

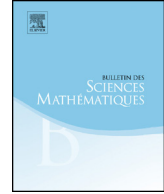


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Maximal multiplier operators in $L^{p(\cdot)}(\mathbb{R}^n)$ spaces [☆]



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ABSTRACT

In this paper we study some estimates of norms in variable exponent Lebesgue spaces for maximal multiplier operators. We will consider the case when multiplier is the Fourier transform of a compactly supported Borel measure.

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1. Introduction

Let $f^\wedge(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx$ be a Fourier transform of f . Given a multiplier $m \in L^\infty(\mathbb{R}^n)$, we define the operators M_t , $t > 0$, by $(M_t f)^\wedge(\xi) = \widehat{f}(\xi)m(t\xi)$ and the maximal multiplier operator

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$$\mathcal{M}_m f(x) := \sup_{t>0} |(M_t f)(x)|$$

which is well defined a priori for a Schwartz functions $S(\mathbb{R}^n)$.

It is well known, if multiplier m satisfies the well known Mihlin–Hörmander condition

$$|\partial^\alpha m(\xi)| \leq C_\alpha |\xi|^\alpha$$

for all (or sufficiently large) multiindices α , if $\mathcal{F}^{-1} f(\xi) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} f(x) dx$ is a reverse Fourier transform, then the multiplier operator $f \mapsto \mathcal{F}^{-1}[m\widehat{f}]$ is bounded in $L^p(\mathbb{R}^n)$ when $1 < p < \infty$ (see [12,15,7,11]). Note that maximal operator \mathcal{M}_m formed by multiplier m with the Mihlin–Hörmander condition in general not bounded on $L^p(\mathbb{R}^n)$. The corresponding example can be found in [4].

We will consider the case when multiplier m is the Fourier transform of a compactly supported Borel measure. In this case the operator M_t , $t > 0$, we can represent as a convolution operator

$$M_t f(x) = \int_S f(x - ty) d\sigma(y),$$

where σ is a compactly supported Borel measure on the set $S \subset \mathbb{R}^n$ and $\widehat{\sigma}(\xi) = m(\xi)$. Obviously we have

$$\mathcal{M}_m f(x) \equiv \mathcal{M}_S f(x) := \sup_{t>0} \left| \int_S f(x - ty) d\sigma(y) \right|.$$

We say that σ is locally uniformly β -dimensional ($\beta > 0$) if $\sigma(B(x, R)) \leq C_\beta R^\beta$, where $B(x, R)$ is a ball of radius $R \leq 1$ centered at x . It is easy to see that a locally uniformly β -dimensional measure must be absolutely continuous with respect to β -dimensional Hausdorff measure μ_β , but such a measure need not exhibit any actual “fractal” behavior. Thus, for example, Lebesgue measure is locally uniformly β -dimensional for any $\beta < n$. We can allow $\beta = 0$ in these definitions, in which case a measure σ is uniformly 0-dimensional if and only if it is finite, and locally uniformly bounded, i.e. $\sigma(B(x, 1))$ is uniformly bounded in x .

Rubio de Francia [16] proved the following

Theorem 1.1. *If $m(\xi)$ is the Fourier transform of a compactly supported Borel measure and satisfies $|m(\xi)| \leq (1 + |\xi|)^{-a}$ for some $a > 1/2$ and all $\xi \in \mathbb{R}^n$, then the maximal operator \mathcal{M}_m maps $L^p(\mathbb{R}^n)$ to itself when $p > \frac{2a+1}{2a}$.*

The case when σ is normalized surface measure on the $(n - 1)$ -dimensional unit sphere was investigated by Stein [17]. According to Stein’s theorem for corresponding maximal operator (spherical maximal operator)

$$\|\mathcal{M}_S f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}$$

holds if $p > n/(n - 1)$, $n \geq 3$, where f is initially taken to be in the class of rapidly decreasing functions. The two-dimensional version of this result was proved by Bourgain [2]. The key feature of the spherical maximal operator is the non-vanishing Gaussian curvature of the sphere. Indeed, one obtains the same L^p bounds if the sphere is replaced by a piece of any hypersurface in \mathbb{R}^n with everywhere non-vanishing Gaussian curvature (see [10]).

Note that for normalized surface measure on the sphere we have $|\widehat{\sigma}(\xi)| \leq C(1 + |\xi|)^{-(n-1)/2}$ and from the Rubio de Francia Theorem follows Stein’s theorem on boundedness spherical maximal operator in $L^p(\mathbb{R}^n)$ (see [16]). More generally, if σ is smooth compactly supported measure in a hypersurface on \mathbb{R}^n with k non-vanishing principal curvatures ($k > 1$), then $|\widehat{\sigma}(\xi)| \leq C(1 + |\xi|)^{-k/2}$ and from the Rubio de Francia Theorem follows Greenleaf’s theorem (see [10,16]).

The main tool used in proving Rubio de Francia’s maximal theorems is the square function technique. Essentially, this says that if the Fourier transform $m(\xi)$ of a compactly supported Borel measure σ has decay of order $-1/2 - \epsilon$; $\epsilon > 0$ i.e.,

$$|m(\xi)| \leq C(1 + |\xi|)^{-1/2-\epsilon} \tag{1}$$

then the maximal operator \mathcal{M}_m is bounded on L^2 . A modified proof of this results due by Iosevich and Sawyer (see Theorem 15 in [13]) shows that the condition (1) can be replaced by more general conditions

$$\left\{ \int_1^2 |m(t\xi)|^2 dt \right\}^{1/2} \leq C(1 + |\xi|)^{-1/2}\gamma(|\xi|),$$

$$\left\{ \int_1^2 |\nabla m(t\xi)|^2 dt \right\}^{1/2} \leq C(1 + |\xi|)^{-1/2}\gamma(|\xi|),$$

where γ is bounded and nonincreasing on $[0, \infty)$, and $\sum_{n=0}^\infty \gamma(2^n) < \infty$.

Our aim of this paper is to study boundedness properties of the Rubio de Francia’s maximal multiplier operator \mathcal{M}_m in variable Lebesgue spaces.

The boundedness of the spherical maximal operator in variable Lebesgue spaces was studied in the papers [8] and [9].

2. The main results

The Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent and the corresponding variable Sobolev spaces $W^{k,p(\cdot)}(\mathbb{R}^n)$ are of interest for their applications to modeling problems in physics, and to the study of variational integrals and partial differential equations with non-standard growth condition (see [6,5]).

We define $\mathcal{P}(\mathbb{R}^n)$ to be the set of all measurable functions $p : \mathbb{R}^n \rightarrow [1, \infty]$. Functions $p \in \mathcal{P}(\mathbb{R}^n)$ are called variable exponents on \mathbb{R}^n . We define $p^- = \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x)$ and $p^+ = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x)$. If $p^+ < \infty$, then we call p a bounded variable exponent.

Let $p \in \mathcal{P}(\mathbb{R}^n)$, $L^{p(\cdot)}(\mathbb{R}^n)$ denote the set of measurable functions f on \mathbb{R}^n such that for some $\lambda > 0$

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

Let $B(x, r)$ denote the open ball in \mathbb{R}^n of radius r and center x . By $|B(x, r)|$ we denote n -dimensional Lebesgue measure of $B(x, r)$. The Hardy–Littlewood maximal operator M is defined on locally integrable function f on \mathbb{R}^n by the formula

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy.$$

In many applications a crucial step has been to show that the Hardy–Littlewood maximal operator is bounded on a variable $L^{p(\cdot)}$ spaces. Note that many classical operators in harmonic analysis such as singular integrals, commutators and fractional integrals are bounded on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ whenever the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Let $\mathcal{B}(\mathbb{R}^n)$ be the class of all functions $p \in \mathcal{P}(\mathbb{R}^n)$ for which the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. This class has been a focus of intense study in recent years. We refer to the books [6] and [5], where several results on maximal, potential and singular integral operators in variable Lebesgue spaces are presented.

We say that a function $p : \mathbb{R}^n \rightarrow (0, \infty)$ is locally log-Hölder continuous on \mathbb{R}^n if there exists $c_1 > 0$ such that

$$|p(x) - p(y)| \leq c_1 \frac{1}{\log(e + 1/|x - y|)}$$

for all $x, y \in \mathbb{R}^n$, $|x - y| \leq 1/2$. We say that $p(\cdot)$ satisfies the log-Hölder decay condition if there exist $p_\infty \in (0, \infty)$ and a constant $c_2 > 0$ such that

$$|p(x) - p_\infty| \leq c_2 \frac{1}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. We say that $p(\cdot)$ is globally log-Hölder continuous in \mathbb{R}^n ($p(\cdot) \in \mathcal{P}_{\log}$) if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition.

If $p : \mathbb{R}^n \rightarrow (1, \infty)$ is globally log-Hölder continuous function in \mathbb{R}^n and $p^- > 1$, then the classical boundedness theorem for the Hardy–Littlewood maximal operator can be extended to $L^{p(\cdot)}$ (see in [5,6]).

Throughout the paper, we denote by c, C, c_1, C_1, c_2, C_2 , etc. positive constant which is independent of the main parameters but which may vary from line to line.

Our main results are the following

Theorem 2.1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Let $m(\xi)$ be the Fourier transform of a compactly supported Borel measure σ and the following conditions be fulfilled:*

- 1) σ is locally β -dimensional, where $0 \leq \beta \leq n$;
- 2) $\left\{ \int_1^2 |m(t\xi)|^2 dt \right\}^{1/2} \leq C(1 + |\xi|)^{-\alpha}$;
- 3) $\left\{ \int_1^2 |\nabla m(t\xi)|^2 dt \right\}^{1/2} \leq C(1 + |\xi|)^{-\alpha}$,

where $\alpha > 1/2$. If $\frac{2\theta p(\cdot)}{2-(1-\theta)p(\cdot)} \in \mathcal{B}(\mathbb{R}^n)$ for some $0 < \theta < \frac{2\alpha-1}{2\alpha-1+2n-2\beta}$, then the maximal operator \mathcal{M}_m maps $L^{p(\cdot)}(\mathbb{R}^n)$ to itself.

Theorem 2.2. *Let $m(\xi)$ be the Fourier transform of a compactly supported Borel measure σ and the following conditions be fulfilled:*

- 1) σ is locally β -dimensional, where $0 \leq \beta \leq n$;
- 2) $\left\{ \int_1^2 |m(t\xi)|^2 dt \right\}^{1/2} \leq C(1 + |\xi|)^{-\alpha}$;
- 3) $\left\{ \int_1^2 |\nabla m(t\xi)|^2 dt \right\}^{1/2} \leq C(1 + |\xi|)^{-\alpha}$,

where $\alpha > 1/2$. If $p(\cdot) \in \mathcal{P}_{\log}$ and

$$\frac{2n + 2\alpha - 2\beta - 1}{n + 2\alpha - \beta - 1} < p_- \leq p_+ < \frac{2n + 2\alpha - 2\beta - 1}{n - \beta},$$

then the maximal operator \mathcal{M}_m maps $L^{p(\cdot)}(\mathbb{R}^n)$ to itself.

Theorem 2.3. *Let $m(\xi)$ be the Fourier transform of a compactly supported Borel measure σ and the following conditions be fulfilled:*

- 1) σ is locally β -dimensional, where $0 \leq \beta \leq n$;
- 2) $\left\{ \int_1^2 |m(t\xi)|^2 dt \right\}^{1/2} \leq C(1 + |\xi|)^{-\alpha}$;
- 3) $\left\{ \int_1^2 |\nabla m(t\xi)|^2 dt \right\}^{1/2} \leq C(1 + |\xi|)^{-\alpha}$,

where $\alpha > 1/2$. If $p(\cdot) \in \mathcal{P}_{\log}$ and

$$\frac{2n + 2\alpha - 2\beta - 1}{n + 2\alpha - \beta - 1} < p_- \leq p_+ < \frac{n + 2\alpha - \beta - 1}{n - \beta} p_-,$$

then the maximal operator \mathcal{M}_m maps $L^{p(\cdot)}(\mathbb{R}^n)$ to itself.

If we take $\beta = 0$, then we will obtain analog of the Rubio de Francia Theorem for variable exponent Lebesgue spaces.

Corollary 2.4. *Suppose $m(\xi)$ is the Fourier transform of a compactly supported Borel measure and satisfies $|m(\xi)| \leq (1 + |\xi|)^{-a}$ for some $a > 1/2$ and all $\xi \in \mathbb{R}^n$. If $p(\cdot) \in \mathcal{P}_{\log}$ and*

$$\frac{2n + 2\alpha - 1}{n + 2\alpha - 1} < p_- \leq p_+ < \frac{n + 2\alpha - 1}{n} p_-.$$

then the maximal operator \mathcal{M}_m maps $L^{p(\cdot)}(\mathbb{R}^n)$ to itself.

Let μ denote a Hausdorff measure on $E \subset [0, 1]$ and ν_r denote the rotationally invariant probability measure on the sphere of radius r . Let

$$\sigma = \int_0^1 \nu_r d\mu(r) \tag{2}$$

denote the corresponding rotationally invariant measure on the set $E_n = \{x \in \mathbb{R}^n : |x| \in E\}$.

If the measure μ is locally α -dimensional ($0 \leq \alpha \leq 1$), then the measure σ is locally uniformly $n - 1 + \alpha$ -dimensional and

$$\left(\int_1^2 |\widehat{\sigma}(t\xi)|^2 dt \right)^{1/2} \leq C(1 + |\xi|)^{-\frac{n-1+\alpha}{2}},$$

moreover, the same estimates hold if $\widehat{\sigma}(t\xi)$ is replaced by $\nabla \widehat{\sigma}(t\xi)$ (see [14]).

Let E denote the Cantor-like subset of $[0, 1]$ consisting of real numbers whose base m , $m > 2$, expansions have only 0's and 1's. Let μ denote the probability measure on E . Note that $\widehat{\mu}(\xi)$ does not tend to 0 as $\xi \rightarrow \infty$ (see e.g. [18]) and for the corresponding measure σ Fourier transform $\widehat{\sigma}(\xi)$ decays only of order $-\frac{n-1}{2}$ at infinity, but square function

$$\left(\int_1^2 |\widehat{\sigma}(t\xi)|^2 dt \right)^{1/2}$$

decays of order $-\frac{n-1+\alpha}{2}$, where $\alpha = \frac{\log 2}{\log m}$ is dimension of E (see [14]).

Corollary 2.5. *Let μ denote a Hausdorff measure on $E \subset [0, 1]$. Suppose μ is locally α -dimensional $0 \leq \alpha < 1$. Let \mathcal{M}_σ be maximal operator corresponding to the measure defined by (2). If $p(\cdot) \in \mathcal{P}_{\log}$ and*

$$\frac{n - \alpha}{n - 1} < p_- \leq p_+ < \frac{n - 1}{1 - \alpha} p_-,$$

then the maximal operator \mathcal{M}_σ maps $L^{p(\cdot)}(\mathbb{R}^n)$ to itself.

3. Proof of main results

Proof of Theorem 2.1. We set $m(\xi) = \widehat{d\sigma}(\xi)$. Obviously $m(\xi)$ is a C^∞ function. To study the maximal multiplier operator $\mathcal{M}_m f(x)$ we decompose the multiplier $m(\xi)$ into radial pieces as follows: we fix a radial C^∞ function φ_0 in \mathbb{R}^n such that $\varphi_0(\xi) = 1$ when $|\xi| \leq 1$ and $\varphi_0(\xi) = 0$ when $|\xi| \geq 2$. For $j \geq 1$ we let

$$\varphi_j(\xi) = \varphi_0(2^{-j}\xi) - \varphi_0(2^{1-j}\xi)$$

and we observe that φ_j is localized near $|\xi| \approx 2^j$. Then we have

$$\sum_{j=0}^\infty \varphi_j = 1.$$

Set $m_j = \varphi_j m$ for all $j \geq 0$. Then m_j are C_0^∞ functions that satisfy

$$m = \sum_{j=0}^\infty m_j.$$

Also, the following estimate is valid:

$$\mathcal{M}_m f \leq \sum_{j=0}^\infty \mathcal{M}_j f$$

where

$$\mathcal{M}_j f(x) = \sup_{t>0} |\mathcal{F}^{-1}(\widehat{f}(\xi)m_j(t\xi))|(x).$$

Note that for any $j \geq 0$ we have (see the proof of Theorem 15 in [13]) the estimate

$$\|\mathcal{M}_j f\|_{L^2} \leq C2^{(1/2-\alpha)j} \|f\|_{L^2} \tag{3}$$

for all $f \in L^2(\mathbb{R}^n)$.

Note also that since $\tilde{p}(\cdot) := \frac{2\theta p(\cdot)}{2-(1-\theta)p(\cdot)} \in \mathcal{B}(\mathbb{R}^n)$ we have the estimate

$$\|\mathcal{M}_j f\|_{\tilde{p}(\cdot)} \leq C2^{j(n-\beta)} \|f\|_{\tilde{p}(\cdot)} \tag{4}$$

for any $j \geq 0$. The proof of estimate (4) is based on the estimate

$$\mathcal{M}_j f(x) \leq C2^{j(n-\beta)} Mf(x), \tag{5}$$

where M is the Hardy–Littlewood maximal operator.

The proof of (5) for specific measure defined by (2) was done in [14]. The proof is based only on the geometric assumption of the measure (assumption 1)). We will prove it for general case for completeness.

To establish (5), it is sufficient to show that for any $M > n$ there is a constant $C_M < \infty$ such that

$$|(\mathcal{F}^{-1}(\varphi_j) * d\sigma)(x)| \leq \frac{C2^{j(n-\beta)}}{(1+|x|)^M}. \tag{6}$$

Using the fact that φ is a Schwartz function, we have for every $N > 0$,

$$|(\mathcal{F}^{-1}(\varphi_j) * d\sigma)(x)| \leq C_N 2^{nj} \int_{\mathbb{R}^n} \frac{d\sigma(y)}{(1+2^j|x-y|)^N}. \tag{7}$$

Let $N > M$. We split the last integral into the regions

$$S_{-1}(x) = \{y \in \mathbb{R}^n : 2^j|x-y| \leq 1\}$$

and for $k > 0$,

$$S_k(x) = \{y \in \mathbb{R}^n : 2^k < 2^j|x-y| \leq 2^{k+1}\}.$$

We obtain the following estimate for the expression $|(\mathcal{F}^{-1}(\varphi_j) * d\sigma)(x)|$

$$\begin{aligned} & \sum_{k=-1}^j \int_{S_k(x)} \frac{C_N 2^{nj} d\sigma(y)}{(1+2^j|x-y|)^N} + \sum_{k=j+1}^{\infty} \int_{S_k(x)} \frac{C_N 2^{nj} d\sigma(y)}{(1+2^j|x-y|)^N} \\ & \leq C'_N 2^{nj} \sum_{k=-1}^j \frac{\sigma(S_k(x))\chi_{B(0,3)}(x)}{2^{kN}} + C_N 2^{nj} \sum_{k=j+1}^{\infty} \frac{\sigma(S_k(x))\chi_{B(0,2^{k+1-j+1})}(x)}{2^{kN}} \\ & =: I + II. \end{aligned} \tag{8}$$

Using the fact that σ is locally uniformly β -dimensional, together with the fact that for $y \in S_k(x)$ we have $|x| \leq 2^{k+1-j} + 1$, we obtain the following estimate

$$I \leq C'_N 2^{nj} \sum_{k=-1}^j \frac{C_\beta 2^{(k+1-j)\beta} \chi_{B(0,3)}(x)}{2^{kN}} \leq C_{N,\beta} 2^{(n-\beta)j} \chi_{B(0,3)}(x). \tag{9}$$

On the other hand

$$\begin{aligned} II &\leq C'_N 2^{nj} \sum_{k=j+1}^\infty C 2^{-kN} \chi_{B(0,2^{k+1-j+1})}(x) \\ &\leq C'_N \sum_{k=j+1}^\infty 2^{nj} 2^{-kN} \frac{(1 + 2^{k-j+2})^M}{(1 + |x|)^M} \\ &\leq C'_M \sum_{k=j+1}^\infty \frac{2^{(k-j)(M-N)}}{2^{k(N+1-n)}} \\ &\leq \frac{C''_M 2^j}{(1 + |x|)^M}, \end{aligned} \tag{10}$$

where we used that $N > M > n$. From (7)–(10) we obtain (6) and consequently (5).

Note that

$$\frac{1}{p(\cdot)} = \frac{1 - \theta}{2} + \frac{\theta}{\tilde{p}(\cdot)},$$

and, therefore,

$$L^{p(\cdot)}(\mathbb{R}^n) = [L^2(\mathbb{R}^n), L^{\tilde{p}(\cdot)}(\mathbb{R}^n)]_\theta = (L^2(\mathbb{R}^n))^{1-\theta} (L^{\tilde{p}(\cdot)}(\mathbb{R}^n))^\theta$$

(where $[X_0, X_1]_\theta$ is a complex interpolation space, see [1] and $X_0^{1-\theta} X_1^\theta$ is a Calderón construction see [3]). Now from (3)–(4) we obtain

$$\|\mathcal{M}_j\|_{L^{p(\cdot)} \rightarrow L^{p(\cdot)}} \leq C \|\mathcal{M}_j\|_{L^2 \rightarrow L^2}^{1-\theta} \|\mathcal{M}_j\|_{L^{\tilde{p}(\cdot)} \rightarrow L^{\tilde{p}(\cdot)}} \leq C' 2^{(1/2-\alpha)(1-\theta)j} 2^{j(n-\beta)\theta}. \tag{11}$$

Using the last estimate we obtain that if $0 < \theta < \frac{2\alpha-1}{2\alpha-1+2n-2\beta}$, then

$$\|\mathcal{M}_m\|_{p(\cdot)} \leq C' \sum_{j=0}^\infty 2^{(1/2-\alpha)(1-\theta)j} 2^{j(n-\beta)\theta} \|f\|_{p(\cdot)} \leq C'' \|f\|_{p(\cdot)}. \quad \square$$

To prove Theorem 2.2 we need the following lemma.

Lemma 3.1. *Suppose $\alpha > 1/2$, $0 \leq \beta \leq n$ and for exponent $p : \mathbb{R}^n \rightarrow (1, +\infty)$ we have*

$$\frac{2n + 2\alpha - 2\beta - 1}{n + 2\alpha - \beta - 1} < p_- \leq p_+ < \frac{2n + 2\alpha - 2\beta - 1}{n - \beta}.$$

Then there exists exponent $\tilde{p} : \mathbb{R}^n \rightarrow (1, +\infty)$ such that $1 < \tilde{p}_- \leq \tilde{p}_+ < \infty$ and $\frac{1}{p(x)} = \frac{1-\theta}{2} + \frac{\theta}{\tilde{p}(x)}$; $x \in \mathbb{R}^n$ for some θ with property $0 < \theta < \frac{2\alpha-1}{2n+2\alpha-2\beta-1}$.

Proof. Note that if $\beta < n$ then

$$1 < \frac{2n + 2\alpha - 2\beta - 1}{n + 2\alpha - \beta - 1} < 2 < \frac{2n + 2\alpha - 2\beta - 1}{n - \beta},$$

and if $\beta = n$, then

$$\frac{2n + 2\alpha - 2\beta - 1}{n + 2\alpha - \beta - 1} = 0 \quad \text{and} \quad \frac{2n + 2\alpha - 2\beta - 1}{n - \beta} = \infty.$$

We have

$$\frac{n - \beta}{2n + 2\alpha - 2\beta - 1} < \inf_{x \in \mathbb{R}^n} \frac{1}{p(x)} \leq \sup_{x \in \mathbb{R}^n} \frac{1}{p(x)} < \frac{n + 2\alpha - \beta - 1}{2n + 2\alpha - 2\beta - 1}.$$

Let $\frac{1}{p(x)} = \frac{1}{2} + r(x)$. By assumption we have

$$\frac{n - \beta}{2n + 2\alpha - 2\beta - 1} - \frac{1}{2} < \inf_{x \in \mathbb{R}^n} r(x) \leq \sup_{x \in \mathbb{R}^n} r(x) < \frac{n + 2\alpha - \beta - 1}{2n + 2\alpha - 2\beta - 1} - \frac{1}{2}. \tag{12}$$

It is easy to see that the equation

$$\frac{1}{p(x)} = \frac{1 - \theta}{2} + \frac{\theta}{\tilde{p}(x)} \tag{13}$$

is equivalent to

$$\frac{1}{2} + \frac{r(x)}{\theta} = \frac{1}{\tilde{p}(x)}. \tag{14}$$

Using (11) we may take small $\delta > 0$ such that

$$\frac{n - \beta}{2n + 2\alpha - 2\beta - 1} - \frac{1}{2} + \delta < \inf_{x \in \mathbb{R}^n} r(x) \leq \sup_{x \in \mathbb{R}^n} r(x) < \frac{n + 2\alpha - \beta - 1}{2n + 2\alpha - 2\beta - 1} - \frac{1}{2} - \delta.$$

Then for θ , $0 < \theta < \frac{2\alpha-1}{2\alpha-1+2\beta}$, where $\theta = \theta < \frac{2\alpha-1}{2\alpha-1+2\beta} - \theta_0$, $\theta_0 > 0$ we have

$$\begin{aligned} \frac{\frac{n-\beta}{2n+2\alpha-2\beta-1} - \frac{1}{2} + \delta}{\frac{2\alpha-1}{2n+2\alpha-2\beta+1} - \theta_0} &< \inf_{x \in \mathbb{R}^n} \frac{r(x)}{\theta} \leq \sup_{x \in \mathbb{R}^n} \frac{r(x)}{\theta} < \frac{\frac{n+2\alpha-\beta-1}{2n+2\alpha-2\beta-1} - \frac{1}{2} - \delta}{\frac{2\alpha-1}{2n+2\alpha-2\beta+1} - \theta_0}, \\ -\frac{1}{2} \frac{\frac{2\alpha-1}{2n+2\alpha-2\beta-1} - 2\delta}{\frac{2\alpha-1}{2n+2\alpha-2\beta-1} - \theta_0} &< \inf_{x \in \mathbb{R}^n} \frac{r(x)}{\theta} \leq \sup_{x \in \mathbb{R}^n} \frac{r(x)}{\theta} < \frac{1}{2} \frac{\frac{2\alpha-1}{2n+2\alpha-2\beta-1} - 2\delta}{\frac{2\alpha-1}{2n+2\alpha-2\beta-1} - \theta_0}. \end{aligned}$$

If we take $\theta_0 < 2\delta$ we obtain

$$-\frac{1}{2} < \inf_{x \in \mathbb{R}^n} \frac{r(x)}{\theta} \leq \sup_{x \in \mathbb{R}^n} \frac{r(x)}{\theta} < \frac{1}{2}. \tag{15}$$

From (13) and (14) we get

$$0 < \inf_{x \in \mathbb{R}^n} \frac{1}{\tilde{p}(x)} \leq \sup_{x \in \mathbb{R}^n} \frac{1}{\tilde{p}(x)} < 1.$$

Consequently we have $1 < \tilde{p}_- \leq \tilde{p}_+ < \infty$. \square

Proof of Theorem 2.2. Using the fact that if $p(\cdot) \in \mathcal{P}_{\log}$ then $\tilde{p}(\cdot) := \frac{2\theta p(\cdot)}{2-(1-\theta)p(\cdot)} \in \mathcal{P}_{\log}$ and by Lemma 3.1 we have $1 < \tilde{p}_- \leq \tilde{p}_+ < \infty$, it follows from [6, Theorem 4.3.8], that $\tilde{p}(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Now the proof of Theorem 2.2 follows from Theorem 2.1. \square

Proof of Theorem 2.3. As by the assumption

$$\frac{2n + 2\alpha - 2\beta - 1}{(n + 2\alpha - \beta - 1)p_-} < \frac{2n + 2\alpha - 2\beta - 1}{(n - \beta)p_+},$$

we can find θ such that

$$\frac{2n + 2\alpha - 2\beta - 1}{(n + 2\alpha - \beta - 1)p_-} < \theta < \min \left(1, \frac{2n + 2\alpha - 2\beta - 1}{(n - \beta)p_+} \right).$$

It is clear, that

$$\frac{2n + 2\alpha - 2\beta - 1}{(n + 2\alpha - \beta - 1)} < \theta p_- < \theta p_+ < \frac{2n + 2\alpha - 2\beta - 1}{(n - \beta)}.$$

It is clear that if $p(\cdot) \in \mathcal{P}_{\log}$ then $\theta p(\cdot) \in \mathcal{P}_{\log}$ and by Theorem 2.2 we get that the operator \mathcal{M}_m is bounded in $L^{\theta p(\cdot)}(\mathbb{R}^n)$. Using the fact that $[L^\infty(\mathbb{R}^n), L^{p(\cdot)\theta}(\mathbb{R}^n)]_\theta = L^{p(\cdot)}(\mathbb{R}^n)$ ($0 < \theta < 1$) and the operator \mathcal{M}_m is bounded in $L^\infty(\mathbb{R}^n)$ and $L^{\theta p(\cdot)}(\mathbb{R}^n)$ we obtain that operator \mathcal{M}_m is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$. \square

Conflict of interest statement

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

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