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Abstract

For solutions of a class of divergence type quasilinear elliptic equations with (p, q)-growth conditions we establish the condition for removability of singularity on manifolds.

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1 Statement of the problem and the main result

The paper is devoted to study of solutions to quasi-linear elliptic equations in the divergence form

$$-\text{div}\mathbb{A}\left(x, \nabla u\right) = a_0\left(x, \nabla u\right), \quad \forall x \in \Omega \setminus \Gamma, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geqslant 3$ and $\Gamma \subset \Omega$ is an open manifold of dimension $s : 1 \leqslant s \leqslant n-2$, belonging to the class C^1. Without loss of generality we assume that $\Gamma \subset \{x_1 = \cdots = x_{n-s} = 0\}$.

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Throughout the paper we assume that $\mathbb{A} = (a_1, \ldots, a_n)$ and a_0 are such that $\mathbb{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, $a_0 : \Omega \times \mathbb{R}^n \to \mathbb{R}$, the functions $\mathbb{A}(\cdot, \xi)$, $a_0(\cdot, \xi)$ are Lebesgue measurable for all $\xi \in \mathbb{R}^n$ and $\mathbb{A}(x, \cdot)$, $a_0(x, \cdot)$ are continuous for almost all $x \in \Omega$. We also assume that the following structural conditions hold:

$$\mathbb{A}\,(x,\xi)\,\xi \geqslant \nu_1\, g\,(|\xi|)\,|\xi|\,,$$
$$|\mathbb{A}(x,\xi)| + |a_0(x,\xi)| \leqslant \nu_2\, g(|\xi|), \tag{1.2}$$

where ν_1, ν_2 are some positive constants. For the function $g \in C(\mathbb{R}^1_+)$ we assume that

$$\left(\frac{t}{\tau}\right)^{p-1} \leqslant \frac{g(t)}{g(\tau)} \leqslant \left(\frac{t}{\tau}\right)^{q-1}, \quad t \geqslant \tau > 0, \tag{1.3}$$

and the constants p, q satisfy the inequalities

$$1 < p < q, \quad p < n - s. \tag{1.4}$$

Some typical examples of the function g are the following:

$$g(t) = t^{p-1} + t^{q-1},$$
$$g(t) = t^{p-1} \ln^a(1+t),\ t > 0,\ q = p + a,$$
$$g(t) = t^{p_1-1} + t^{p_2-1} \ln^a(1+t),\ t > 0,\ p = \min\{p_1, p_2\},\ q = \max\{p_1, p_2\} + a.$$

The operator from (1.1) with such properties has so called nonstandard growth or (p, q) growth following P. Marcellini [10], G.M. Lieberman [8], V.V. Zhikov [24]. In the case $p = q$ the behaviour of solutions of (1.1) has been well understood. A model example of (1.1) if $p = q$ is the following equation involving p-Laplacian

$$-\triangle_p u = |\nabla u|^{p-1} \quad \text{in} \ \ \Omega \setminus \Gamma, \quad p > 1. \tag{1.5}$$

The study of quasilinear equations with nonstandard growth conditions of different types were motivated by problems from mathematical modeling of the behaviour of electrorheological fluids (see [16]), nonlinear elasticity, and others. In particular, during the last decade a wide literature has been devoted to the study of regularity properties of equations with the following model representatives:

$$\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = 0, \tag{1.6}$$

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right) = 0. \tag{1.7}$$

A survey of the result and references to the original sources can be found for e.g. in [1, 3, 5, 9, 10, 7, 2, 4, 14, 22].

Local properties for solutions to equation (1.1) under the conditions (1.3), (1.4) were obtained in [6, 8, 13, 12]. Our main interest in this paper is to establish under what conditions a solution of (1.1) with (1.2), (1.3) can be extended to the whole domain Ω such that a singularity on the manifold Γ can be removed.

It is well known that the necessary and sufficient condition for a harmonic function u to have a removable singularity at a point $\{x_0\}$ is that $u(x) = o\left(|x - x_0|^{2-n}\right)$ as $x \to x_0$. From the celebrated paper by J. Serrin [17] it is known the behaviour of positive solutions of quasilinear equations in the neighborhood of an isolated singularitiy. In particular, an extension of the result for the Laplace equation to a wide class of nonlinear equations has been done under the appropriate assumptions on the coefficients. This work started the series of studies of removability of isolated singularities and singularities on the manifolds for different classes of nonlinear elliptic and parabolic equations (see [23] for the survey of the relevant results). The problem of removability of singularities for anisotropic operators which model representative is (1.6), has been also considered in several papers (see [15], [22]).

The question of removability of isolated singularity for solution to equation (1.1) was studied in [11]. Since we deal here with the singularities on the smooth manifold Γ of dimension s, we recall some known results first.

For quasilinear elliptic equations, Serrin's condition to have such singularity removable requires the following behaviour of positive solutions

$$u(x) = O\left((d(x,\Gamma))^{-\frac{n-p-s}{p-1}+\delta}\right), \quad \delta > 0, \quad 1 < p < n-s, \tag{1.8}$$

where $d(x,\Gamma)$ is a distance from a point x to the manifold Γ, (see [17], [18]). According to [21], the sharp condition for the removability of singularities of sign changing solutions on the manifold Γ as for equations type (1.5) as for more general quasilinear equations has the view

$$u(x) = o\left((d(x,\Gamma)^{-\frac{n-p-s}{p-1}}\right), \quad 1 < p < n-s. \tag{1.9}$$

Before formulation of our main result, let us remind the notion of a weak solution of (1.1). By $W^{1,G}(\Omega)$ we denote the class of functions which are weakly differentiable in Ω with

$$\int_\Omega G(|\nabla u|)\, dx < \infty,$$

where $G(t) = tg(t)$.

Definition 1.1. A function $u(x)$ is said to be a weak solution of equation (1.1) in $\Omega \setminus \Gamma$, if for any function $\psi \in C^1(\overline{\Omega})$ vanishing in the neighborhood of Γ, there is an inclusion $u\psi \in W^{1,G}(\Omega)$ and the integral identity

$$\int\limits_\Omega \left(\mathbb{A}\,(x, \nabla u)\, \nabla(\psi\varphi) - a_0\,(x, \nabla u)\, \psi\varphi\,\right) dx = 0 \tag{1.10}$$

holds for any $\varphi \in W_0^{1,G}(\Omega)$.

Definition 1.2. We say that a weak solution u of (1.1) has a removable singularity on the manifold Γ if $u(x)$ can be extended to Γ so that its extension $\tilde{u}$ belongs to $W^{1,G}(\Omega)$ and satisfies the equation (1.1) in Ω.

Denote by $x' = (x_1, \dots, x_{n-s})$, $x'' = (x_{n-s+1}, \dots, x_n)$. For every R_0, $H_0 > 0$ we define the following sets

$$D(R_0, H_0) = \{x : |x'| < R_0, \ |x''| < H_0\},$$

$$D_1(R_0) = \{x : |x'| < R_0\}, \quad D_2(H_0) = \{x : |x''| < H_0\}.$$

We can assume that R_0, H_0 are sufficiently small such that

$$D(R_0, H_0) \subset \Omega, \quad \Gamma \subset D\left(R_0, \frac{H_0}{2}\right) \bigcap \{x' = 0\}.$$

Next we define a number $m(r)$, which characterizes some local behaviour of the weak solution $u(x)$ in the neighborhood of the manifold Γ:

$$m(r) := \operatorname{ess\,sup} \{|u(x)| : \ x \in D(R_0, H_0) \setminus D(r, H_0)\}. \tag{1.11}$$

The regularity result from G.M. Lieberman [8] yields that $m(r) < \infty$ for any $r > 0$.

The main result of the paper is the following theorem.

Theorem 1.1. *Let u be a weak solution to* (1.1) *in $\Omega \setminus \Gamma$ and the conditions* (1.2)–(1.4) *be fulfilled. Assume also that:*

$$\lim_{r \to 0} g\left(\frac{m(r)}{r}\right) r^{n-s-1} = 0, \quad 1 \le s \le n-2. \tag{1.12}$$

Then the singularity of u on the manifold Γ is removable.

The sharpness of the proposed condition (1.12) is approved by estimations of a fundamental solution to equation (1.1) which were shown in [11].

A few words about a technique applied here. Our approach is an extension of the method of pointwise and integral estimates of potential type solutions, developed by I.V. Skrypnik in [19], [20] and it is based on some sharp pointwise estimates of nonlinear capacity potentials. The same ideas were used recently in [22] for another operator with nonstandard growth conditions. Namely, the sufficient condition of removability of singularity on the manifold Γ was obtained for solutions of anisotropic equation type (1.6).

The rest of the paper is devoted to the proof of the above theorem and it is organized as follows. The auxiliary integral estimates for a gradient of a weak solution are established in Subsection 2.1. Integral estimates of the solution are proved in Subsection 2.2. We show the bondedness of solution in Subsection 2.3. Finally, Subsection 2.4 concludes the proof on the main theorem.

2 Proof of the Theorem 1.1

2.1 Integral estimates for the gradient of solutions

In this Subsection we derive the auxiliary integral estimates of weak solutions to equation (1.1).

Now we take a nonnegative cut-off function $\tau \in C^{\infty}(\mathbb{R}^1)$ satisfying the following conditions

$$\tau(t) \equiv 0 \ \text{ for } \ |t| \leqslant 1, \quad \tau(t) \equiv 1 \ \text{ for } \ |t| \geqslant 2, \quad 0 \leqslant \frac{d\tau(t)}{dt} \leqslant 2 \ \text{ for } \ t \in \mathbb{R}^1_+.$$

We fix a point $|\xi''| \leq \frac{H_0}{2}$, and for every $r > 0, h > 0$ we set

$$\psi_r(x') := \tau\left(r^{-1}|x'|\right), \quad \varsigma_h(x'') := 1 - \tau\left(h^{-1}|x'' - \xi''|\right).$$

For every r such that $0 < r \leq R_0$ we set

$$u_r(x) := (u(x) - m(r))_+, \quad E(r) := \{x \in D(R_0, H_0) : \ u(x) > m(r)\}.$$

In what follows γ stands for a generic constant that depends on the known parameters only and may vary from line to line. By the know parameters we understand the numbers ν_1, ν_2, n, s, p, q, R_0, H_0.

Lemma 2.1. *Let $u(x)$ be a weak solution of equation (1.1) and all conditions of Theorem 1.1 are satisfied. Then there exists a positive constant c_1 depending on the known parameters only, such that the following inequality is valid*

$$\int_{E(\rho)} G\left(|\nabla u|\right) \, \psi_r^q \, \varsigma_h^{q+1} \, dx \leqslant c_1 m(r) \, h^s \left(\mu(r) + 1\right), \tag{2.1}$$

for any r, ρ, h satisfying

$$0 < r < \rho \leq R_0, \quad 0 < h \leq \frac{H_0}{2}, \quad \rho \leq h.$$

Here $\mu(r) := g\left(\frac{m(r)}{r}\right) r^{n-s-1}$.

Proof Without lost of generality we assume that $\lim\limits_{r \to 0} m(r) = \infty$.

The next inequality will be used in the sequel ant it is an immediate consequence of (1.3), namely

$$g(a)b \leqslant \varepsilon \, a \, g(a) + b \, g\left(\frac{b}{\varepsilon}\right), \quad \text{for any} \quad \varepsilon, \, a, \, b > 0. \tag{2.2}$$

We test the integral identity (1.10) by the following functions:

$$\varphi(x) = u_\rho(x)\psi_r^{q-1}(x)\varsigma_h^{q+1}(x), \quad \psi(x) = \psi_r(x').$$

Using structural inequalities (1.2), we derive:

$$\int_{E(\rho)} G(|\nabla u|)\,\psi_r^q\,\varsigma_h^{q+1}\,dx \leqslant \gamma\,r^{-1}\int_{E(\rho)\bigcap K(r)} g(|\nabla u|)\,u_\rho\,\psi_r^{q-1}\varsigma_h^{q+1}\,dx$$
$$+\,\gamma h^{-1}\int_{E(\rho)} g(|\nabla u|)u_\rho\psi_r^q\varsigma_h^q\,dx + \gamma\int_{E(\rho)} g(|\nabla u|)u_\rho\psi_r^q\varsigma_h^{q+1}\,dx,$$

where $K(r) := \{x' : r < |x'| < 2r\}$. By (2.2) we have

$$\int_{E(\rho)} G(|\nabla u|)\,\psi_r^q\,\varsigma_h^{q+1}\,dx \leqslant \gamma\,r^{-1}\int_{E(\rho)\bigcap K(r)} g\left(\frac{u_\rho}{r}\right)\,u_\rho\,\varsigma_h^{q+1}\,dx$$
$$+\,\gamma h^{-1}\int_{E(\rho)} g\left(\frac{u_\rho}{h}\right)u_\rho\psi_r^q\varsigma_h\,dx + \gamma\int_{E(\rho)} g(u_\rho)u_\rho\psi_r^q\varsigma_h^{q+1}\,dx, \quad (2.3)$$

Therefore, using the definition of $m(r)$, we obtain

$$r^{-1}\int_{E(\rho)\bigcap K(r)} g\left(\frac{u_\rho}{r}\right)\,u_\rho\varsigma_h^q dx \leqslant \gamma\,m(r)\,h^s\,g\left(\frac{m(r)}{r}\right)r^{n-s-1}. \quad (2.4)$$

Using the condition (1.12), the inclusion $E(\rho)\subset D(\rho,H_0)$ and (1.3), we deduce

$$h^{-1}\int_{E(\rho)} g\left(\frac{u_\rho}{h}\right)\,u_\rho\,\psi_r^q\,\varsigma_h\,dx \leqslant \gamma\,m(r)h^{s-1}\int_{r<|x'|<\rho}\left(\frac{|x'|}{h}\right)^{p-1}|x'|^{1+s-n}\,dx'$$
$$\leqslant \gamma\,m(r)h^s\,\left(\frac{\rho}{h}\right)^p \leq \gamma\,m(r)h^s. \quad (2.5)$$

Analogously,

$$\int_{E(\rho)} g\left(u_\rho\right)\,u_\rho\,\psi_r^q\,\varsigma_h^{q+1}\,dx \leqslant \gamma\,m(r)h^s\int_{r<|x'|<\rho}|x'|^{p+s-n}\,dx' \leqslant \gamma\,m(r)h^s\rho^p. \quad (2.6)$$

Thus, collecting (2.3)-(2.6), we derive the desired inequality (2.1). Lemma 2.1 is proved.

For any θ, ρ such that $0<\theta\rho<\rho\leqslant R_0$ we set

$$E(\theta\rho,\rho) := \{x\in E(\rho):\ u(x)\leqslant m(\theta\rho)\}, \quad u^{(\theta\rho)}(x) := \min\{u_\rho(x),\, m(\theta\rho)-m(\rho)\}.$$

Let

$$\Phi(t) := \frac{1}{t}\int_0^t g(\tau)\,d\,\tau,$$

by (1.3) it is easy to see that

$$\frac{g(t)}{q}\leqslant\Phi(t)\leqslant\frac{g(t)}{p} \quad \text{and} \quad \frac{p-1}{p}\frac{g(t)}{t}\leqslant\Phi'(t)\leqslant\frac{q-1}{q}\frac{g(t)}{t}.$$

Lemma 2.2. *Let $u(x)$ be a weak solution of equation (1.1) and the conditions of Theorem 1.1 are satisfied. Then there exists a positive constant c_2, depending on the known parameters only, such that the following inequality is valid*

$$\int_{E(\theta\rho,\rho)} g^{\lambda}\left(\frac{u_\rho}{\rho}\right) G(|\nabla u|)\, u_\rho^{-1}\, \psi_r^q\, \varsigma_h^{q+1}\, dx \leqslant c_2(\theta\,\rho)^{-\lambda(n-s-1)} h^s \mu_1(r)$$
$$+\, c_2 g^{\frac{\lambda q}{q-1}}\left(\frac{m(\theta\rho)-m(\rho)}{\rho}\right)\int_{E(\theta\rho)} g^{-\frac{\lambda}{q-1}}\left(\frac{u_\rho}{\rho}\right) u_\rho^{-1} G(|\nabla u|)\, \psi_r^q\, \varsigma_h^{q+1}\, dx$$
$$+\, c_2\, h^{-1}\left(\frac{\rho}{h}\right)^{p-1}\int_{E(\rho)} g^{1+\lambda}\left(\frac{u_\rho}{\rho}\right) \psi_r^q\, \varsigma_h\, dx, \quad (2.7)$$

for any $0<\theta<1$, $0<\lambda<1$, $0<r<\frac{\theta\rho}{2}<\rho\leqslant R_0$.
Here $\mu_1(r) := \left(g\left(\frac{m(r)}{r}\right) r^{n-s-1}\right)^{\frac{1}{q}}$.

Proof Test the integral identity (1.10) by the following functions:

$$\varphi(x)=\Phi^{\lambda}\left(\frac{u^{(\theta\rho)}}{\rho}\right)\psi_r^{q-1}\varsigma_h^{q+1}, \quad \psi(x)=\psi_r(x').$$

Using structural inequalities (1.2), we derive

$$\int_{E(\theta\rho,\rho)} g^{\lambda}\left(\frac{u_\rho}{\rho}\right) u_\rho^{-1} G(|\nabla u|)\, \psi_r^q\, \varsigma_h^{q+1}\, dx$$
$$\leqslant \gamma\, r^{-1}\int_{E(\rho)\bigcap K(r)} g(|\nabla u|)\, g^{\lambda}\left(\frac{u^{(\theta\rho)}}{\rho}\right)\psi_r^{q-1}\,\varsigma_h^{q+1}\, dx$$
$$+\,\gamma\,(1+h^{-1})\int_{E(\rho)} g(|\nabla u|)\, g^{\lambda}\left(\frac{u^{(\theta\rho)}}{\rho}\right)\psi_r^q\,\varsigma_h^q\, dx := I_1+I_2. \quad (2.8)$$

First we consider I_1. Applying (2.2) with $\varepsilon = r m^{-1}(r)\left(g\left(\frac{m(r)}{r}\right) r^{n-s-1}\right)^{\frac{1}{q}}\psi_r$, using (1.12) and Lemma 2.1, we obtain

$$I_1 \leqslant \gamma\, r^{-1} g^{\lambda}\left(\frac{m(\theta\rho)}{\rho}\right)\int_{E(\rho)} \varepsilon G(|\nabla u|)\, \psi_r^{q-1}\,\varsigma_h^{q+1}\, dx$$
$$+\,\gamma\, r^{-1} g^{\lambda}\left(\frac{m(\theta\rho)}{\rho}\right)\int_{E(\rho)\bigcap K(r)} g\left(\frac{1}{\varepsilon}\right)\psi_r^{q-1}\,\varsigma_h^{q+1}\, dx \leqslant \gamma(\theta\rho)^{-\lambda(n-s-1)} h^s \mu_1(r). \quad (2.9)$$

To estimate I_2 we decompose the set $E(\rho)$ as $E(\rho)=E(\theta\rho,\rho)\bigcup E(\theta\rho)$. Applying

(2.2) and using the fact that $u_\rho \geqslant m(\theta\rho) - m(\rho)$ for any $x \in E(\theta\rho)$, we derive

$$I_2 - \frac{1}{4}\int_{E(\theta\rho,\rho)} g^{\lambda}\left(\frac{u_\rho}{\rho}\right) u_\rho^{-1}\, G(|\nabla u|)\, \psi_r^q\, \varsigma_h^{q+1}\, dx$$

$$\leqslant \gamma\, h^{-1}\int_{E(\theta\rho,\rho)} g^{\lambda}\left(\frac{u_\rho}{\rho}\right) g\left(\frac{u_\rho}{h}\right) \psi_r^q\, \varsigma_h\, dx$$

$$+ \gamma g^{\frac{\lambda q}{q-1}}\left(\frac{m(\theta\rho)-m(\rho)}{\rho}\right)\int_{E(\theta\rho)} g^{-\frac{\lambda}{q-1}}\left(\frac{u_\rho}{\rho}\right) u_\rho^{-1}\, G(|\nabla u|)\, \psi_r^q\, \varsigma_h^{q+1}\, dx$$

$$+ \gamma\, g^{\lambda}\left(\frac{m(\theta\rho)-m(\rho)}{\rho}\right) h^{-1}\int_{E(\theta\rho)} g\left(\frac{u_\rho}{h}\,\frac{g^{\frac{\lambda}{q-1}}\left(\frac{u_\rho}{\rho}\right)}{g^{\frac{\lambda}{q-1}}\left(\frac{m(\theta\rho)-m(\rho)}{\rho}\right)}\right)\psi_r^q\, \varsigma_h\, dx$$

$$\leqslant \gamma g^{\frac{\lambda q}{q-1}}\left(\frac{m(\theta\rho)-m(\rho)}{\rho}\right)\int_{E(\theta\rho)} g^{-\frac{\lambda}{q-1}}\left(\frac{u_\rho}{\rho}\right) u_\rho^{-1}\, G(|\nabla u|)\, \psi_r^q\, \varsigma_h^{q+1}\, dx$$

$$+ \gamma\, h^{-1}\left(\frac{\rho}{h}\right)^{p-1}\int_{E(\rho)} g^{1+\lambda}\left(\frac{u_\rho}{\rho}\right)\psi_r^q\, \varsigma_h\, dx. \quad (2.10)$$

Combining estimates (2.8)–(2.10), we derive the required inequality (2.7).

Lemma 2.3. *Let $u(x)$ be a weak solution of equation (1.1) and all conditions of Theorem 1.1 are satisfied. Then there exists a positive constant c_3 depending on known parameters only, such that the following inequality is valid*

$$\int_{E(\theta\rho)} g^{-\frac{\lambda}{q-1}}\left(\frac{u_\rho}{\rho}\right) u_\rho^{-1} G(|\nabla u|)\, \psi_r^q\, \varsigma_h^{q+1}\, dx$$

$$\leqslant c_3\, g^{-\frac{\lambda}{q-1}}\left(\frac{m(\theta\rho)-m(\rho)}{\rho}\right) h^s\, \mu_1(r)$$

$$+ c_3\, h^{-1}\left(\frac{\rho}{h}\right)^{p-1} g^{-\frac{\lambda q}{q-1}}\left(\frac{m(\theta\rho)-m(\rho)}{\rho}\right)\int_{E(\rho)} g^{1+\lambda}\left(\frac{u_\rho}{\rho}\right)\psi_r^q\, \varsigma_h\, dx. \quad (2.11)$$

Proof Test the integral identity (1.10) by the following functions:

$$\varphi(x) = \left(\Phi^{-\frac{\lambda}{q-1}}\left(\frac{m(\theta\rho)-m(\rho)}{\rho}\right) - \Phi^{-\frac{\lambda}{q-1}}\left(\frac{\max(u_\rho, m(\theta\rho)-m(\rho))}{\rho}\right)\right)\psi_r^{q-1}\varsigma_h^{q+1},$$

$$\psi(x) = \psi_r(x').$$

Using structural inequalities (1.2), we derive

$$\int_{E(\theta\rho)} g^{-\frac{\lambda}{q-1}}\left(\frac{u_\rho}{\rho}\right) u_\rho^{-1} G(|\nabla u|)\, \psi_r^q\, \varsigma_h^{q+1}\, dx$$

$$\leqslant \gamma\, r^{-1} g^{-\frac{\lambda}{q-1}}\left(\frac{m(\theta\rho)-m(\rho)}{\rho}\right)\int_{E(\rho)\bigcap K(r)} g(|\nabla u|)\, \psi_r^{q-1}\, \varsigma_h^{q+1} dx$$

$$+\gamma\,(1+h^{-1}) g^{-\frac{\lambda}{q-1}}\left(\frac{m(\theta\rho)-m(\rho)}{\rho}\right)\int_{E(\theta\rho)} g\left(\frac{u_\rho}{h}\,\frac{g^{\frac{\lambda}{q-1}}\left(\frac{u_\rho}{\rho}\right)}{g^{\frac{\lambda}{q-1}}\left(\frac{m(\theta\rho)-m(\rho)}{\rho}\right)}\right)\psi_r^q\, \varsigma_h\, dx$$

$$\leqslant \gamma\, r^{-1} g^{-\frac{\lambda}{q-1}}\left(\frac{m(\theta\rho)-m(\rho)}{\rho}\right)\int_{E(\rho)\bigcap K(r)} g(|\nabla u|)\, \psi_r^{q-1}\, \varsigma_h^{q+1} dx$$

$$+\gamma\, h^{-1}\left(\frac{\rho}{h}\right)^{p-1} g^{-\frac{\lambda q}{q-1}}\left(\frac{m(\theta\rho)-m(\rho)}{\rho}\right)\int_{E(\rho)} g^{1+\lambda}\left(\frac{u_\rho}{\rho}\right)\psi_r^q\, \varsigma_h\, dx. \quad (2.12)$$

The first term in the right-hand side of (2.12) was already estimated in (2.9), therefore we derive the required inequality (2.11).

The next statement is a direct consequence of the Lemmas 2.2,2.3.

Lemma 2.4. *Let $u(x)$ be a weak solution of equation (1.1) and all conditions of Theorem 1.1 are satisfied. Then there exists a positive constant c_4 depending on known parameters only, such that the following inequality is valid*

$$\int_{E(\theta\rho,\rho)} g^{\lambda}\left(\frac{u_\rho}{\rho}\right) u_\rho^{-1} G(|\nabla u|)\, \psi_r^q\, \varsigma_h^{q+1}\, dx$$

$$\leqslant c_4\,(\theta\rho)^{-\lambda(n-s-1)} h^s \mu_1(r) + c_4\, h^{-1}\left(\frac{\rho}{h}\right)^{p-1}\int_{E(\rho)} g^{1+\lambda}\left(\frac{u_\rho}{\rho}\right)\psi_r^q\, \varsigma_h\, dx, \quad (2.13)$$

where $0<\theta<1$, $0<\lambda<1$, $0<r<\frac{\theta\rho}{2}<\rho\leqslant R_0$, $\rho\leqslant h$ and $\mu_1(r)$ was defined in Lemma 2.2.

2.2 Integral estimates of the solutions

Let $1<\alpha<\frac{n-s}{n-s-1}$ and we set

$$I(\rho,h) := \rho^{(n-s)\frac{\alpha-1}{\alpha}}\int_{D_2(H_0)} dx''\left(\int_{D_1(R_0)}\left(g^{1+\lambda}\left(\frac{u_\rho}{\rho}\right)\psi_r^q\, \varsigma_h^{q+1}\right)^{\alpha} dx'\right)^{\frac{1}{\alpha}}. \quad (2.14)$$

Lemma 2.5. *Let $u(x)$ be a weak solution of equation (1.1) and all conditions of Theorem 1.1 are satisfied. Then there exists a positive constant c_5 depending on known parameters only, such that the following inequality is valid*

$$I(\rho,h)\leqslant 2^{(1+\lambda)(q-1)}\theta^{-(n-s)\frac{\alpha-1}{\alpha}} I(\theta\rho,h) + c_5\left(\frac{\rho}{h}\right)^{\frac{\lambda}{q-1}} I(\rho,2h) + c_5 F(r,\rho,h), \quad (2.15)$$

where $0<\theta<1$, $0<\lambda<\min\left(1,\frac{1}{n-s},\frac{p}{q}(q-1)\right)$, $0<r<\frac{\theta\rho}{2}<\rho\leqslant R_0$, $\rho\leqslant h$.

Here $F(r,\rho,h):=h^s\rho(\theta\rho)^{-\lambda(n-s)}(r^\lambda+\mu_1(r))$, *and* $\mu_1(r)$ *was defined in Lemma 2.2.*

Proof Let $\chi(E(\theta\rho,\rho))$, $\chi(E(\theta\rho))$ denote the characteristic functions of the sets $E(\theta\rho,\rho)$, $E(\theta\rho)$ respectively. We will estimate $I(\rho,h)$ using the inequality

$$u_\rho=u_\rho\chi(E(\theta\rho,\rho))+(u_{\theta\rho}+m(\theta\rho)-m(\rho))\chi(E(\theta\rho))\leqslant u_{\theta\rho}\chi(E(\theta\rho))+u^{(\theta\rho)},\ \ x\in E(\rho).$$

Hence

$$I(\rho,h)\leqslant 2^{(1+\lambda)(q-1)}\theta^{-(n-s)\frac{\alpha-1}{\alpha}+(1+\lambda)(p-1)}I(\theta\rho,h)+\gamma\rho^{(n-s)\frac{\alpha-1}{\alpha}}I_3,$$

where

$$I_3:=\int_{D_2(H_0)}dx''\left(\int_{D_1(R_0)}\left(\Phi^{1+\lambda}\left(\frac{u^{(\theta\rho)}}{\rho}\right)\psi_r^q\,\varsigma_h^{q+1}\right)^\alpha dx'\right)^{\frac{1}{\alpha}}.$$

Using the Hölder inequality, Sobolev's imbedding theorem, Lemma 2.4 and inequality (2.2) with $0<\varepsilon<1$, where ε will be chosen later, we obtain

$$\begin{aligned}\rho^{(n-s)\frac{\alpha-1}{\alpha}}I_3&\leqslant\gamma\,\rho\int_{D_2(H_0)}dx''\left(\int_{D_1(R_0)}\left(\Phi^{1+\lambda}\left(\frac{u^{(\theta\rho)}}{\rho}\right)\psi_r^q\,\varsigma_h^{q+1}\right)^{\frac{n-s}{n-s-1}}dx'\right)^{\frac{n-s-1}{n-s}}\\&\leqslant\gamma\rho\int_{E(\theta\rho,\rho)}g^{1+\lambda}\left(\frac{u_\rho}{\rho}\right)u_\rho^{-1}|\nabla u|\,\psi_r^q\,\varsigma_h^{q+1}\,dx\\&\quad+\gamma\rho r^{-1}\int_{E(\rho)\bigcap K(r)}g^{1+\lambda}\left(\frac{u^{(\theta\rho)}}{\rho}\right)\psi_r^{q-1}\,\varsigma_h^{q+1}\,dx\\&\leqslant\gamma\rho\varepsilon^{1-q}\int_{E(\theta\rho,\rho)}g^\lambda\left(\frac{u_\rho}{\rho}\right)u_\rho^{-1}G(|\nabla u|)\,\psi_r^q\,\varsigma_h^{q+1}\,dx+\gamma\,\varepsilon\int_{E(\rho)}g^{1+\lambda}\left(\frac{u_\rho}{\rho}\right)\psi_r^q\,\varsigma_h^{q+1}\,dx\\&\quad+\gamma\rho g^{1+\lambda}\left(\frac{m(\theta\rho)}{\rho}\right)h^sr^{n-s-1}\leqslant\gamma\rho\varepsilon^{1-q}(\theta\rho)^{-\lambda(n-s-1)}h^s\mu_1(r)\\&\quad+\gamma\left(\varepsilon+\left(\frac{\rho}{h}\right)^p\varepsilon^{1-q}\right)\int_{E(\rho)}g^{1+\lambda}\left(\frac{u_\rho}{\rho}\right)\psi_r^q\,\varsigma_h\,dx+\gamma\rho(\theta\rho)^{-\lambda(n-s)}h^sr^\lambda.\end{aligned}\tag{2.16}$$

From the fact that $\{\varsigma_h\neq 0\}\subseteq\{\varsigma_{2h}=1\}$ we obtain

$$\int_{E(\rho)}g^{1+\lambda}\left(\frac{u_\rho}{\rho}\right)\psi_r^q\,\varsigma_h\,dx\leqslant\gamma I(\rho,2h).$$

We choose ε such that $\varepsilon=\left(\frac{\rho}{h}\right)^{\frac{\lambda}{q-1}}$, then the required inequalities (2.15) follows from (2.16), this proves Lemma 2.5.

We fix λ with the following condition

$$0<\lambda<\min\left(1,\ \frac{1}{n-s},\ \frac{p(q-1)}{q},\ \frac{n-s-\alpha(n-s-1)}{\alpha(n-s)}\right).$$

Theorem 2.1. *Let $u(x)$ be a weak solution of equation (1.1) and all conditions of Theorem 1.1 are satisfied. Then there exist positive numbers c_6, c_7 depending on the known parameters only, such that for any*

$$0 < r < \frac{\theta\rho}{2} < \rho \leqslant R_0, \quad \rho \leqslant c_6 h, \tag{2.17}$$

the following inequality is valid

$$I(\rho, h) \leqslant c_7 h^n \rho^{1-\lambda(n-s-1)} + c_7 F_1(r, \rho, h), \tag{2.18}$$

where

$$F_1(r, \rho, h) := h^s \rho^{1-\lambda(n-s-1)} \left(g\left(\frac{m(r)}{r}\right) r^{n-s-1} \right)^{1+\lambda} + h^s \rho^{1-\lambda(n-s)}(r^\lambda + \mu_1(r)),$$

and $\mu_1(r)$ was defined in Lemma 2.2.

Proof Let

$$A := 2^{(1+\lambda)(q-1)} \theta^{-(n-s)\frac{\alpha-1}{\alpha}}, \quad B := c_5 c_6^{\frac{\lambda}{q-1}}.$$

We choose some integers N_1, N_2 such that

$$2r < \rho\, \theta^{N_1} \leqslant \frac{2r}{\theta}, \quad \frac{H_0}{2} < h 2^{N_2} \leqslant H_0. \tag{2.19}$$

Thus the inequality (2.15) can be rewritten in the form

$$I(\rho, h) \leqslant A\, I(\theta\rho, h) + B\, I(\rho, 2h) + c_5\, F(r, \rho, h).$$

From this we deduce

$$\begin{aligned} I(\rho, h) \leqslant (2A)^{N_1} &\sum_{j=0}^{N_2-1} (2B)^j I(2r, 2^j h) \\ &+ (2B)^{N_2} \sum_{i=0}^{N_1-1} (2A)^i I(\theta^i \rho, H_0) + \gamma \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} A^i B^j F(r, \theta^i \rho, 2^j h). \end{aligned} \tag{2.20}$$

Let us estimate the terms in the right-hand side of (2.20). By our choice of λ, we have

$$I(2r, 2^j h) \leqslant \gamma (2^j h)^s g^{1+\lambda} \left(\frac{m(r)}{r}\right) r^{n-s}.$$

We choose $c_6 < 1$ from the condition

$$2^{s+1} B = 2^{s+1} c_5 c_6^{\frac{\lambda}{q-1}} \leqslant \frac{1}{2},$$

then the previous inequality yields

$$(2A)^{N_1} \sum_{j=0}^{N_2-1} (2B)^j I(2r, 2^j h) \leqslant \gamma (2A)^{N_1} h^s r^{1-\lambda(n-s-1)} \left(g\left(\frac{m(r)}{r}\right) r^{n-s-1} \right)^{1+\lambda}. \tag{2.21}$$

By (2.19) we have

$$(2A)^{N_1} \leqslant \gamma \left(\frac{\rho}{r}\right)^{(n-s)\frac{\alpha-1}{\alpha}+\frac{(1+\lambda)(q-1)}{\log_2 \frac{1}{\theta}}}.$$

Choosing $0<\theta<1$ from the condition

$$\frac{(1+\lambda)(q-1)}{\log_2 \frac{1}{\theta}} \leqslant \frac{n-s-(n-s-1)\alpha}{\alpha} - \lambda(n-s-1),$$

we conclude from (2.21) that

$$(2A)^{N_1} \sum_{j=0}^{N_2-1} (2B)^j I(2r, 2^j h) \leqslant \gamma h^s \rho^{1-\lambda(n-s-1)} \left(g\left(\frac{m(r)}{r}\right) r^{n-s-1}\right)^{1+\lambda}. \tag{2.22}$$

Using (1.12) we have

$$I(\theta^i \rho, H_0) \leqslant \gamma(\theta^i\rho)^{1-\lambda(n-s-1)}.$$

This inequality ensures that

$$(2B)^{N_2} \sum_{i=0}^{N_1-1} (2A)^i I(\theta^i \rho, H_0)$$
$$\leqslant \gamma (2B)^{N_2} \rho^{1-\lambda(n-s-1)} \sum_{i=0}^{N_1-1} 2^{(1+\lambda)(q-1)i} \theta^{i\left(\frac{n-s-\alpha(n-s-1)}{\alpha}-\lambda(n-s-1)\right)}. \tag{2.23}$$

Choosing θ, c_6 small enough, so that

$$2^{(1+\lambda)(q-1)} \theta^{\frac{n-s-\alpha(n-s-1)}{\alpha}-\lambda(n-s-1)} \leqslant \frac{1}{2}, \quad 2B = 2c_5 c_6^{\frac{\lambda}{q-1}} \leqslant 2^{-n},$$

we derive from (2.23) that

$$(2B)^{N_2} \sum_{i=0}^{N_1-1} (2A)^i I(\theta^i\rho, H_0) \leqslant \gamma h^n \rho^{1-\lambda(n-s-1)}. \tag{2.24}$$

Finally, we have

$$A^i B^j F(r, \theta^i\rho, 2^j h) \leqslant \gamma A^i B^j (\theta^i \rho)^{1-\lambda(n-s)} (2^j h)^s (r^\lambda + \mu_1(r)). \tag{2.25}$$

First, we choose c_6 from the condition

$$2^s B = 2^s c_5 c_6^{\frac{\lambda}{q-1}} \leqslant \frac{1}{2},$$

and then we take $\theta:\ 0<\theta<1$ satisfying

$$A\theta^{1-\lambda(n-s)} = 2^{(1+\lambda)(q-1)} \theta^{\frac{n-s-(n-s-1)\alpha}{\alpha}-\lambda(n-s)} \leqslant \frac{1}{2}.$$

Finally, from (2.25) we conclude that

$$\sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} A^i B^j F(r, \theta^i \rho, 2^j h) \leqslant \gamma \rho^{1-\lambda(n-s)} h^s (r^\lambda + \mu_1(r)). \tag{2.26}$$

Combining estimates (2.20)–(2.26), we arrive at the required inequality (2.18).

2.3 Boundedness of the solutions

In this section we introduce the proof of Theorem 1.1 applying the Moser's iteration technique.

We fix $\rho > 0$ and for any $j = 1, 2, \dots, J = \left[\frac{\ln \frac{R_0}{\rho}}{\ln \frac{1}{\theta}}\right] + 1$, define the sequence of numbers $\rho_j = R_0\theta^j$. Let x_0 be an arbitrary point in $D(R_0, H_0) \setminus D(\rho_j, H_0)$. For $l = 0, 1, 2, \dots$ we set

$$R_l := (1-\theta)\rho_j \left(1 - \frac{1}{2} + \frac{1}{2^{l+1}}\right), \quad \overline{R}_l := \frac{1}{2}(R_l + R_{l+1}),$$

$$Q_{R_l}(x_0) := \left\{x : \ |x' - x_0'| \leqslant R_l, \ |x'' - x_0''| \leqslant c_6^{-1} R_l\right\},$$

where θ, c_6 were defined in Theorem 2.1.

Now we introduce the sequence of nonnegative cut-off functions $\xi_l \in C_0^\infty(B_{\overline{R}_l}(x_0))$ such that $\xi_l(x) \equiv 1$ for $x \in B_{R_{l+1}}(x_0)$, and $|\nabla \xi_l| \leqslant \gamma\, 2^l \rho_j^{-1}$.

We test the integral identity (1.10) by the following functions:

$$\varphi(x) = \Phi^m \left(\frac{u_{\rho_{j-1}}}{\rho_j}\right) \xi_l^{k-1}, \quad \psi(x) = \xi_l(x),$$

for any m, $k > 0$. After some easy computations, using structural conditions (1.2) and (2.2), we deduce

$$\int_{B_{\overline{R}_l}(x_0)} \Phi^m \left(\frac{u_{\rho_{j-1}}}{\rho_j}\right) u_{\rho_{j-1}}^{-1} G(|\nabla u|)\, \xi_l^k \, dx \\ \leqslant \gamma\,(m+k)^q 2^{\gamma\, l} \rho_j^{-1} \int_{B_{\overline{R}_l}(x_0)} \Phi^{m+1} \left(\frac{u_{\rho_{j-1}}}{\rho_j}\right) \xi_l^{k-q}\, dx.$$

Using the last inequality and the Sobolev imbedding theorem, we derive

$$\begin{aligned}
&\int_{B_{\overline{R}_l}(x_0)} \Phi^m \left(\frac{u_{\rho_{j-1}}}{\rho_j}\right) \xi_l^k \, dx \\
&\leqslant \gamma 2^{\gamma\, l}(m+k)^\gamma \Big(\rho_j^{-1} \int_{B_{\overline{R}_l}(x_0)} \Phi^{m\frac{n-1}{n}} \left(\frac{u_{\rho_{j-1}}}{\rho_j}\right) |\nabla u|\, \xi_l^{k\frac{n-1}{n}}\, dx \\
&\quad + \rho_j^{-1} \int_{B_{\overline{R}_l}(x_0)} \Phi^{m\frac{n-1}{n}} \left(\frac{u_{\rho_{j-1}}}{\rho_j}\right) \xi_l^{k\frac{n-1}{n}-1}\, dx\Big)^{\frac{n}{n-1}} \\
&\leqslant \gamma 2^{\gamma\, l}(m+k)^\gamma \Big(\rho_j^{-1} \int_{B_{\overline{R}_l}(x_0)} \Phi^{m\frac{n-1}{n}-1} \left(\frac{u_{\rho_{j-1}}}{\rho_j}\right) u_{\rho_j}^{-1} G(|\nabla u|)\, \xi_l^{k\frac{n-1}{n}}\, dx \\
&\quad + \rho_j^{-1} \int_{B_{\overline{R}_l}(x_0)} \Phi^{m\frac{n-1}{n}} \left(\frac{u_{\rho_{j-1}}}{\rho_j}\right) \xi_l^{k\frac{n-1}{n}-1}\, dx\Big)^{\frac{n}{n-1}} \\
&\leqslant \gamma 2^{\gamma\, l}(m+k)^\gamma \left(\rho_j^{-1} \int_{B_{\overline{R}_l}(x_0)} \Phi^{m\frac{n-1}{n}} \left(\frac{u_{\rho_{j-1}}}{\rho_j}\right) \xi_l^{k\frac{n-1}{n}-1}\, dx\right)^{\frac{n}{n-1}}. \quad (2.27)
\end{aligned}$$

For every l we define numbers

$$m_l := (1+\lambda)\left(\frac{n}{n-1}\right)^l, \quad k_l = (q+1-n)\left(\frac{n}{n-1}\right)^l - n,$$

$$I_l := \left(\rho_j^{-n}\int_{B_{\overline{R}_l}(x_0)} \Phi^{m_l}\left(\frac{u_{\rho_{j-1}}}{\rho_j}\right)\xi_l^{k_l}\,dx\right)^{\left(\frac{n-1}{n}\right)^l}.$$

Applying the Moser iteration method, from the last inequality we obtain

$$\operatorname{ess\,sup}\left\{g^{1+\lambda}\left(\frac{u_{\rho_{j-1}}}{\rho_j}\right), \quad x \in B_{\frac{1-\theta}{2}\rho_j}(x_0)\right\} \leqslant \gamma\rho_j^{-n}\int_{E(\rho_{j-1})} g^{1+\lambda}\left(\frac{u_{\rho_{j-1}}}{\rho_j}\right)dx. \quad (2.28)$$

Since x_0 is an arbitrary point in $D(R_0, H_0)\backslash D(\rho_j, H_0)$, from (2.28) we derive

$$g^{1+\lambda}\left(\frac{m(\rho_j)-m(\rho_{j-1})}{\rho_j}\right) \leqslant \gamma\rho_j^{-n}\int_{E(\rho_{j-1})} g^{1+\lambda}\left(\frac{u_{\rho_{j-1}}}{\rho_j}\right)dx.$$

Using Theorem 2.1 with $\rho = \rho_{j-1}$, $h = c_6^{-1}\rho_{j-1}$, we obtain

$$g^{1+\lambda}\left(\frac{m(\rho_j)-m(\rho_{j-1})}{\rho_j}\right) \leqslant \gamma\rho_j^{1-\lambda(n-s-1)} + \gamma F_1(r, \rho_{j-1}, c_6^{-1}\rho_{j-1}) \quad (2.29)$$

Passing to the limit in (2.29) as $r \to 0$ and using (1.12), we have

$$g^{1+\lambda}\left(\frac{m(\rho_j)-m(\rho_{j-1})}{\rho_j}\right) \leqslant \gamma\rho_j^{1-\lambda(n-s-1)}.$$

Iterating the last inequality, we obtain

$$g^{1+\lambda}(m(\rho)) \leqslant \gamma g^{1+\lambda}(m(R_0)) + \gamma, \quad (2.30)$$

for any $\rho \leqslant \frac{R_0}{2}$. This proves the boundedness of the solution.

2.4 End of the proof of Theorem 1.1

Let K be a compact subset of the domain Ω. Let $\eta \in C_0^\infty(\Omega)$ be such that $\eta(x) \equiv 1$ for $x \in K$. We test (1.10) by the function $\varphi = u\,\eta^q\psi_r^{q-1}$, $\psi = \psi_r$. Using (1.2), (2.2), the boundedness of u, and passing to the limit as $r \to 0$, we get

$$\int_K G(|\nabla u|)\,dx \leqslant \gamma. \quad (2.31)$$

Let $\varphi \in W_0^{1,G}(\Omega)$. Test (1.10) by $\varphi\,\psi_r$ and using (2.31) and the boundedness of solution, we pass to the limit as $r \to 0$. Finally, we obtain the required integral identity with an arbitrary $\varphi \in W_0^{1,G}(\Omega)$ and $\psi \equiv 1$. Thus, Theorem 1.1 is proved.

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