On diffuse interface models of binary mixtures of compressible fluids

Eduard Feireisl based on joint work with H.Abels (Regensburg)

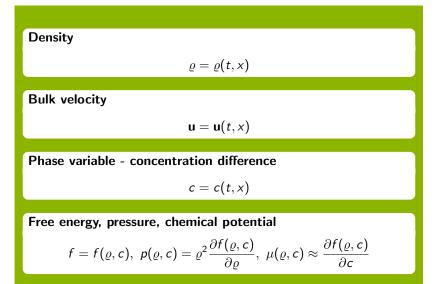
Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

ECM 2016, Berlin, 18 July - 22 July 2016

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078

《口》 《聞》 《臣》 《臣》 三臣

Basic fields in diffuse interface modeling



Model by Anderson, McFadden and Wheeler

Mass conservation - equation of continuity

 $\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = \mathbf{0}$

Momentum equation

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \rho(\rho, c)$$

= $\operatorname{div}_x \mathbb{S}(c, \nabla_x \mathbf{u}) - \operatorname{div}_x \left(\nabla_x c \otimes \nabla_x c - \frac{|\nabla_x c|^2}{2} \mathbb{I} \right)$
 $\mathbb{S}(c, \nabla_x \mathbf{u}) = \nu(c) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \eta(c) \operatorname{div}_x \mathbf{u} \mathbb{I}$

Cahn–Hilliard equation

$$\partial_t(\varrho c) + \operatorname{div}_x(\varrho c \mathbf{u}) = \Delta \mu, \ \varrho \mu = \varrho \frac{\partial f(\varrho, c)}{\partial c} - \Delta c$$

Model by Lowengrub and Truskinovsky

Mass conservation - equation of continuity

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = \mathbf{0}$$

Momentum equation

$$\begin{split} \partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \rho(\varrho, c) \\ = -\operatorname{div}_x\left(\varrho \nabla_x c \otimes \nabla_x c - \varrho \frac{|\nabla_x c|^2}{2} \mathbb{I}\right) \end{split}$$

Cahn-Hilliard equation

$$\partial_t(\varrho c) + \operatorname{div}_x(\varrho c \mathbf{u}) = \Delta \mu, \ \varrho \mu = \varrho \frac{\partial f(\varrho, c)}{\partial c} - \operatorname{div}_x(\varrho \nabla_x c)$$

Existence of weak solution - viscous case

Basic assumptions

$$p(\varrho,c) = p_e(\varrho) + \varrho H(c), \ p_e(\varrho) pprox \varrho^\gamma, \ \gamma > rac{3}{2}$$

Global-in-time weak solutions, H.Abels, EF [Indiana Univ. Math. J. 2008]

The model by Anderson, McFadden and Wheeler (viscous model) admits global-in-time weak solutions for any finite energy initial data

<ロト <部ト <注下 <注下 = 1

Existence of weak solutions - inviscid case

Basic assumption

$$f(\varrho, c) = H(c) + \log(\varrho) \left(lpha_1 rac{1-c}{2} + lpha_2 rac{1+c}{2}
ight)$$

Global-in-time weak solutions, EF [DCDS(S) 2016]

The model by Lowengrub and Truskinovsky (inviscid model) admits *infinitely many* global-in-time weak solutions for any initial data

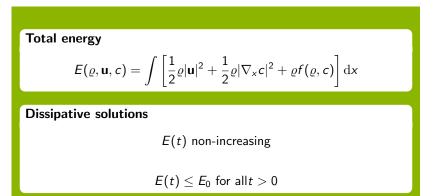
$$\varrho_0, \ \mathbf{u}_0, \ c_0 \in C^3, \ \varrho_0 > 0.$$

<ロト <部ト <注下 <注下 = 1

200

The solutions satisfy $\rho > 0$ (no-vacuum)

Total energy - dissipative solutions



▲ロト ▲母ト ▲ヨト ▲ヨト ヨー のへで

Existence of dissipative solutions

Theorem [EF, IM Preprint 2-2015 (to appear)]

Let $\varrho_0 \in C^3$ be given. Then for a dense (in L^2) set of $c_0 \in C^3$, there exists $u_0 \in L^\infty$ such that the model by Lowengrub and Truskinovsky (inviscid model) admits *infinitely many* global-in-time *dissipative* weak solutions

Abstract formulation

Variable coefficients "Euler system"

$$\begin{split} \partial_t \mathbf{v} + \operatorname{div}_x \left(\frac{(\mathbf{v} + \mathbf{H}[\mathbf{v}]) \odot (\mathbf{v} + \mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} + \mathbb{M}[\mathbf{v}] \right) &= 0\\ \operatorname{div}_x \mathbf{v} &= 0, \end{split}$$

Kinetic energy

$$\frac{1}{2}\frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} = E[\mathbf{v}]$$

Data

$$\mathbf{v}(0,\cdot)=\mathbf{v}_0,\ \mathbf{v}(T,\cdot)=\mathbf{v}_T$$

▲ロト ▲御ト ▲ヨト ▲ヨト ヨー わへで

Abstract operators

Boundedness

b maps bounded sets in $L^\infty((0,T)\times\Omega;R^N)$ on bounded sets in $C_b(Q,R^M)$

Continuity

$$b[\mathbf{v}_n] o b[\mathbf{v}]$$
 in $C_b(Q; R^M)$ (uniformly for $(t, x) \in Q$)

whenever

$$\mathbf{v}_n \rightarrow \mathbf{v}$$
 in $C_{ ext{weak}}([0, T]; L^2(\Omega; R^N))$

Causality

$$\mathbf{v}(t, \cdot) = \mathbf{w}(t, \cdot)$$
 for $0 \le t \le \tau \le T$ implies $b[\mathbf{v}] = b[\mathbf{w}]$ in $[(0, \tau] \times \Omega]$

Subsolutions

Field equations, differential constraints

$$\partial_t \mathbf{v} + \operatorname{div}_{\mathbf{x}} \mathbb{F} = \mathbf{0}, \ \operatorname{div}_{\mathbf{x}} \mathbf{v} = \mathbf{0}$$
$$\mathbf{v}(\mathbf{0}, \cdot) = \mathbf{v}_{\mathbf{0}}, \ \mathbf{v}(\mathcal{T}, \cdot) = \mathbf{v}_{\mathcal{T}}$$

Non-linear constraint

$$\mathbf{v} \in C(Q; R^N), \ \mathbb{F} \in C(Q; R_{\mathrm{sym},0}^{N imes N}),$$

$$\frac{N}{2}\lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{H}[\mathbf{v}])\otimes(\mathbf{v}+\mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} - \mathbb{F} + \mathbb{M}[\mathbf{v}]\right] < E[\mathbf{v}]$$

・ロト ・母ト ・ヨト ・ヨト ・ヨー うへで

Subsolution relaxation

Algebraic inequality

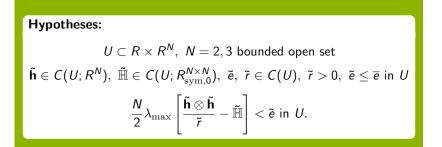
$$\frac{1}{2} \frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} \leq \frac{N}{2} \lambda_{\max} \left[\frac{(\mathbf{v} + \mathbf{H}[\mathbf{v}]) \otimes (\mathbf{v} + \mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} - \mathbb{F} + \mathbb{M}[\mathbf{v}] \right] \\ < E[\mathbf{v}]$$

Solutions

$$\begin{aligned} \frac{1}{2} \frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} &= E[\mathbf{v}] \\ \Rightarrow \\ \mathbb{F} &= \frac{(\mathbf{v} + \mathbf{H}[\mathbf{v}]) \odot (\mathbf{v} + \mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} + \mathbb{M}[\mathbf{v}] \end{aligned}$$

◆ロト ◆母ト ◆臣ト ◆臣ト 三臣 - のへで

Oscillatory lemma



▲ロト ▲母ト ▲ヨト ▲ヨト ヨー のへで

Conclusion:

$$\begin{split} \mathbf{w}_n &\in C_c^{\infty}(U; R^N), \ \mathbb{G}_n \in C_c^{\infty}(U; R_{\mathrm{sym},0}^{N \times N}), \ n = 0, 1, \dots \\ \partial_t \mathbf{w}_n + \operatorname{div}_x \mathbb{G}_n &= 0, \ \operatorname{div}_x \mathbf{w}_n = 0 \ \operatorname{in} \ R \times R^N, \\ \frac{N}{2} \lambda_{\max} \left[\frac{(\tilde{\mathbf{h}} + \mathbf{w}_n) \otimes (\tilde{\mathbf{h}} + \mathbf{w}_n)}{\tilde{r}} - (\tilde{\mathbb{H}} + \mathbb{G}_n) \right] < \tilde{\mathbf{e}} \ \operatorname{in} \ U, \\ \mathbf{w}_n &\to 0 \ \text{weakly in} \ L^2(U; R^N) \\ \lim_{n \to \infty} \int_U \frac{|\mathbf{w}_n|^2}{\tilde{r}} \ \mathrm{d}x \mathrm{d}t \geq \Lambda(\overline{\mathbf{e}}) \int_U \left(\tilde{\mathbf{e}} - \frac{1}{2} \frac{|\tilde{\mathbf{h}}|^2}{\tilde{r}} \right)^2 \ \mathrm{d}x \mathrm{d}t \end{split}$$

Eduard Feireisl based on joint work with H.Abels (Regens Diffuse interface models

Basic ideas of proof

Localization

Localizing the result of DeLellis and Széhelyhidi to "small" cubes by means of scaling arguments

Linearization

Replacing all continuous functions by their means on any of the "small" cubes

Eliminating singular sets

Applying Whitney's decomposition lemma to the non-singular sets (e.g. out of the vacuum $\{h = 0\}$)

Energy and other coefficients depending on solutions

Applying compactness of the abstract operators in C

Results

Result (A)

The set of subsolutions is non-empty \Rightarrow there exists infinitely many weak solutions of the problem with the same initial data

Initial energy jump

$$\frac{1}{2} \frac{|\mathbf{v}_{\mathbf{0}} + \mathbf{H}[\mathbf{v}_{\mathbf{0}}]|^2}{h[\mathbf{v}_{\mathbf{0}}]} \leq \liminf_{t \to 0} \frac{1}{2} \frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]}$$

Result (B)

The set of subsolutions is non-empty \Rightarrow there exists a dense set of times such that the values $\mathbf{v}(t)$ give rise to non-empty subsolution set with

$$\frac{1}{2} \frac{|\mathbf{v}_0 + \mathbf{H}[\mathbf{v}_0]|^2}{h[\mathbf{v}_0]} \equiv \liminf_{t \to 0} \frac{1}{2} \frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]}$$