathcal{G}(\tilde{\mathbb{Q}}) \right] \ \mathrm{d}x$$

Relative energy inequality, I

Relative energy

$$\left[\mathcal{E}\left(\mathbf{v},\mathbb{P},\mathbb{Q}\;\middle|\;\tilde{\mathbf{v}},\tilde{\mathbb{P}},\tilde{\mathbb{Q}}\right)\right]_{t=0}^{\tau}$$

$$= E(\mathbf{v},\mathbb{P},\mathbb{Q}) + E(\tilde{\mathbf{v}},\tilde{\mathbb{P}},\tilde{\mathbb{Q}}) - \int_\Omega \left[\mathbf{v}\cdot\tilde{\mathbf{v}} + \mathbb{P}:\tilde{\mathbb{P}} + \nabla_x\mathbb{Q}:\nabla_x\tilde{\mathbb{Q}}\right]\ \mathrm{d}x$$

$$-\int_\Omega \left[\partial G(\tilde{\mathbb{Q}}):(\mathbb{Q}-\tilde{\mathbb{Q}}) + 2\mathcal{G}(\tilde{\mathbb{Q}})\right]\ \mathrm{d}x$$

Relative energy inequality, II

$$
\begin{aligned}
&\left[\mathcal{E} \left(\mathbf{v}, \mathbb{P}, \mathbb{Q} \ \middle| \tilde{\mathbf{v}}, \tilde{\mathbb{P}}, \tilde{\mathbb{Q}} \right) \right]_{t=0}^{t=\tau} + \nu \int_0^\tau \int_\Omega |\nabla_x \mathbf{v}|^2 \ \mathrm{d}x\mathrm{d}t \\
&\leq \left[E(\tilde{\mathbf{v}}, \tilde{\mathbb{P}}, \tilde{\mathbb{Q}}) \right]_{t=0}^{t=\tau} \\
&- \int_0^\tau \int_\Omega \left[\mathbf{v} \cdot \partial_t \tilde{\mathbf{v}} - \mathbf{v} \cdot \nabla_x \mathbf{v} \cdot \tilde{\mathbf{v}} - \nu \nabla_x \mathbf{v} : \nabla_x \tilde{\mathbf{v}} + \left(\nabla_x \mathbb{Q} \odot \nabla_x \mathbb{Q} \right) : \nabla_x \tilde{\mathbf{v}} \right] \ \mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left[\mathbb{P} : \partial_t \tilde{\mathbb{P}} + (\mathbf{v} \cdot \mathbb{P}) : \nabla_x \tilde{\mathbb{P}} + \Delta_x \mathbb{Q} : \tilde{\mathbb{P}} - \partial G(\mathbb{Q}) : \tilde{\mathbb{P}} \right] \ \mathrm{d}x \ \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\mathbb{Q} : \partial_t \Delta_x \tilde{\mathbb{Q}} - \mathbf{v} \cdot \nabla_x \mathbb{Q} : \Delta_x \tilde{\mathbb{Q}} + \mathbb{P} : \Delta_x \tilde{\mathbb{Q}} \right] \ \mathrm{d}x \ dt \\
&- \int_0^\tau \int_\Omega \left[\mathbb{Q} : \partial_t \partial \mathcal{G}(\tilde{\mathbb{Q}}) - \mathbf{v} \cdot \nabla_x \mathbb{Q} : \partial \mathcal{G}(\tilde{\mathbb{Q}}) + \mathbb{P} : \partial \mathcal{G}(\tilde{\mathbb{Q}}) \right] \ \mathrm{d}x \ dt \\
&- \int_0^\tau \int_\Omega \partial_t \left(2\mathcal{G}(\tilde{\mathbb{Q}}) - \partial \mathcal{G}(\tilde{\mathbb{Q}}) : \tilde{\mathbb{Q}} \right) \ \mathrm{d}x \ \mathrm{d}t
\end{aligned}
$$

Weak-strong uniqueness

Weak-strong uniqueness

Weak and strong solutions emanating from the same initial data coincide as long as the latter exists

However, weak solutions are (not known) to exist...

Admissible weak solutions

Admissibility principle 1

“Smooth” weak solutions are strong (classical) solutions

Admissibility principle 2 (weak-strong uniqueness)

Weak and strong solution coincide as long as the latter exists

Observation

Local existence of strong solutions implies:
Principle 2 $\Rightarrow$ Principle 1

Weak solutions with a defect measure

Equation of continuity

$$\mathrm{div}_x \mathbf{v} = 0$$

Momentum balance

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = \nu \Delta \mathbf{v} - \mathrm{div}_x (\overline{\nabla_x \mathbb{Q} \times \nabla_x \mathbb{Q}})$$

$$= \nu \Delta \mathbf{v} - \mathrm{div}_x (\nabla_x \mathbb{Q} \times \nabla_x \mathbb{Q}) + \boxed{\mathrm{div}_x \mathbb{M}}$$

Director field equation

$$\partial_t \mathbb{Q} + \mathbf{v} \cdot \nabla_x \mathbb{Q} = \mathbb{P}$$

$$\partial_t \mathbb{P} + \mathbf{v} \cdot \nabla_x \mathbb{P} = \Delta \mathbb{Q} + \mathcal{F}(\mathbb{Q}) - \lambda \mathbb{Q}$$

Energy dissipation defect

Energy inequality

$$\left[E(\mathbf{v}, \mathbb{P}, \mathbb{Q}) \right]_{t=0}^{t=\tau} + \nu \int_0^\tau \int_\Omega |\nabla_x \mathbf{v}|^2 \, \mathrm{d}x + \boxed{D}(\tau) \leq 0$$

Dissipation defect

$$|\mathbb{M}|_{\mathcal{M}([0,\tau]\times\Omega)} \leq cD(\tau)$$

Weak-strong uniqueness

Weak-strong uniqueness

Dissipative solution with a defect measure coincides with the strong solution starting from the same initial data as long as the latter one exists