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Abstract

In the set $B_R \setminus \overline{\mathcal{F}}$ we consider the Dirichlet problem for the anisotropic p-Laplace-type equation. Here $\mathcal{F}$ is an open set of diameter d, $B_R \subset \mathbb{R}^n$ is an open ball of radius $R = R(n, d, p)$, d is small enough. We derive the poitnwise estimates for the solution of this problem in terms of the diameter of the set $\mathcal{F}$ and the distance from the point to the set $\mathcal{F}$.

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1 Statement of the problems and the main results

In the present paper we consider the following anisotropic elliptic operator

$$Au(x) := \sum_{i=1}^{n} \frac{d}{dx_i} a_i(x, u_x), \tag{1.1}$$

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namely, for a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, we suppose that the functions $a_i(x,\xi)$ are defined on $(\Omega, \mathbb{R}^n)$, satisfy the Caratheodory conditions and there exist some positive constants ν_1, ν_2 such that the following structure conditions are valid:

$$\begin{aligned}
&\sum_{i=1}^{n} a_i(x,\xi)\xi_i \geq \nu_1 \sum_{i=1}^{n} |\xi_i|^{p_i},\\
&|a_i(x,\xi)| \leq \nu_2 \left(\sum_{j=1}^{n} |\xi_j|^{p_j}\right)^{1-\frac{1}{p_i}}, \quad i = 1,\dots n,\\
&\sum_{i=1}^{n}(a_i(x,\xi) - a_i(x,\eta))(\xi_i - \eta_i) > 0, \quad \forall\, \xi,\eta \in \mathbb{R}^n, \quad \xi \neq \eta,
\end{aligned} \tag{1.2}$$

where the numbers p_i are such that

$$1 < p_1 \leq p_2 \leq \dots \leq p_n < \frac{n-1}{n-p}p, \quad \frac{1}{p} := \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}. \tag{1.3}$$

Define the domain Ω of the special type $\Omega = \mathcal{A}_R \setminus \mathcal{F}$, $\mathcal{F} \subset \mathcal{B}_d$, so that

$$\mathcal{B}_d := \left\{x : |x_i| < d^{\frac{p}{p_i}}\ k^{-\frac{p}{p_i}+1},\ i = 1,\dots,n\right\}, \tag{1.4}$$

$$\mathcal{A}_R := \left\{x : |x_i| < R^{\frac{p}{p_i}}\ \left(\frac{R}{d}\right)^{\frac{n-p}{p-1}\frac{p-p_i}{p_i}} k^{-\frac{p}{p_i}+1},\ i = 1,\dots,n\right\}, \tag{1.5}$$

where k, d, R are some fixed positive numbers.

In the further investigations we use the anisotropic Sobolev spaces

$$W_0^{1,\overline{p}}(\Omega) := \left\{v \in W_0^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega), \quad i = \overline{1,n}\right\},$$

$$W^{1,\overline{p}}(\Omega) := \left\{v \in W^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega), \quad i = \overline{1,n}\right\},$$

where the numbers p_i satisfy (1.3).

In the present paper we establish the local behavior of solution to the boundary value problem for (1.1) in Ω. We understand the solution in the following weak sense. Let $\psi : \mathbb{R}^1 \to \mathbb{R}^1$, $\psi \in C^\infty(\mathbb{R}^1)$ such that $\psi(t) = 1$ for $t \leq 1$, $\psi(t) = 0$ for $t > 3/2$. For any $k \in \mathbb{R}^1$ we denote by u the function such that $u(x) - k\psi\left(\frac{|x|}{d}\right) \in W_0^{1,\overline{p}}(\Omega)$ and satisfying the following integral identity

$$\sum_{i=1}^{n}\int\limits_{\Omega} a_i(x,u_x)\frac{\partial \phi}{\partial x_i}dx = 0, \tag{1.6}$$

for any function $\phi(x) \in W_0^{1,\overline{p}}(\Omega)$.

The existence of weak solutions in the sense of this definition can be proved by the theory of the monotone operators using some standard tools. The problem of type (1.6) is so-called "model boundary value problem", which play a crucial role in study of necessary condition of regularity of a boundary point, removable singularity of solutions, asymptotic behaviour of sequence of solutions in perforated domains.

This boundary value problem is closely related to I.V.Skrypnik's method of homogenization of the Dirichlet non–linear elliptic problems in non–periodic strongly perforated domains in the case when the perforations are some small disjoint components, so called the domains with a fine–grained boundary (see [9]). In the framework of this method, we construct an asymptotic expansion of the solution to the corresponding non–linear problem in the perforated domain in terms of the model problem type (1.6). Knowing its behavior, one can obtain the homogenized problem.

In the case of the linear elliptic operators, the solutions of the Dirichlet problem type (1.6) are very well known. Namely, in the case of the Laplace operator function u is a capacity potential of set $\mathcal{F}$ in reference to the ball B_R. The aim of the present paper is to derive some sharp a priory estimates for the appropriate "anisotropic" capacity potential u defined by (1.6).

The problem of the study of the local behaviour of solutions to nonlinear problems has a long history (see, for example, [10] and references therein). In particular, the anisotropic operators are of great interests by many researchers during the last decades starting from [3]. The simplest model example of operator (1.1) is the following

$$A_0 u(x) := \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \right). \tag{1.7}$$

It was proved in [1] that fundamental solution of A_0 belongs to the anisotropic Sobolev space $u \in W^{1,\overline{q}}(\Omega)$, where the real q_i have to satisfy to the following conditions

$$1 < q_i < n(p-1)p^{-1}(n-1)^{-1}p_i, \quad i = \overline{1,n}.$$

The further analysis of fundamental solution to operator (1.7), its existence and a priory estimates was made in [2].

In the case $1 < p_1 = p_2 = \cdots = p_n = p \leq n$, operator A_0 is the usual p-Laplacian which has the source type (fundamental) solution of the form

$$u(x) = |x|^{-\frac{n-p}{p-1}}, \quad p < n.$$

Due to [9], the point-wise estimate of the corresponding p-Laplace potential of the type (1.6) has the following view

$$u(x) \leq c\, k \left(\frac{d}{|x|} \right)^{\frac{n-p}{p-1}},$$

for some positive constant c depending on n, p only.

The study of entire local behaviour of solution to the Dirichlet problem for nonlinear elliptic equations is a long-established topic in PDE and our present research contains some extensions for the case of anisotropic elliptic operators. Let's now formulate our main result. For a positive ρ we define a number

$$m(\rho) = \text{ess sup} \left\{ |u(x)| : x \in \mathcal{A}\left(\frac{5}{4}\rho\right) \setminus \mathcal{A}\left(\frac{3}{4}\rho\right) \right\}. \tag{1.8}$$

By usual tools, using Moser's iterations, we have

$$m(\rho) \leq C_1 \tag{1.9}$$

where C_1 is a positive constant depending on $n, p_1, \ldots, p_n, \nu_1, \nu_2$ only.

The main result is the following.

Theorem 1.1. *Let $u(x)$ be a solution of problem (1.6) and the condition (1.2), (1.3) are satisfied. Let $2d < \rho < R$, then we have*

$$m(\rho) \leq C_2 k \left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}}. \tag{1.10}$$

with some positive constant C_2 depending on $n, p_1, \ldots, p_n, \nu_1, \nu_2$ only.

To prove the main theorem we apply the modification of Moser's iteration method used for the estimates of the maximum of solutions to the quasilinear elliptic equations (see [4]) and follow the ideas of [7, 8, 5, 6]. The sharpness of estimate (1.10) is shown by the next statement.

Theorem 1.2. *Let all conditions of Theorem 1.1 are satisfied. We suppose also that there exists a constant $\alpha \in (0,1)$, such that*

$$\text{meas}\ \mathcal{F} \geq \alpha\, \text{meas}\ \mathcal{B}_d. \tag{1.11}$$

then the following inequality is valid

$$m(\rho) \geq C_2^{-1} k \left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}}, \tag{1.12}$$

for every $\rho : 0 < \rho < R$, where the constant C_2 was defined in (1.10).

The next statement is a direct consequence of Theorem 1.1 and it is very useful, in particular, in the homogenization problem for the anisotropic elliptic operators type (1.1).

Corollary 1.1. *Let $u(x)$ is a solution to problem (1.6) and the conditions (1.2), (1.3) are satisfied. Then for every point $x^{(0)} \in \Omega$ the following estimate is valid*

$$|u(x_0)| \leq C_3 k \frac{d^{\frac{n-p}{p-1}}}{\left(\sum_{i=1}^{n} |x_i^{(0)}|^{\frac{p_i(p-1)}{p(p-1)+(n-p)(p-p_i)}}\ d^{\frac{p_i(n-p)}{p(p-1)+(n-p)(p-p_i)}}\right)^{\frac{n-p}{p-1}}}. \tag{1.13}$$

with a positive constant C_3 depending on $n, p_1, \ldots, p_n, \nu_1, \nu_2$ only.

Proof In case $\rho > 2d$, where

$$\rho = \sum_{i=1}^{n} |x_i^{(0)}|^{\frac{p_i(p-1)}{p(p-1)+(n-p)(p-p_i)}} \; d^{\frac{p_i(n-p)}{p(p-1)+(n-p)(p-p_i)}} \tag{1.14}$$

the estimate (1.13) is a consequence of inequality (1.10). If $\rho < 2d$ we have from (1.9) the following inequality

$$|u(x_0)| \le C_1 \le C_1 \frac{2^{\frac{n-p}{p-1}} \; d^{\frac{n-p}{p-1}}}{\left(\sum\limits_{i=1}^{n} |x_i^{(0)}|^{\frac{p_i(p-1)}{p(p-1)+(n-p)(p-p_i)}} \; d^{\frac{p_i(n-p)}{p(p-1)+(n-p)(p-p_i)}} \right)^{\frac{n-p}{p-1}}}, \tag{1.15}$$

this proves inequality (1.13) and Corollary 1.1.

The paper is organized as follows. First, in Section 2 we obtain some preliminary integral estimates for the solution of the boundary value problem (1.6). Then in Section 3 we obtain the estimate for the solution from above. Finally, in Section 4 we show the sharpeness of the estimate (1.10).

Remark 1.1. Later on, by γ we will denote some different positive constants depending on $n, p_1, \dots, p_n, \nu_1, \nu_2$ only.

2 Auxiliary estimates

Lemma 2.1. *Let $u(x)$ be a solution of problem (1.6) and the condition (1.2), (1.3) are satisfied. Then the following inequalities are valid*

$$\sum_{i=1}^{n} \int\limits_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \le \gamma \, k^p d^{n-p}, \tag{2.1}$$

$$\sum_{i=1}^{n} \int\limits_{\Omega} \left| \frac{\partial u_\mu}{\partial x_i} \right|^{p_i} dx \le \gamma \, \mu \, k^{p-1} d^{n-p}, \tag{2.2}$$

where $u_\mu(x) = \min\{u(x), \mu\}$, for every $\mu : 0 < \mu \le k$.

Proof Test the integral identity (1.6) by function $\phi = u - k\varphi(x)$, where $\varphi \in C^\infty(R^n)$, $0 \le \varphi(x) \le 1$, $\varphi(x) = 1$ for $x \in \mathcal{B}_d$, $\varphi(x) = 0$ for $x \notin \mathcal{B}_{2d}$, such that $\left| \frac{\partial \varphi}{\partial x_i} \right| \le \gamma \; d^{-\frac{p}{p_i}} k^{\frac{p}{p_i}-1}$.

Using the conditions (1.2) and Young's inequality we get

$$\sum_{i=1}^{n} \int\limits_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \le \gamma \sum_{i=1}^{n} k^{p_i} \int\limits_{\mathcal{B}_{2d}} \left| \frac{\partial \varphi}{\partial x_i} \right|^{p_i} dx \le \gamma \, d^{n-p} k^p,$$

this proves estimate (2.1). To show (2.2), we test the integral identity (1.6) by function $u_\mu(x) - \frac{\mu}{k}u(x)$, finally we get

$$\nu_1 \sum_{i=1}^{n} \int\limits_{\Omega} \left|\frac{\partial u_\mu}{\partial x_i}\right|^{p_i} dx \le \nu_2 \frac{\mu}{k} \sum_{i=1}^{n} \int\limits_{\Omega} \left(\sum_{j=1}^{n} \left|\frac{\partial u}{\partial x_j}\right|^{p_j} \right)^{1-\frac{1}{p_i}} \left|\frac{\partial u}{\partial x_i}\right| dx \le$$

$$\le \nu_2 \frac{\mu}{k} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \int\limits_{\Omega} \left|\frac{\partial u}{\partial x_j}\right|^{p_j} dx \right)^{1-\frac{1}{p_i}} \left(\int\limits_{\Omega} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx \right)^{\frac{1}{p_i}}.$$

Then using inequality (2.1) we obtain the required estimate (2.2). Lemma 2.1 is proved.

3 Proof of Theorem 1.1

For any $j = 1, 2, \dots,$ we define some sequences of numbers

$$\rho_j^{(1,i)} = \frac{\rho^{\frac{p}{p_i}}}{2} \left(\frac{\rho}{d}\right)^{\frac{n-p}{p-1}\frac{p-p_i}{p_i}} \left(1 + 2^{-j}\right) k^{-\frac{p}{p_i}+1},$$

$$\rho_j^{(2,i)} = \frac{\rho^{\frac{p}{p_i}}}{2} \left(\frac{\rho}{d}\right)^{\frac{n-p}{p-1}\frac{p-p_i}{p_i}} \left(3 - 2^{-j}\right) k^{-\frac{p}{p_i}+1},$$

where $i = 1, \dots, n$, and ρ is defined by (1.14). Denote by $\mathcal{D}_j$ the following sequence of domains

$$\mathcal{D}_j := \left\{ x \in \Omega : \rho_j^{(1,i)} \le |x_i| \le \rho_j^{(2,i)},\ i = 1, \dots, n \right\}.$$

Let's consider the sequences of functions $\varphi_j(x)$, such that $\varphi_j(x) = 1$ for $x \in \mathcal{D}_j$, $\varphi_j(x) = 1$ for $x \in \mathcal{D}_{j+1}$ and such that $0 \le \varphi_j(x) \le 1$,

$$\left|\frac{\partial \varphi_j(x)}{\partial x_i}\right| \le \gamma 2^{\gamma j} \rho^{-\frac{p}{p_i}} \left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}\frac{p-p_i}{p_i}} k^{\frac{p}{p_i}-1},$$

for any $j = 1, 2, \dots,$ $i = 1, \dots, n$.

We test the integral identity (1.6) by function $\phi(x) = u(x)|u(x)|^l \varphi_j^s(x)$, where l, s are some positive arbitrary numbers. Using conditions (1.2) and Young's inequality, we get

$$\begin{aligned} &\sum_{i=1}^{n} \int\limits_{\Omega} |u(x)|^l \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \varphi_j^s(x)\, dx \le \\ &\le \gamma (l+s)^{p_n} \sum_{i=1}^{n} \rho^{-p} \left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}(p-p_i)} m_{j+1}^{p_i} \int\limits_{\Omega} |u(x)|^l \varphi_j^{s-p_n}(x)\, dx, \end{aligned} \tag{3.1}$$

where $m_j := \text{ess sup}\, \{|u(x)| : x \in \mathcal{D}_j\}$.

From inequality (3.1) applying the imbedding theorem (Lemma 5.1 from the Appendix) and Moser's iterations, we get

$$m_j^{p+n} \le \gamma 2^{\gamma j} \left(\sum_{i=1}^{n} \rho^{-p} \left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}(p-p_i)} k^{p-p_i} m_{j+1}^{p_i} \right)^{\frac{n}{p}} \int\limits_{\mathcal{D}_{j+1}} |u(x)|^p dx. \tag{3.2}$$

Taking into account that for $x \in \mathcal{D}_{j+1}$ the estimate $u(x) \le m_{j+1}$ is valid and using Lemma 2.1, we have

$$\begin{aligned} &\int\limits_{\mathcal{D}_{j+1}} |u(x)|^p dx = \int\limits_{\mathcal{D}_{j+1}} |u_{m_{j+1}}(x)|^p dx \le \gamma 2^{\gamma j} \rho^p \left(\int\limits_{\Omega} |u_{m_{j+1}}(x)|^{\frac{np}{n-p}} dx \right)^{\frac{n-p}{n}} \\ &\le \gamma 2^{\gamma j} \rho^p \prod_{i=1}^{n} \left(\int\limits_{\Omega} \left| \frac{\partial u_{m_{j+1}}(x)}{\partial x_i} \right|^{p_i} dx \right)^{\frac{p}{np_i}} \le \gamma 2^{\gamma j} \rho^p \prod_{i=1}^{n} (m_{j+1} k^{p-1} d^{n-p})^{\frac{p}{np_i}} \\ &\le \gamma 2^{\gamma j} k^{p-1} m_{j+1} \rho^p \, d^{n-p}. \end{aligned} \tag{3.3}$$

From (3.2), (3.3) the following estimate follows

$$m_j^{p+n} \le \gamma 2^{\gamma j} k^{p-1} \left(\sum_{i=1}^{n} \rho^{-p} \left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}(p-p_i)} k^{p-p_i} m_{j+1}^{p_i} \right)^{\frac{n}{p}} m_{j+1} \rho^p d^{n-p}. \tag{3.4}$$

We denote by $y_j := k^{-1} m_j \left(\frac{\rho}{d}\right)^{\frac{n-p}{p-1}}$. From (3.4) we derive

$$y_j^{p+n} \le \gamma 2^{\gamma j} \left(\sum_{i=1}^{n} y_{j+1}^{p_i} \right)^{\frac{n}{p}} y_{j+1}. \tag{3.5}$$

By iterations of inequality (3.5), we get the estimate

$$k^{-1} m_1 \left(\frac{\rho}{d}\right)^{\frac{n-p}{p-1}} = y_1 \le \gamma, \tag{3.6}$$

which proves inequality (1.10). This finishes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

Let define a cut-off function $\xi \in C^\infty(R^n)$, $0 \le \xi(x) \le 1$ such that $\xi(x) = 1$, for $x \in \mathcal{A}\left(\frac{5}{4}\rho\right) \setminus \mathcal{A}\left(\frac{3}{4}\rho\right)$, $\xi(x) = 0$ for $x \notin \mathcal{A}\left(\frac{7}{8}\rho\right) \setminus \mathcal{A}\left(\frac{5}{8}\rho\right)$, and the following estimate is valid

$$\left| \frac{\partial \xi}{\partial x_i} \right| \le \gamma \rho^{-\frac{p}{p_i}} \left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}\frac{p-p_i}{p_i}} k^{\frac{p}{p_i}-1},$$

for every $i = 1, \dots, n$.

Testing the integral identity (1.6) by the function

$$\phi(x) = u(x) - k\xi^{p_n}(x),$$

we have

$$\sum_{i=1}^{n} \int\limits_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \le \gamma k \sum_{i=1}^{n} \rho^{-\frac{p}{p_i}} \left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}\frac{p-p_i}{p_i}} k^{\frac{p}{p_i}-1} \int\limits_{\mathcal{A}\left(\frac{5}{8}\rho\right)\setminus\mathcal{A}\left(\frac{7}{8}\rho\right)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i-1} dx \le$$

$$\le \gamma k \sum_{i=1}^{n} \rho^{-\frac{p}{p_i}} \left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}\frac{p-p_i}{p_i}} k^{\frac{p}{p_i}-1}$$

$$\times \left(\int\limits_{\mathcal{A}\left(\frac{5}{8}\rho\right)\setminus\mathcal{A}\left(\frac{7}{8}\rho\right)} u^{-\alpha} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \right)^{\frac{p_i-1}{p_i}} \left(\int\limits_{\mathcal{A}\left(\frac{5}{8}\rho\right)\setminus\mathcal{A}\left(\frac{7}{8}\rho\right)} u^{\alpha(p_i-1)} dx \right)^{\frac{1}{p_i}}. \tag{4.1}$$

To estimate the first integral term on the right-hand side of (4.1) we need some additional estimates. To derive them, we define the cut-off function $\varsigma \in C^\infty(R^n)$, $0 \le \varsigma(x) \le 1$, such that $\varsigma(x) = 1$, $x \in \mathcal{A}\left(\frac{5}{8}\rho\right) \setminus \mathcal{A}\left(\frac{7}{8}\rho\right)$, $\varsigma(x) = 0$ $x \notin \mathcal{A}\left(\frac{7}{4}\rho\right) \setminus \mathcal{A}\left(\frac{9}{16}\rho\right)$, and the following estimate is valid

$$\left| \frac{\partial \varsigma}{\partial x_i} \right| \le \gamma \rho^{-\frac{p}{p_i}} \left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}\frac{p-p_i}{p_i}} k^{\frac{p}{p_i}-1},\ i = 1, \dots, n.$$

Now we test the integral identity (1.6) by the function $\phi = u^{-\alpha+1}\varsigma^{p_n}(x)$, $0 < \alpha < 1$. As a result we obtain

$$\sum_{i=1}^{n} \int\limits_{\mathcal{A}\left(\frac{5}{8}\rho\right)\setminus\mathcal{A}\left(\frac{7}{8}\rho\right)} u^{-\alpha} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx$$
$$\le \gamma \sum_{i=1}^{n} \rho^{-p} \left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}(p-p_i)} k^{p-p_i} \int\limits_{\mathcal{A}\left(\frac{7}{4}\rho\right)\setminus\mathcal{A}\left(\frac{9}{16}\rho\right)} u^{-\alpha+p_i} dx. \tag{4.2}$$

We denote by $M(\rho)$, $y(\rho)$ the following numbers

$$M(\rho) := \text{ess sup} \left\{ u(x), x \in \mathcal{A}\left(\frac{7}{4}\rho\right) \setminus \mathcal{A}\left(\frac{9}{16}\rho\right) \right\},$$

$$y(\rho) := M(\rho) k^{-1} \left(\frac{\rho}{d}\right)^{\frac{n-p}{p-1}}.$$

From (4.1), (4.2) it follows that

$$\sum_{i=1}^{n} \int\limits_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \le \gamma k^p d^{n-p} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} y^{p_j}(\rho) \right)^{\frac{p_i-1}{p_i}}. \tag{4.3}$$

Using condition (1.11), from (4.3) we derive

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{n} y^{p_j}(\rho)\right)^{\frac{p_i-1}{p_i}} \geq \gamma. \tag{4.4}$$

This proves the required inequality (1.12) and Theorem 1.2.

5 Appendix

Lemma 5.1 ([3]). *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain, function $v \in W_0^{1,1}(\Omega)$, and*

$$\sum_{i=1}^{n} \int\limits_{\Omega} |v(x)|^{\alpha_i} \left|\frac{\partial v}{\partial x_i}\right|^{p_i} dx < \infty, \ \alpha_i \geq 0, \quad p_i \geq 1.$$

If $1 < p < n$, p is defined by (1.3), then $v \in L_q(\Omega)$, $q = \dfrac{np}{n-p}\left(1 + \dfrac{1}{n}\sum\limits_{i=1}^{n}\dfrac{\alpha_i}{p_i}\right)$ and the following inequality holds

$$\|v\|_{L_q(\Omega)} \leq K_3 \prod_{i=1}^{n}\left(\int\limits_{\Omega} |v(x)|^{\alpha_i}\left|\frac{\partial v}{\partial x_i}\right|^{p_i} dx\right)^{\frac{1}{np_i\left(1+\frac{1}{n}\sum\limits_{k=1}^{n}\frac{\alpha_k}{p_k}\right)}}, \tag{5.1}$$

where the constant K_3 depends on n, α_i, p_i, $i = 1, \ldots, n$ only.

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