INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

A variational approach to nonlinear electro-magneto-elasticity: convexity conditions and existence theorems

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Preprint No. 35-2016

PRAHA 2016

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Abstract Electro- or magneto-sensitive elastomers are smart materials whose mechanical properties change instantly by the application of an electric or magnetic fields. The paper analyses the convexity conditions (quasiconvexity, polyconvexity, ellipticity) of the free energy of such materials. These conditions are treated within the framework of the general $\mathscr{A}$-quasiconvexity theory for the constraints

$$\operatorname{curl} F = 0, \quad \operatorname{div} d = 0, \quad \operatorname{div} b = 0, \tag{$*$}$$

where F is deformation gradient, d is the electric displacement and b is the magnetic induction. If the energy depends separately only on F, or on d, or on b, the $\mathscr{A}$-quasiconvexity reduces, respectively, to Morrey's quasiconvexity, polyconvexity and ellipticity conditions or to convexity in d or in b. In the present case, the simultaneous occurrence of F, d, and b leads to the cross-phenomena: mechanic-electric, mechanic-magnetic, and electro-magnetic.

The main results of the paper are:

- In dimension 3 there are 32 linearly independent scalar $\mathscr{A}$-affine functions (and 15 in dimension 2) corresponding to the constraints $(*)$.
- Therefore, an energy function $\psi(F, d, b)$ is $\mathscr{A}$-polyconvex if and only if it is of the form

$$\psi(F, d, b) = \Phi(F, \operatorname{cof} F, \det F, d, b, Fd, Fb)$$

  where Φ is a convex function (of 31 scalar variables). Apart from the expected terms F, $\operatorname{cof} F$, $\det F$, d, and b, we have the cross-effect terms Fd, Fb (and in dimension 2 also $d \times b$).
- An existence theorem is proved for a state of minimum energy for a system consisting of an $\mathscr{A}$-polyconvex electro-magneto-elastic solid plus the vacuum electromagnetic field outside the body.
- Broad sufficient conditions are given for $\mathscr{A}$-polyconvexity of isotropic bodies. The commonly used isotropic electro-elastic or magneto-elastic invariants are $\mathscr{A}$-polyconvex except for the biquadratic ones. The paper determines their $\mathscr{A}$-quasiconvex envelope.
- A complete analysis is given of the $\mathscr{A}$-convexity concepts for electro-magneto-rheological fluids.

⋆ Acknowledgement I thank M. Itskov for fruitful discussions. The support of the institutional research plan RVO 67985840 is gratefully acknowledged.

Keywords Electromechanical and magnetomechanical interactions, finite strain, constitutive equations, energy methods, variational principles, instabilities, $\mathscr{A}$-quasiconvexity, $\mathscr{A}$-polyconvexity, isotropy, fluidity

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1 Introduction

Electro- or magneto-sensitive elastomers are smart materials whose mechanical properties change instantly by the application of an electric or magnetic field. The sensitivity to the electromagnetic fields is due to the manufacturing process in which some metallic electro- or magneto- sensitive inclusions (such as alumina particles or iron powder) are deposited in an elastomeric (usually rubber) matrix. If the fabrication process is conducted under the external electric or magnetic fields, it produces an alignment of the inclusions and consequently an anisotropy; the latter is combined with large deformations of the matrix. One is thus faced with full nonlinear couplings of the mechanical response with the electric and magnetic fields and also with an indirect magneto-electric coupling.

As is well-known, for large deformations the well-posedness questions play important role.

For nonlinear elastostatics Ball [1] showed that Morrey's quasiconvexity condition [32–33] has a direct relevance for the behavior of the body; moreover, recognized the importance of Morrey's sufficient condition for quasiconvexity [33; Theorem 4.4.10], for which he introduced the term polyconvexity. He showed that the polyconvexity is compatible with the realistic constraint for the energy function ψ, viz.,

$$\psi(F) \to \infty \quad \text{as} \quad \det F \to 0 \tag{1.1}$$

and leads to a satisfactory existence theorems in nonlinear elasticity under realistic assumptions. These convexity conditions are known to be one of the main guiding principles for the formation of the nonlinear constitutive equations [51–52, 58, 22–23] and [24].

In this paper I extend the quasiconvexity and polyconvexity notions to the completely coupled problem in electro-magneto-rheological elastomers. There are new issues beyond the purely mechanical case, as we shall see below. The polyconvexity condition, called E-polyconvexity in this paper, is described explicitly, see (2.4) and Theorem 6.5, below. Efficient sufficient conditions are given below for isotropic solids a fluids which show that E-polyconvexity is satisfied by many constitutive proposals commonly used in the literature (but not by all). Furthermore, an equilibrium existence result under the Dirichlet boundary data is proved for an E-polyconvex solid plus the surrounding vacuum electromagnetic field (Theorem 9.3, below).

Our choice of the basic variables in the constitutive equations are the deformation gradient F, the (lagrangean) electric displacement d and the (lagrangean) magnetic induction b.* Among the new issues that we encounter is that the electromagnetic variables satisfy

$$\operatorname{div} d = 0, \quad \operatorname{div} b = 0 \tag{1.2}$$

identically as a counterpart of

$$\operatorname{curl} F = 0 \tag{1.3}$$

for the deformation gradient F. We make a full use of (1.2) and (1.3) by adopting the convexity theory under differential constraints known as the $\mathscr{A}$-quasiconvexity theory [5, 15, 37–38].

In the absence of electromagnetic phenomema the $\mathscr{A}$-quasiconvexity under the constraint (1.3) reduces to the afore-mentioned Morrey's quasiconvexity. The convexity conditions for the electric or magnetic phenomena in rigid bodies (no deformation) have been studied in [59, 5, 44, 15]. These works show that the $\mathscr{A}$-quasiconvexity under $(1.2)_1$ or under $(1.2)_2$ reduces to the ordinary convexity.

The quasiconvexity for combinations of mechanical and magnetic phenomena has been discussed in [26] and [25], but ignoring the constraint $(1.2)_2$, which substantially reduces the class of quasiconvex and polyconvex energies. The paper [16] briefly mentions, as an example, a combination of mechanical and magnetic phenomena in 2 dimensions within a different framework, but without any further development.

Note After the research presented in this paper had been completed, the author became aware of the recent papers by Gil & Ortigosa [19] and Ortigosa & Gil1, 2016 [41–42].** The authors postulate, under the name multi-variable convexity, the same condition as the E-polyconvexity mentioned above. Their motivation for the multi-variable convexity comes from its consistency with the electro-magneto-elastic ellipticity condition and from guaranteeing the reality of speeds of infinitesimal plane waves (see [19; Section 4, Remark 4] and [41; Subsection 4.1], respectively). Despite the obvious overlap, the details of the developments and motivations of [19, 41], [42] versus the present work are different: the former group of papers concentrates mainly on the formal variational aspects and numerical implementation, while here I subordinate the E-polyconvexity to the general concepts of $\mathscr{A}$-quasiconvexity theory (in Theorem 6.5 and elsewhere) and prove results related to that (such as the existence

* A future paper [57] will analyze the convexity conditions under different choices of electromagnetic variables.

** I thank M. Itskov for drawing my attention to these papers.

theorem of Section 9). In particular, the electro-magneto-elastic ellipticity condition figures as a consequence of the $\mathscr{A}$-quasiconvexity (Subsection 2.2, below) and the non-trivial proof of the form of the E-polyconvexity presented in Section 8 has no counterpart in [19, 41–42].

The paper is organized as follows. Section 2 provides a detailed but informal descriptions of the results of the paper. Section 3 gives a survey of the equilibrium and constitutive equations for the static electro-magneto-elasticity. Formal aspects of the variational principle of the electro-magneto-elasticity (the total energy, its first and second variations and the variational derivation of the equilibrium equations) are treated in Section 4. The optional Section 5 introduces the $\mathscr{A}$-quasiconvexity in the general case. A specialization of the $\mathscr{A}$-quasiconvexity to electro-magneto-elasticity is provided in Section 6 under the name E-quasiconvexity. This central section can be read independently of Section 5 since independent definitions are given therein. The next two sections 7 and 8 provide the proofs of the results presented in Section 6. Sections 9, 10, and 11 establish the existence theorem and treat the isotropic materials and fluids, respectively. The remaining sections are appendices. Section 12 calculates E-quasiconvex envelopes of certain biquadratic isotropic invariants. Section 13 summarizes some results on the classical rank 1 convexity needed in our proofs. Section 14 collects the results on the weak convergence necessary for the existence theorem. Finally, Section 15 describes the notation and presents the basic definitions from the ordinary convexity.

2 Survey of main results

With the exception of Section 5, we work in the space dimensions $n = 2$ or 3.* Recall from the introduction that the variables in the constitutive equations are the deformation gradient $F \in \mathbb{M}^{n\times n}$, the referential electric displacement $d \in \mathbb{R}^n$ and the magnetic induction $b \in \mathbb{R}^n$. Throughout the section, $\psi : \mathbb{M}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the energy function of an electro-magneto-elastic body and (F, d, b) is an element of $\mathbb{M}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n$.

2.1 E-quasiconvexity The function ψ is said to be E-quasiconvex if the inequality

$$\int_Q \psi(F + \varphi(x), d + \delta(x), b + \beta(x))\, dx \geq \psi(F, d, b) \tag{2.1}$$

holds on the unit cube $Q = (0, 1)^n$ for each constant values of F, d and b and for each triplet (φ, δ, β) of smooth functions on $\mathbb{R}^n$, periodic with respect to Q, and satisfying

$$\operatorname{curl} \varphi = 0, \quad \operatorname{div} \delta = 0, \quad \operatorname{div} \beta = 0 \quad \text{on} \quad \mathbb{R}^n$$

and

$$\int_Q \varphi\, dx = 0, \quad \int_Q \delta\, dx = 0, \quad \int_Q \beta\, dx = 0.$$

The E-quasiconvexity inequality is difficult to verify (even in the absence of d and b). The exception is the quadratic case to be now mentioned.

* The reader is referred to Section 15 for the notation employed throughout the paper.

2.2 E-ellipticity and the quadratic case We say that a twice continuously differentiable ψ is E-elliptic at (F, d, b) if

$$\mathrm{D}^2\psi(F, d, b)[(\xi \otimes \eta, \delta, \beta), (\xi \otimes \eta, \delta, \beta)] \geq 0 \tag{2.2}$$

for every ξ, δ, β, $\eta \in \mathbb{R}^n$ such that

$$\delta \cdot \eta = \beta \cdot \eta = 0 \quad \text{and} \quad \eta \neq 0.$$

Equation (2.2) reads in detail

$$\psi_{,FF}(\xi \otimes \eta, \xi \otimes \eta) + \psi_{,DD}(\delta, \delta) + \psi_{,bb}(\beta, \beta)$$
$$+ 2\psi_{,Fd}(\xi \otimes \eta, \delta) + 2\psi_{,Fb}(\xi \otimes \eta, \beta) + 2\psi_{,db}(\delta, \eta) \geq 0.$$

The E-ellipticity is a consequence of the E-quasiconvexity. Conversely, it implies the E-quasiconvexity if ψ is a quadratic function, i.e., if

$$\psi(F, d, b) = C[(F, d, b), (F, d, b)]$$

for every (F, d, b) in the domain of ψ, where $C[\cdot, \cdot]$ is a symmetric quadratic form. This can be restated equivalently in terms of the second variation $\delta^2\,\mathsf{E}(\sigma)$ of the total energy [see (4.1) and (4.3)]. Namely, the E-ellipticity of ψ at (F, d, b) is equivalent to the nonnegativity of the second variation $\delta^2\,\mathsf{E}(\sigma)$ at the homogeneous state with data (F, d, b) under the Dirichlet boundary conditions, i.e.,

$$\delta^2\,\mathsf{E}(\sigma)[\kappa, \delta, \beta] \geq 0$$

for every triplet of infinitely differentiable functions $\kappa, \delta, \beta : \mathbb{R}^n \to \mathbb{R}^n$ which vanish outside Ω.

2.3 E-polyconvexity The E-polyconvexity is a sufficient condition for E-quasiconvexity and more importantly, it enables to prove an existence theorem under the constraint (1.1), which is impossible under the mere quasiconvexity.

2.3.1 *The logic* The notion of E-polyconvexity is based on Jensen's inequality and on the idea of E-quasiaffine function.

- Jensen's inequality is stated in a special form: *If* $\Phi : \mathbb{R}^m \to \bar{\mathbb{R}}$ *is a convex lowersemicontinuous function then*
$$\int_Q \Phi(z(x))\,dx \geq \Phi\Big(\int_Q z(x)\,dx\Big)$$
for any measurable map $z : Q \to \mathbb{R}^m$.
- A function $\psi = \psi(F, d, b)$ is said to be E-quasiaffine if the inequality (2.1) holds with the equality sign for all choices of objects occurred there. The main point about E-quasiaffine is that in contrast to E-quasiconvex functions, they are easy to describe, see §2.3.2.
- A function ψ is said to be E-polyconvex if there exists a convex lowersemicontinuous function $\Phi : \mathbb{R}^m \to \bar{\mathbb{R}}$ and E-quasiaffine functions $\psi_i = \psi_i(F, d, b)$, $i = 1, \dots, m$, such that
$$\psi(F, d, b) = \Phi\big(\psi_1(F, d, b), \dots, \psi_m(F, d, b)\big)$$
for all F, d, b.

To verify that each E-polyconvex function is E-quasiconvex, one takes (F, d, b) and (φ, δ, β) as in Subsection 2.1 and applies Jensen's inequality to

$$z(x) = \big(\psi_1(F + \varphi(x), d + \delta(x), b + \beta(x)), \dots, \psi_m(F + \varphi(x), d + \delta(x), b + \beta(x))\big)$$

to show that ψ satisfies (2.1).

2.3.2 *Explicit forms of* E*-quasiaffine and* E*-polyconvex functions* It turns out that in dimension 3 there are 32 linearly independent scalar E-quasiaffine functions of (F, d, b), viz.,

$$1, \quad F, \quad \operatorname{cof} F, \quad \det F, \quad d, \quad b, \quad Fd, \quad Fb; \tag{2.3}$$

(in dimension 2 the list reads 1, F, $\det F$, d, b, Fd, Fb, $d \times b$ and represents 15 scalar functions).* A general E-quasiaffine function is a linear combination of the functions in (2.3). Note that the quantities Fd and Fb occurring in (2.3) are related to the spatial (eulerian) electric displacement and the spatial magnetic induction D, B by the formulas $Fd = D/\rho$, $Fb = B/\rho$ where ρ is the mass density, as follows from $(3.6)_{1,2}$ (below).

Therefore, ψ is E-polyconvex if and only if it is of the form

$$\psi(F, d, b) = \Phi(F, \operatorname{cof} F, \det F, d, b, Fd, Fb) \tag{2.4}$$

where Φ is a convex function (of 31 scalar variables). Apart from the expected terms F, $\operatorname{cof} F$, $\det F$, d, b, which follow from the separate $\mathscr{A}$-quasiconvexity with respect to F, d, and b, we have the cross-effect terms Fd, Fb (and in dimension 2 also $d \times b$).

2.4 Existence theorem The power and consistency of the E-polyconvexity is demonstrated by proving an existence theorem for states of minimum total energy of an E-polyconvex electro-magneto-elastic solid plus the energy of the vacuum electromagnetic field in the exterior of the body. Currently, the proof is available only for the Dirichlet boundary data for the deformation. Apart from the standard growth conditions for F and $\operatorname{cof} F$, the theorem needs the coercivity conditions for d and b to manage the nonlinear terms Fd and Fb in (2.4). The div-curl lemma is employed to do that.

2.5 Symmetries In practice, the E-polyconvexity condition (2.4) is combined with different types of material symmetry to obtain more concrete forms of the constitutive equations. In the present paper, this is demonstrated on isotropic bodies and fluids.

2.5.1 *Isotropy* The following sufficient condition is provided for $\mathscr{A}$-polyconvexity of isotropic bodies: the energy ψ is polyconvex, isotropic and satisfies the principle of objectivity if it has the form

$$\psi(F, d, b) = \Theta(v_1, v_2, v_3, v_1 v_2, v_2 v_3, v_1 v_3, d^{\parallel}, b^{\parallel}, d^{\times}, b^{\times}, v_1 v_2 v_3) \tag{2.5}$$

where

$$v_1, v_2, v_3 \text{ are the singular values of } F,$$

$$d^{\parallel} = |d|, \quad b^{\parallel} = |b|, \quad d^{\times} = |Fd|, \quad b^{\times} = |Fb|$$

and where Θ is a convex function of 11 scalar variables such that $\Theta(z_1, \dots, z_{11})$ nondecreasing in the variables $z_1, \dots, z_{10}$ and symmetric under the permutations of z_1, z_2, z_3 and of z_4, z_5, z_6, see Theorem 10.1, below, which actually provides a somewhat more general sufficient condition. Condition (2.5) extends the "seven variables theorem" for isotropic polyconvex elastic materials [1; Theorem 5.2] and covers a very large class of polyconvex energies of isotropic electro-magneto-elastic bodies, see Subsections 10.2–10.4.

* This result, one of the main results of this paper, is proved in Section 8 by a direct calculation based on the Λ_{E}-affine functions. The paper [55] proves a general result which yields the above as a special case.

2.5.2 *Isotropic invariants* One popular way to satisfy the symmetry requirement imposed by the isotropy is to express the energy as functions of a minimal family of isotropic invariants. This way already occurred in the basic paper by Toupin [60] (who used a slightly different set of independent variables) and employed numerously in the literature since then. A standard list of isotropic invariants is

$$I_1, \quad I_2, \quad I_3, \quad K_1^{\mathrm{e}}, \quad K_2^{\mathrm{e}}, \quad K_3^{\mathrm{e}}, \quad K_1^{\mathrm{m}}, \quad K_2^{\mathrm{m}}, \quad K_3^{\mathrm{m}}, \quad M^{\mathrm{em}}, \tag{2.6}$$

where

$$I_1 = \operatorname{tr}(C), \quad I_2 = \frac{1}{2}\big((\operatorname{tr} C)^2 - \operatorname{tr}(C^2)\big), \quad I_3 = \det(C),$$
$$C = F^{\mathrm{T}}F, \quad K_i^{\mathrm{e}} = d \cdot C^{i-1}d, \quad K_i^{\mathrm{m}} = b \cdot C^{i-1}b, \quad M^{\mathrm{em}} = d \cdot b,$$

where i ranges the set $\{1, 2, 3\}$. We note that in the literature these invariants are used either in a purely electro-elastic or a purely magneto-elastic context, with the corresponding sublists of the full list (2.6). The invariant M^{em} thus does not occur at all. The invariants

$$I_1, \quad I_2, \quad I_3, \quad K_1^{\mathrm{e}}, \quad K_2^{\mathrm{e}}, \quad K_1^{\mathrm{m}}, \quad K_2^{\mathrm{m}} \tag{2.7}$$

are each of the format (2.5) and thus they are E-polyconvex (actually, more strongly, even their square roots are E-polyconvex). An example of an E-polyconvex energy expressed through these invariants is a "Mooney–Rivlin magnetoelastic solid" [43]

$$\psi = \frac{1}{4}\mu(0)\big[(1+\gamma)(I_1 - 3) + (1-\gamma)(I_2 - 3)\big] + \alpha K_1^{\mathrm{m}} + \beta K_2^{\mathrm{m}}$$

where $\mu(0)$ is the shear modulus, $\alpha \geq 0, \beta \geq 0$ magnetoelastic coupling parameters, and γ an additional parameter, with $|\gamma| \leq 1$. Of course, many other proposals occur in the literature, some E-polyconvex and others not.

Contrary to the invariants in (2.7), the invariants

$$K_3^{\mathrm{e}}, \quad K_3^{\mathrm{m}} \quad \text{and} \quad M^{\mathrm{em}}$$

are not E-polyconvex. A computational way of verifying this is to show that they violate the ellipticity condition. A better way is to show that the E-quasiconvex envelopes $\mathsf{Q}K_3^{\mathrm{e}}, \mathsf{Q}K_3^{\mathrm{m}}, \mathsf{Q}M^{\mathrm{em}}$, defined by (12.1) (below), do not coincide with the original functions. Indeed, it will be shown that

$$\mathsf{Q}K_3^{\mathrm{e}} = |Fd|^4, \quad \mathsf{Q}K_3^{\mathrm{m}} = |Fb|^4, \quad \mathsf{Q}M^{\mathrm{em}} = -\infty$$

for every (F, d, b).

2.5.3 *Fluidity* The free energy of electro-magneto-elastic fluids has a representation

$$\psi(F, d, b) = \tau(\rho, Fd, Fb) \equiv \bar{\tau}(\rho, D, B)$$

where $\rho = 1 / \det F$ is the mass density and τ and $\bar{\tau}$ are some functions of the indicated arguments, satisfying

$$\tau(\rho, Qd^{\#}, Qb^{\#}) = \tau(\rho, d^{\#}, b^{\#}), \quad \bar{\tau}(\rho, QD, QB) = \bar{\tau}(\rho, D, B)$$

for every proper orthogonal tensor Q, every $\rho > 0$ and every $d^{\#}, b^{\#}, D, B \in \mathbb{R}^n$. It turns out that for energies ψ of that format,

$$\text{E-quasiconvexity} \iff \text{E-polyconvexity} \iff \text{E-ellipticity}$$

and these conditions are satisfied $\iff$

the function $(v, d^{\#}, b^{\#}) \mapsto \tau(1/v, d^{\#}, b^{\#})$ is convex on $(0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$.

Note that the variable v is the specific volume. Section 11 gives a broad sufficient condition for the E-polyconvexity for fluids. An important particular case of electro-magneto-elastic fluids are the electrorheological fluids [21; Chapter 8].

3 Equilibrium and constitutive equations for electro-magneto-elasticity

The coupling between electricity, magnetism and nonlinear elasticity is well studied since the sixties of the last century, as illustrated by the book expositions [62, 4, 21, 30, 13–14, 27] and others. Our situation is purely static, so that only the static form of Maxwell's equations and the mechanical equilibrium of forces govern the behavior of the body.

3.1 Equilibrium equations

3.1.1 *Actual ("eulerian") configuration* The basic electromagnetic variables are the electric and magnetic fields, the electric displacement and the magnetic induction, denoted, respectively, by E, H, D, B. The mechanical variables are the Cauchy stress tensor T, the density of the body force g, and the actual density of mass ρ. The equilibrium equations are

$$\operatorname{Div} D = 0, \quad \operatorname{Div} B = 0, \quad \operatorname{Curl} E = 0, \quad \operatorname{Curl} H = 0 \quad \text{on} \quad \mathbb{R}^n, \tag{3.1}$$

$$\operatorname{Div} T + \rho g = 0 \quad \text{on} \quad \omega \tag{3.2}$$

where Curl and Div denote the curl and divergence with respect to the actual position and ω is the actual configuration of the body. In Section 4, the equilibrium equations will be derived from a variational principle. The equations (3.1) and (3.2) are assumed to hold in the weak sense, which then includes the well-known jump conditions for the electromagnetic variables on the boundary of the body. This is not repeated here. Furthermore, below we shall consider only the Dirichlet boundary conditions for the deformation; thus there is no equation for the surface traction on the boundary. Outside ω we have the ether relations

$$E = D, \quad H = B; \tag{3.3}$$

on ω, we have the constitutive relations for E and H to be discussed below.

3.1.2 *Referential ("lagrangian") configuration* We denote by $\Omega \subset \mathbb{R}^n$ the reference configuration of the body and by $y : \Omega \to \mathbb{R}^n$ the deformation. We prescribe the Dirichlet boundary conditions on $\partial\Omega$, i.e.,

$$y = \tilde{y} \quad \text{on} \quad \partial\Omega \tag{3.4}$$

where $\tilde{y} : \partial\Omega \to \mathbb{R}^n$ is a given function. We assume that $\tilde{y}$ can be extended to an equally denoted injective function on $\mathbb{R}^n \sim \mathrm{cl}\,\Omega$ such that $\det \nabla\tilde{y} > 0$ on $\mathbb{R}^n \sim \Omega$. For notational convenience we define the deformation gradient $F : \mathbb{R}^n \to \mathbb{M}^{n\times n}_+$ by

$$F = \begin{cases} \nabla y & \text{on} \quad \Omega, \\ \nabla\tilde{y} & \text{on} \quad \mathbb{R}^n \sim \mathrm{cl}\,\Omega. \end{cases} \tag{3.5}$$

We now use the classical Piola transformation [20], [29; Chapter I, §§7.18–7.20] to introduce the referential (lagrangean) quantities by

$$\begin{aligned} &e = F^{\mathrm{T}}E, \quad h = F^{\mathrm{T}}H, \quad d = (\mathrm{cof}\,F)^{\mathrm{T}}D, \quad b = (\mathrm{cof}\,F)^{\mathrm{T}}B && \text{on} \quad \mathbb{R}^n, \\ &S = T\,\mathrm{cof}\,F && \text{on} \quad \Omega, \end{aligned} \tag{3.6}$$

where, of course the spatial variables $E, \dots, T$ are now expressed as functions of the referential variable. The referential forms of the equilibrium equations read

$$\mathrm{div}\,d = 0, \quad \mathrm{div}\,b = 0, \quad \mathrm{curl}\,e = 0, \quad \mathrm{curl}\,h = 0, \quad \text{on} \quad \mathbb{R}^n, \tag{3.7}$$

$$\mathrm{div}\,S + g = 0 \qquad \text{on} \quad \Omega, \tag{3.8}$$

where curl and div denote the referential forms of the curl and divergence, i.e., the same differential operators as Curl and Div, but with the derivatives with respect to the actual position replaced by the derivatives with respect to referential position. The ether relations (3.3) read in terms of the referential variables as

$$e = F^{\mathrm{T}}Fd/\det F, \quad h = F^{\mathrm{T}}Fb/\det F \tag{3.9}$$

outside Ω.

3.2 Constitutive relations

3.2.1 *The energy and potential relations* The density of the free energy function is

$$\psi = \psi(F, d, b)$$

with $\det F > 0$. We assume that ψ is twice continuously differentiable throughout the domain

$$\mathbb{D}^n_+ := \mathbb{M}^{n\times n}_+ \times \mathbb{R}^n \times \mathbb{R}^n \tag{3.10}$$

which is a subset of

$$\mathbb{D}^n := \mathbb{M}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n. \tag{3.11}$$

We have the potential relations

$$S = \mathrm{D}_F\psi, \quad e = \mathrm{D}_d\psi, \quad h = \mathrm{D}_b\psi. \tag{3.12}$$

3.2.2 *Objectivity* We assume that ψ satisfies

$$\psi(QF, d, b) = \psi(F, d, b) \tag{3.13}$$

for all $(F, d, b) \in \mathbb{D}^n_+$ and all $Q \in \mathrm{SO}(n)$. A standard argument shows that (3.13) implies the symmetry of the stress,

$$SF^{\mathrm{T}} = FS^{\mathrm{T}}, \quad T^{\mathrm{T}} = T.$$

3.2.3 *The symmetry group* The material symmetry (isotropy, crystal classes, etc.) is one of the basic guiding principles for the creation of realistic constitutive equations. For a general electro-magneto-elastic material with the energy function ψ, we define the symmetry group $\mathscr{G}$ of the material as the set of all $H \in \mathbb{M}^{n\times n}_{+}$ such that

$$\psi(F \operatorname{cof} H^{\mathrm{T}}, (\operatorname{cof} H)^{-\mathrm{T}} d, (\operatorname{cof} H)^{-\mathrm{T}} b) = \psi(F, d, b) \tag{3.14}$$

for all $(F, d, b) \in \mathbb{D}^{n}_{+}$. If $\mathscr{G} \subset \mathrm{SL}(n)$ then (3.14) reduces to

$$\psi(FH^{-1}, Hd, Hb) = \psi(F, d, b) \tag{3.15}$$

and if even $\mathscr{G} \subset \mathrm{SO}(n)$ then

$$\psi(FR^{\mathrm{T}}, Rd, Rb) = \psi(F, d, b)$$

for each $R \in \mathscr{G}$. An electro-magneto-elastic body is said to be an isotropic solid if $\mathscr{G} = \mathrm{SO}(n)$ [13; Section 5.9] and a fluid if $\mathscr{G} = \mathrm{SL}(n)$ [13; Section 5.12]. The present paper provides a detailed specialization of the convexity conditions for these two symmetries; other symmetries, such as the transverse isotropy and the crystallographic symmetries, will be treated elsewhere [56].

3.3 Example (Non-interacting matter) Consider an elastic body Ω in the external electromagnetic field but suppose that there is no field-matter interaction. Therefore, the energy splits into the sum

$$\psi(F, d, b) = \psi_1(F) + \psi_2(F, d, b)$$

of the elastic energy $\psi_1(F)$ and of the energy of the vacuum electromagnetic field $\psi_2(F, d, b)$. In the reference configuration Ω, ψ_2 is given by

$$\psi_2(F, d, b) = \frac{1}{2}(\det F)^{-1}\big(|Fd|^2 + |Fb|^2\big). \tag{3.16}$$

Indeed, passing from the reference variable x to the spatial variable $y = y(x)$ and employing the transformation rules (3.6) we obtain the vacuum energy of the electromagentic field, i.e.,

$$\frac{1}{2}\int_{\Omega}(\det F)^{-1}\big(|Fd|^2 + |Fb|^2\big)\,dx = \frac{1}{2}\int_{\omega}\big(|D|^2 + |B|^2\big)\,dy$$

where dx and dy are the referential and actual elements of volume if $n = 3$ or those of area if $n = 2$, D and B are the spatial electric displacement and magnetic induction and $\omega = y(\Omega)$ is the actual configuration of the body. The potential relations $(3.12)_{2,3}$ yield the ether relations (3.9); the stress relation $(3.12)_1$ yields

$$S = S_1 + S_2 \quad \text{where} \quad S_1(F) = \mathrm{D}_F\psi_1(F), \quad S_2(F, d, b) = \mathrm{D}_F\psi_2(F, d, b)$$

where S_1 is the elastic stress while a calculation shows that S_2 is given by

$$S_2(F, d, b) = (\det F)^{-1}\Big(Fd \otimes d + Fb \otimes b - \frac{1}{2}F^{-\mathrm{T}}\big(|Fd|^2 + |Fb|^2\big)\Big).$$

Let us show that

$$\operatorname{div} S_2 = 0 \tag{3.17}$$

for any deformation y of Ω and any vector fields d and b that satisfy $(3.7)_{1,2}$ on Ω. Indeed, passing to the spatial stress $T_2 = S_2 \operatorname{cof} F^{-1}$, we obtain the vacuum Maxwell tensor

$$T_2 = D \otimes D + B \otimes B - \frac{1}{2}\left(|D|^2 + |D|^2\right)\mathrm{I}$$

whose spatial divergence is known to vanish as a consequence of $(3.1)_{1,2}$:

$$\operatorname{Div} T_2 = 0.$$

The referential form (3.17) then follows by Piola's transformation. The equilibrium equation (3.8) with the total stress S then reduces to the equilibrium for the elastic stress

$$\operatorname{div} S_1 + g = 0 \quad \text{on} \quad \Omega.$$

We thus summarize that the total stress S is different form zero even in the (idealized) absence of matter as a consequence of the geometric factors in (3.16); however, its divergence identically vanishes.

3.4 Remark (*"The ether is an* E*-polyconvex fluid"*) We note that the referential form of the vaccum energy (3.16) satisfies the symmetry relation (3.15) for every $H \in \mathrm{SL}(n)$; thus according to the definition in §3.2.3, ψ_2 represents a fluid. Furthermore, we shall show below that ψ_2 is E-polyconvex. Indeed each of the terms $(\det F)^{-1}|Fd|^2/2$ and $(\det F)^{-1}|Fb|^2/2$ constituting ψ_2 figures on the list of E-polyconvex functions (10.3) (the third and fourth members on the last line of (10.3) with $\beta = 2$, $\gamma = 1$).

4 Variational principle

This section presents a preliminary analysis of a variational principle for an electro-magneto-elastic body. We consider a state of minimum energy of the system consisting of an elastic body Ω interacting with the electromagnetic field inside Ω and the vacuum electromagnetic field in its exterior. Section 9 treats the same minimum principle under natural, weakened assumptions on y, d, b which ensure the existence of a minimizer. As already mentioned, the proof is currently available on for the Dirichlet data for the deformation. Even though the considerations to be presented in this section can be carried out for the general boundary conditions, we assume the Dirichlet data for notational simplicity also here.

4.1 The system and its states We assume that the reference configuration Ω is bounded and has class C^2 boundary $\partial\Omega$. We denote by $\Omega^{\mathrm{c}} = \mathbb{R}^n \sim (\Omega \cup \partial\Omega)$ the complement of the body and by n the outer normal to $\partial\Omega$.

By a state we mean any triplet $\sigma = (y, d, b)$ of maps

$$y : \Omega \to \mathbb{R}^n, \quad d : \mathbb{R}^n \to \mathbb{R}^n, \quad b : \mathbb{R}^n \to \mathbb{R}^n;$$

these represent the deformation of the body and the referential electric displacement and magnetic induction, respectively. We assume that

(i) y is twice continuously differentiable with y and its derivatives up to the order 2 having continuous extensions to the closure cl Ω of Ω, and with $\det \nabla y > 0$ on cl Ω;
(ii) y satisfies Dirichlet's boundary condition (3.4);
(iii) e and b are continuously differentiable in Ω and in Ω^c with d and b and their derivatives having continuous extensions from Ω to cl Ω and from Ω^c to cl Ω^c.
(iv) e and b satisfy

$$\begin{aligned} &\operatorname{div} d = 0, \qquad && \operatorname{div} b = 0 \qquad && \text{on } \Omega \cup \Omega^c, \\ &[\![d]\!] \cdot n = 0, \qquad && [\![b]\!] \cdot n = 0 \qquad && \text{on } \partial\Omega, \end{aligned}$$

where $[\![\cdot]\!]$ the jump across $\partial\Omega$.

We denote by $\mathfrak{S}$ the set of all states.

4.2 The total energy The total energy of a state $\sigma = (y, d, b) \in \mathfrak{S}$ is defined by

$$\mathsf{E}(\sigma) = \int_\Omega \psi(\nabla y, d, b)\, dx - \int_\Omega g \cdot y\, dx + \frac{1}{2} \int_{\Omega^c} J^{-1}\big(|Fd|^2 + |Fb|^2 \big)\, dx \tag{4.1}$$

where the deformation gradient outside Ω is defined by (3.5) using the extension $\tilde{y}$ on Ω^c which is fixed, and $J = \det F$. Following [9; Chapter 8], we note that the last term in (4.1) is independent of the choice of the fictitious 'deformation' $\tilde{y}$ since it can be transformed into the vacuum energy

$$\frac{1}{2} \int_{\omega^c} \big(|D|^2 + |B|^2 \big)\, dy$$

as in Example 3.3, where $\omega^c = \mathbb{R}^n \sim y(\Omega)$ is the exterior of the actual configuration $y(\Omega)$.

The set $\delta\,\mathfrak{S}$ of admissible variations of state is the set of triplets (κ, δ, β) of infinitely differentiable functions on $\mathbb{R}^n$ with values in $\mathbb{R}^n$ such that

$$\kappa = 0 \quad \text{on} \quad \Omega^c, \quad \operatorname{div} \delta = 0, \quad \operatorname{div} \beta = 0 \quad \text{in} \quad \mathbb{R}^n \tag{4.2}$$

and δ, β vanish outside some (varying) bounded subset of $\mathbb{R}^n$. If $\sigma = (y, d, b)$ is a state, we define the first and second variations $\delta\,\mathsf{E}(\sigma)[\cdot]$ and $\delta^2\,\mathsf{E}(\sigma)[\cdot]$ of energy at σ as linear and quadratic functionals on $\delta\,\mathfrak{S}$ by

$$\left.\begin{aligned} \delta\,\mathsf{E}(\sigma)[\kappa, \delta, \beta] &= \int_\Omega \mathrm{D}\psi(\nabla y, d, b)[\nabla\kappa, \delta, \beta]\, dx - \int_\Omega g \cdot \kappa\, dx \\ &\quad + \frac{1}{2} \int_{\Omega^c} J^{-1}\big((Fd \cdot F\delta) + (Fb \cdot F\beta) \big)\, dx, \\ \delta^2\,\mathsf{E}(\sigma)[\kappa, \delta, \beta] &= \int_\Omega \mathrm{D}^2\psi(\nabla y, d, b)[(\nabla\kappa, \delta, \beta), (\nabla\kappa, \delta, \beta)]\, dx \\ &\quad + \int_{\Omega^c} J^{-1}\big(|F\delta|^2 + |F\beta|^2 \big)\, dx. \end{aligned}\right\} \tag{4.3}$$

4.3 Equilibrium states A state $\sigma = (y, d, b) \in \mathfrak{S}$ is said to be an equilibrium state if $\mathsf{E}(\sigma) < \infty$ and

$$\mathsf{E}(\sigma) \le \mathsf{E}(\bar{\sigma})$$

for all $\bar{\sigma} \in \mathfrak{S}$. Necessary conditions for the minimum are, standardly,

$$\delta\, \mathsf{E}(\sigma)[\kappa, \delta, \beta] = 0, \quad \delta^2\, \mathsf{E}(\sigma)[\kappa, \delta, \beta] \geq 0 \tag{4.4}$$

for each $(\kappa, \delta, \beta) \in \delta\, \mathfrak{S}$. Moreover, $(4.4)_1$ is equivalent to the equilibrium conditions

$$\left.\begin{array}{lll} \operatorname{div} S + g = 0 & & \text{in} \quad \Omega, \\ \operatorname{curl} e = 0, & \operatorname{curl} h = 0 & \text{in} \quad \Omega \cup \Omega^{\mathrm{c}}, \\ [\![e]\!] \times n = 0, & [\![h]\!] \times n = 0 & \text{on} \quad \partial\Omega, \end{array}\right\} \tag{4.5}$$

where the associated stress S on Ω and the electric and magnetic fields e and h on the entire space are given by

$$\left\{\begin{array}{lll} S = \mathrm{D}_F \psi(F, d, b) & & \text{on} \quad \Omega \\ e = \mathrm{D}_d \psi(F, d, b), & h = \mathrm{D}_b \psi(F, d, b) & \text{on} \quad \Omega \\ e = J^{-1} F^{\mathrm{T}} F d, & h = J^{-1} F^{\mathrm{T}} F b & \text{on} \quad \Omega^{\mathrm{c}}. \end{array}\right.$$

To derive (4.5), note that if $(\kappa, \delta, \beta) \in \delta\, \mathfrak{S}$, then

$$\delta\, \mathsf{E}(\sigma)[\kappa, \delta, \beta] = \int_\Omega (S \cdot \nabla\kappa - g \cdot \kappa)\, dx + \int_{\mathbb{R}^n} (e \cdot \delta + h \cdot \beta)\, dx = 0 \tag{4.6}$$

by $(4.4)_1$. By $(4.2)_{2,3}$ we may write $\delta = \operatorname{curl} \pi$, $\beta = \operatorname{curl} \rho$; inserting this in to (4.6) and integrating by parts we obtain

$$\int_\Omega (-\operatorname{div} S - g) \cdot \kappa\, dx + \int_{\mathbb{R}^n} (\pi \cdot \operatorname{curl} e + \rho \cdot \operatorname{curl} h)\, dx = 0.$$

The arbitrariness of κ, π, and ρ then gives (4.5). □

5 $\mathscr{A}$-quasiconvexity: the general case

Our treatment of the convexity properties for the electro-magneto-elasticity is based on the $\mathscr{A}$-quasiconvexity theory, which includes the associated notions $\mathscr{A}$-quasiaffinity, $\mathscr{A}$-polyconvexity, Λ-convexity and Λ-ellipticity [5, 15, 37–38]. This section discusses these notions from a general point of view; the specialization to electro-magneto-elastic materials is the subject of the succeeding sections.

5.1 The differential operator $\mathscr{A}$ and the characteristic cone Λ The following dimensions will be needed in the subsequent discussion:

$$\begin{aligned} n &= \text{the number of independent variables, } x = (x_1, \ldots, x_n), \\ d &= \text{the number of dependent variables, } u = (u_1, \ldots, u_d), \\ l &= \text{the number of differential constrains.} \end{aligned}$$

Let $Q = (0, 1)^n$ be the unit cube, let $C^\infty_{\mathrm{per}}(\mathbb{R}^n, \mathbb{R}^d)$ denote the set of all infinitely differentiable Q-periodic maps $u : \mathbb{R}^n \to \mathbb{R}^d$. We shall consider the first–order differential constraint $\mathscr{A} v = 0$ on a map $v \in C^\infty(\mathbb{R}^n, \mathbb{R}^d)$ where

$$\mathscr{A} v = \sum_{i=1}^n A^{(i)} v_{,i}$$

with $A^{(i)} \in \mathrm{Lin}(\mathbb{R}^d, \mathbb{R}^l)$. For each $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$ define

$$\mathbb{A}(\eta) = \sum_{i=1}^{n} \eta_i A^{(i)},$$

which is an element of $\mathrm{Lin}(\mathbb{R}^d, \mathbb{R}^l)$, and make the standing assumption that the rank of $\mathbb{A}(\eta)$ is the same for all $\eta \neq 0$. We define the characteristic cone

$$\Lambda = \{u \in \mathbb{R}^d : \mathbb{A}(\eta)u = 0 \ \text{ for some } \ \eta \in \mathbb{R}^n,\ \eta \neq 0\}.$$

5.2 Definition A continuous function $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ is said to be
(i) $\mathscr{A}$-quasiconvex if

$$\int_Q f(u + v(x))\, dx \geq f(u)$$

for all $u \in \mathbb{R}^d$ and all $v \in C^\infty_{\mathrm{per}}(\mathbb{R}^n, \mathbb{R}^d)$ such that $\mathscr{A}v = 0$ on $\mathbb{R}^n$ and $\int_Q v\, dx = 0$;
(ii) $\mathscr{A}$-quasiaffine if it takes only finite values and both f and $-f$ are $\mathscr{A}$-quasiconvex;
(iii) Λ-convex if

$$f(tu_1 + (1-t)u_2) \leq tf(u_1) + (1-t)f(u_2)$$

for every $t \in (0, 1)$ and $u_1, u_2 \in \mathbb{R}^d$ such that $u_2 - u_1 \in \Lambda$;
(iv) Λ-affine if it takes only finite values and both f and $-f$ are Λ-convex.

If f is continuously differentiable then the Λ-convexity is equivalent to the Λ-ellipticity

$$\mathrm{D}^2 f(u)(l, l) \geq 0$$

for every $u \in \mathbb{R}^n$ and $l \in \Lambda$.

5.3 Theorem ([15; Proposition 3.4]) *If $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ is a continuous $\mathscr{A}$-quasiconvex function then f is Λ-convex; consequently, if f is $\mathscr{A}$-quasiaffine then f is Λ-affine.*

The following weak sequential lower semicontinuity theorem is the main motivation for the $\mathscr{A}$-quasiconvexity. We refer to Section 14 (below) for our conventions about the weak convergence. The weak sequential lower semicontinuity is the basic ingredient of the direct method of the calculus of variations. It should be also noted that for the proof of the existence of the minimizer in electro-magneto-elasticity in Theorem 9.3 (below) the sequential lower semicontinuity theorem cannot be used as the hypothesis (5.1) is inconsistent with the requirement $\psi(F, d, b) \to \infty$ for $\det F \to 0$.

5.4 Theorem ([15; Theorem 3.7]) *Let $1 \leq p < \infty$ and suppose that $f : \Omega \times \mathbb{R}^d \to [0, \infty)$ is a Carathéodory integrand such that $f(x, \cdot)$ is $\mathscr{A}$-quasiconvex and*

$$0 \leq f(x, u) \leq a(x)(1 + |u|^p) \tag{5.1}$$

for all $x \in \Omega$ and $u \in \mathbb{D}^d$ where $a : \Omega \to [0, \infty)$ is a bounded function. If u and u_k belong to $L^p(\Omega, \mathbb{R}^d)$ and satisfy

$$u_k \rightharpoonup u \quad \text{ in } L^p(\Omega, \mathbb{R}^d) \quad \text{ and } \quad \mathscr{A}u_k \to 0 \ \text{ in } \ W^{-1,p}(\Omega, \mathbb{R}^d)$$

then

$$\liminf_{k \to \infty} \int_\Omega f(x, u_k(x))\, dx \geq \int_\Omega f(x, u(x))\, dx.$$

5.5 Definition A continuous function $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ is said to be $\mathscr{A}$-polyconvex if there exists a finite number of $\mathscr{A}$-quasiaffine functions $f_1, \dots, f_m$ and a convex lowersemicontinuous function $\Phi : \mathbb{R}^m \to \bar{\mathbb{R}}$ such that

$$f(u) = \Phi(f_1(u), \dots, f_m(u))$$

for each $u \in \mathbb{R}^d$.

5.6 Theorem ([5; Corollary 2.5]) *Any $\mathscr{A}$-polyconvex function is $\mathscr{A}$-quasiconvex.*

6 A specialization to electro-magneto-elasticity: E-quasiconvexity

We apply the formalism of the preceding section with $n = 2$ or 3, and with the identifications

$$v = (F, d, b)$$

where F is the deformation gradient, d the electric displacement and b the magnetic induction. In view of the constraint $\det F > 0$, we apply the $\mathscr{A}$-quasiconvexity notions to functions f defined on the domain $\mathbb{D}^n_+$, see (3.10). To obtain an agreement with the general theory of the preceding section, where only functions f defined on the entire $\mathbb{R}^d$ have been considered, we tacitly extend $f : \mathbb{D}^n_+ \to \bar{\mathbb{R}}$ to the entire $\mathbb{D}^n$ from (3.11) by setting f equal to ∞ on $\mathbb{D}^n \sim \mathbb{D}^n_+$.

The functions v of Section 5 will be identified with the triples $(\varphi, \delta, \beta) : \mathbb{R}^n \to \mathbb{D}^n$ and the operator $\mathscr{A}$ with

$$\mathscr{A}(\varphi, \delta, \beta) = (\operatorname{curl} \varphi, \operatorname{div} \delta, \operatorname{div} \beta). \tag{6.1}$$

Here curl of $\varphi = [\varphi_{ij}]^n_{i,j=1}$ is defined by

$$(\operatorname{curl} \varphi)_{il} = \sum_{j,k=1}^{3} \varepsilon_{ljk} \varphi_{ij,k}, \qquad (\operatorname{curl} \varphi)_i = \sum_{j,k=1}^{2} \varepsilon_{jk} \varphi_{ij,k},$$

in dimensions $n = 3$ and $n = 2$, respectively, where $i, l = 1, 2, 3$ or $i = 1, 2$ and ε_{ijk} and ε_{ij} are the three– and two– dimensional permutation symbols.

To determine the characteristic cone $\Lambda \equiv \Lambda_{\mathrm{E}}$ corresponding to the system (6.1), we replace the partial derivatives $\nabla\varphi$, $\nabla\delta$, $\nabla\beta$ in (6.1) by the tensor products $\varphi \otimes \eta$, $\delta \otimes \eta$, $\beta \otimes \eta$ where $\eta \in \mathbb{R}^n$ is an arbitrary nonzero vector. This transforms (6.1) into

$$\varphi \times \eta = 0, \quad \delta \cdot \eta = 0, \quad \beta \cdot \eta = 0; \tag{6.2}$$

noting that $(6.2)_1$ is satisfied in and only if $\varphi = \xi \otimes \eta$ for some $\xi \in \mathbb{R}^n$, one obtains

$$\Lambda_{\mathrm{E}} = \{(\xi \otimes \eta, \delta, \beta) \in \mathbb{D}^n : \xi,\ \delta,\ \beta,\ \eta \in \mathbb{R}^n,\ \delta \cdot \eta = \beta \cdot \eta = 0,\ \eta \neq 0\}. \tag{6.3}$$

We now specialize the general definitions of Section 5 to the present case.

6.1 Definition A continuous function $f : \mathbb{D}^n \to \bar{\mathbb{R}}$ is said to be
(i) E-quasiconvex at $(F, d, b) \in \mathbb{D}^n$ if

$$\int_Q f(F + \varphi(x), d + \delta(x), b + \beta(x))\, dx \geq f(F, d, b)$$

for each triplet $(\varphi, \delta, \beta) \in C^\infty_{\mathrm{per}}(\mathbb{R}^n, \mathbb{D}^n)$ satisfying

$$\operatorname{curl}\varphi = 0, \quad \operatorname{div}\delta = \operatorname{div}\beta = 0 \quad \text{on} \quad \mathbb{R}^n \quad \text{and} \quad \int_Q (\varphi, \delta, \beta)\, dx = 0; \tag{6.4}$$

(ii) E-quasiconvex if it is E-quasiconvex at every point of $\mathbb{D}^n$;
(iii) E-quasiaffine if f takes only finite values, and both f and $-f$ are E-quasiconvex;
(iv) Λ_{E}-convex if

$$f(F + t\xi \otimes \eta, d + t\delta, b + t\beta) \le (1-t)f(F, d, b) + tf(F + \xi \otimes \eta, d + \delta, b + \beta)$$

for every $t \in (0, 1)$ and $(F, d, b) \in \mathbb{D}^n$ and every ξ, δ, β, $\eta \in \mathbb{R}^n$ such that

$$\delta \cdot \eta = \beta \cdot \eta = 0 \quad \text{and} \quad \eta \neq 0; \tag{6.5}$$

(v) Λ_{E}-affine if f takes only finite values, and both f and $-f$ are Λ_{E}-convex.

6.2 Proposition *Let $f : \mathbb{D}^n_+ \to \mathbb{R}$ be twice continuously differentiable.*
(i) *If f is* E*-quasiconvex at $(F, d, b) \in \mathbb{D}^n_+$ then f is elliptic at (F, d, b), i.e.,*

$$\mathrm{D}^2\psi(F, d, b)[(\xi \otimes \eta, \delta, \beta), (\xi \otimes \eta, \delta, \beta)] \ge 0 \tag{6.6}$$

for every ξ, δ, β, $\eta \in \mathbb{R}^n$ satisfying (6.5);
(ii) *if f is quadratic, i.e., if*

$$f(F, d, b) = C[(F, d, b), (F, d, b)]$$

for every $(F, d, b) \in \mathbb{D}^n$ and some symmetric bilinear form C then f is E*-quasiconvex at some point $\Leftrightarrow$ f is* E*-quasiconvex $\Leftrightarrow$ f is* E*-elliptic at some point $\Leftrightarrow$ f is* E*-elliptic at every point of $\mathbb{D}^n$.*

The main results of this paper are the following theorem and Theorem 6.5, below.

6.3 Theorem *A continuous function $f : \mathbb{D}^n \to \mathbb{R}$ is* E*-quasiaffine $\Leftrightarrow$ f is Λ_{E}-affine $\Leftrightarrow$ f is a linear combination, with constant coefficients, of the following functions*:

$$\left.\begin{array}{llllllllll} 1, & F, & & \det F, & d, & b, & Fd, & Fb, & d \times b & \text{if} \quad n = 2, \\ 1, & F, & \operatorname{cof} F, & \det F, & d, & b, & Fd, & Fb & & \text{if} \quad n = 3. \end{array}\right\} \tag{6.7}$$

Thus there are 15 linearly independent E-quasiaffine functions if $n = 2$ and 32 linearly independent E-quasiaffine functions if $n = 3$, including constants. The proof of Theorem 6.3 is deferred to Section 8.

6.4 Definition A continuous function $f : \mathbb{D}^n \to \bar{\mathbb{R}}$ is said to be E-polyconvex if there exists a finite number of E-quasiaffine functions $f_1, \ldots, f_m$ and a convex lowersemicontinuous function $\Phi : \mathbb{R}^m \to \bar{\mathbb{R}}$ such that

$$f(F, d, b) = \Phi(f_1(F, d, b), \ldots, f_m(F, d, b))$$

for each $(F, d, b) \in \mathbb{D}^n$.

Theorem 6.3 has the following corollary.

6.5 Theorem *A continuous function $f : \mathbb{D}^n \to \bar{\mathbb{R}}$ is* E*-polyconvex if and only is f is of the following form*:

$$f(F,d,b) = \Phi(\wedge(F,d,b))$$

for every $(F,d,b) \in \mathbb{D}^n_+$, where we abbreviate

$$\wedge(F,d,b) = \begin{cases} (F, \det F, d, b, Fd, Fb, d \times b) & \text{if} \quad n = 2, \\ (F, \operatorname{cof} F, \det F, d, b, Fd, Fb) & \text{if} \quad n = 3 \end{cases}$$

and where Φ is a convex lowersemicontinuous function on

$$\tilde{\mathbb{D}}^n = \begin{cases} \mathbb{M}^{2\times 2} \times \mathbb{R} \times (\mathbb{R}^2)^4 \times \mathbb{R} & \text{if} \quad n = 2, \\ \mathbb{M}^{3\times 3} \times \mathbb{M}^{3\times 3} \times \mathbb{R} \times (\mathbb{R}^3)^4 & \text{if} \quad n = 3. \end{cases}$$

Thus Φ is a function of 14 and 31 scalar variables, respectively.

7 Proof of Proposition 6.2

Noting that Item (i) of Proposition 6.2 follows from (6.3) and Theorem 5.3, we see that only Item (ii) needs a proof, and for this it suffices to prove that the ellipticity (6.6) implies that f is E-quasiconvex at 0.

The proof is essentially the same as van Hove's original proof [63] in the gradient case; the details are presented here only to explain the rôle of the side conditions (6.5). Thus our goal is to prove that the ellipticity condition (6.6) implies that

$$\int_Q C[(\varphi,\delta,\beta),(\varphi,\delta,\beta)]\,dx \geq 0$$

for each triplet $(\varphi,\delta,\beta) \in C^\infty_{\mathrm{per}}(\mathbb{R}^n,\mathbb{D}^n)$ satisfying (6.4). If $\hat{\varphi}_k$, $\hat{\delta}_k$, $\hat{\beta}_k$ are the Fourier coefficients defined by

$$\hat{\varphi}_k = \int_Q \varphi(x)\,\mathrm{e}^{2\pi \mathrm{i} k\cdot x}\,dx, \quad k \in \mathbb{Z}^n,$$

and similarly for $\hat{\delta}_k, \hat{\beta}_k$, then (6.4) provide

$$\hat{\varphi}_k \times k = 0, \quad k\cdot\hat{\delta}_k = 0, \quad k\cdot\hat{\beta}_k = 0, \quad \hat{\varphi}_0 = 0, \quad \hat{\delta}_0 = 0, \quad \hat{\beta}_0 = 0. \tag{7.1}$$

Relation $(7.1)_1$ is equivalent to

$$\hat{\varphi}_k = \hat{\xi}_k \otimes k \tag{7.2}$$

for some $\hat{\xi}_k \in \mathbb{C}^n$. By Parseval's equality and the last three equations in (7.1),

$$\int_Q C[(\varphi,\delta,\beta),(\varphi,\delta,\beta)]\,dx = \sum_{k\in\mathbb{Z}^n,\,k\neq 0} C[(\hat{\xi}_k\otimes k,\hat{\delta}_k,\hat{\beta}_k),(\hat{\xi}^*_k\otimes k,\hat{\delta}_k,\hat{\beta}^*_k)]$$

where $*$ denotes the complex conjugation. Each member of the sum on the right–hand side is nonnegative by the ellipticity, which is applicable as we have $(7.1)_{2,3}$ and (7.2). □

8 Proof of Theorem 6.3

Recall from Definition 6.1 that a continuous function $f : \mathbb{D}^n \to \mathbb{R}$ is E-quasiaffine if

$$\int_Q f(F+\varphi, d+\delta, b+\beta)\, dx = f(F,d,b) \tag{8.1}$$

for each triplet $(\varphi, \delta, \beta) \in C^\infty_{\mathrm{per}}(\mathbb{R}^n, \mathbb{D}^n)$ satisfying

$$\operatorname{curl}\varphi = 0, \quad \operatorname{div}\delta = \operatorname{div}\beta = 0 \quad \text{on} \quad \mathbb{R}^n \quad \text{and} \quad \int_Q (\varphi, \delta, \beta)\, dx = 0, \tag{8.2}$$

and that f is Λ_{E}-affine if

$$f(F+t\xi\otimes\eta, d+t\delta, b+t\beta) = (1-t)f(F,d,b) + tf(F+\xi\otimes\eta, d+\delta, b+\beta) \tag{8.3}$$

for every $t \in (0,1)$ and $(F,d,b) \in \mathbb{D}^n$ and every ξ, δ, β, $\eta \in \mathbb{R}^n$ such that

$$\delta\cdot\eta = \beta\cdot\eta = 0 \quad \text{and} \quad \eta \neq 0. \tag{8.4}$$

The proof of Theorem 6.3 is divided into several lemmas. We start with the analysis of the *separate* Λ_{E}-affinity with respect to the variables F, d, and b. The cross effects will be analyzed subsequently. We refer to Section 13 for the rank 1 affinity which underlies Item (i) of the following result and many points in the subsequent treatment.

8.1 Lemma *Let $f : \mathbb{D}^n \to \mathbb{R}$ be a Λ_{E}-affine function. Then*

(i) *for each d, $b \in \mathbb{R}^n$ the function $f(\cdot, d, b)$ is rank 1 affine, i.e., it is a linear combination, with coefficients depending on d, b, of the functions occurring in* (13.2);

(ii) *for each $F \in \mathbb{M}^{n\times n}$ the function $f(F,\cdot,\cdot)$ is a linear combination, with coefficients depending on F, of the functions*

$$\begin{array}{llll} 1, & d, & b, & d\times b \quad \text{if} \quad n=2, \\ 1, & d, & b & \quad\quad\quad\quad \text{if} \quad n=3. \end{array}$$

Proof (i): Fixing $d, b \in \mathbb{R}^n$, taking $\delta = \beta = 0$ in (8.3) and denoting $g(\cdot) = f(\cdot, d, b)$ we obtain Inequality (13.1) with the equality sign for every $t \in (0,1)$, every and $F \in \mathbb{M}^{n\times n}$ and every ξ, $\eta \in \mathbb{R}^n$. Thus $f(\cdot, d, b)$ is rank 1 affine and Lemma 13.2 yields the assertion.

(ii): Employing (8.3) with $\xi = \beta = 0$ and noting that there always exists an $\eta \in \mathbb{R}^n$, $\eta \neq 0$, such that $\delta\cdot\eta = \beta\cdot\eta = 0$, we obtain

$$f(F, d+t\delta, b) = (1-t)f(F,d,b) + tf(F, d+\delta, b)$$

for every F, d, b and δ. Thus $f(F,\cdot,b)$ is affine and hence

$$f(F,d,b) = \Delta(F,b) + \varepsilon(F,b)\cdot d,$$

for each $(F,d,b) \in \mathbb{D}^n$, where $\Delta(F,b) \in \mathbb{R}$ and $\varepsilon(F,b) \in \mathbb{R}^n$. Repeating the same argument for $\delta = 0$, β arbitrary, we obtain

$$f(F,d,b) = \Gamma(F,d) + \zeta(F,d)\cdot b,$$

$(F,d,b) \in \mathbb{D}^n$, where $\Gamma(F,d) \in \mathbb{R}$ and $\zeta(F,d) \in \mathbb{R}^n$. Thus

$$\Gamma(F,d)+\zeta(F,d)\cdot b=\Delta(F,b)+\varepsilon(F,b)\cdot d.$$

Since the left–hand side is affine in b at any fixed d, we see that the functions Δ and ε must be affine functions of b as well, i.e.,

$$\Delta(F,b)=c_2(F)\cdot b+c_4(F),\quad \varepsilon(F,b)=c_1(F)+A(F)b,$$

$b\in\mathbb{R}^n$, where $c_2(F)$, $c_1(F)\in\mathbb{R}^n$, $c_4(F)\in\mathbb{R}$ and $A(F)\in\mathbb{M}^{n\times n}$. Hence

$$f(F,d,b)=c_1(F)\cdot d+c_2(F)\cdot b+A(F)b\cdot d+c_4(F). \tag{8.5}$$

To complete the proof, we return to (8.3), this time with $\xi=0$, so that we have

$$f(F,d+t\delta,b+t\beta)=(1-t)f(F,d,b)+tf(F,d+\delta,b+\beta) \tag{8.6}$$

for every $t\in(0,1)$, every $(F,d,b)\in\mathbb{D}^n$ and every δ, β, $\eta\in\mathbb{R}^n$ such that (8.4) holds. This gives

$$A(F)(b+t\beta)\cdot(d+t\delta)=(1-t)A(F)b\cdot d+tA(F)(b+\beta)\cdot(d+\delta).$$

The left–hand side contains a quadratic term (i.e., the coefficient of t^2) which is equal to $A(F)\beta\cdot\delta$ and hence we have to have

$$A(F)\beta\cdot\delta=0 \tag{8.7}$$

for every δ, β such that $\delta\cdot\eta=\beta\cdot\eta=0$ for some $\eta\neq 0$.

If $n=3$, then for a given pair (δ,β) there always exists a $\eta\neq 0$ such that $\delta\cdot\eta=\beta\cdot\eta=0$. Hence (8.6) asserts that $f(F,\cdot,\cdot)$ is affine. Thus the bilinear term $A(F)b\cdot d$ in (8.5) must vanish and hence $f(F,\cdot,\cdot)$ is of the form asserted in (ii).

If $n=2$ then for a given pair (δ,β) there exists a $\eta\neq 0$ such that $\delta\cdot\eta=\beta\cdot\eta=0$ if and only if δ and β are parallel, i.e., $\delta\times\beta=0$. Thus (8.7) requires

$$A(F)b\cdot d=c_3(F)(d\times b)$$

for all d, $b\in\mathbb{R}^2$ and some $c_3(F)\in\mathbb{R}$. Then (8.5) gives the asserted form. □

We are about to pass to the cross effects. In view of the results of Lemma 8.1 it suffices to consider functions of very special forms considered in Lemmas 8.2–8.5, as explained in the proof of Lemma 8.6.

8.2 Lemma *Let $f:\mathbb{D}^n\to\mathbb{R}$ be given by*

$$f(F,d,b)=\Omega(F)\cdot d$$

$(F,d,b)\in\mathbb{D}^n$ where Ω is a linear transformation from $\mathbb{M}^{n\times n}$ into $\mathbb{R}^n$, written $F\mapsto\Omega(F)$. Then f is Λ_{E}-affine if and only if f is of the form

$$f(F,d,b)=Fd\cdot c$$

for all $(F,d,b)\in\mathbb{D}^n$ and some $c\in\mathbb{R}^n$.

Proof Writing the equality (8.3) with the choice $F=0$, $d=b=0$, we obtain

$$t^2\Omega(\xi\otimes\eta)\cdot\delta=t\Omega(\xi\otimes\eta)\cdot\delta \tag{8.8}$$

for every $t\in(0,1)$, every ξ, δ, $\eta\in\mathbb{R}^n$ such that

$$\delta\cdot\eta=0,\quad \eta\neq 0. \tag{8.9}$$

Thus $\Omega(\xi \otimes \eta) \cdot \delta = 0$ for every $\xi, \delta, \eta \in \mathbb{R}^n$ such that (8.9) holds. Consequently,

$$\Omega(\xi \otimes \eta) = m(\xi)\eta$$

$\xi, \eta \in \mathbb{R}^n$ where $m(\xi) \in \mathbb{R}$. The linearity in ξ requires $m(\xi) = c \cdot \xi$ for some $c \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^n$; thus $\Omega(\xi \otimes \eta) = (c \cdot \xi)\eta = (\xi \otimes \eta)^{\mathrm{T}} c$ for all $\xi, \eta \in \mathbb{R}^n$. Since every $A \in \mathbb{M}^{n \times n}$ is a sum of tensor products $\xi \otimes \eta$, the linearity of $\Omega(\cdot)$ yields $\Omega(A) = A^{\mathrm{T}} c$ for all $A \in \mathbb{M}^{n \times n}$. Hence

$$f(F, d, b) = \Omega(A) \cdot d = F^{\mathrm{T}} c \cdot d \equiv Fd \cdot c.$$

This completes the proof of the direct implication; the proof of the converse implication is straightforward and the details are omitted. □

8.3 Lemma *Let $n = 3$ and let $f : \mathbb{D}_3 \to \mathbb{R}$ be given by*

$$f(F, d, b) = \Psi(\operatorname{cof} F) \cdot d$$

$(F, d, b) \in \mathbb{D}^n$ where Ψ is a linear transformation from $\mathbb{M}^{3\times 3}$ into $\mathbb{R}^3$, written $A \mapsto \Psi(A)$. Then f is Λ_{E}-affine if and only if $f = 0$ identically.

Proof We apply (8.8) with $F = \mathrm{I}$, $d = 0$ and $\xi, \delta, \eta \in \mathbb{R}^3$ as in (8.9). Using the formula

$$\operatorname{cof}(\mathrm{I} + \xi \otimes \eta) = \big((1 + \xi \cdot \eta)\mathrm{I} - \eta \otimes \xi\big)$$

one finds that (8.3) is equivalent to

$$t^2 \Psi\big((\xi \cdot \eta)\mathrm{I} - t\eta \otimes \xi\big) \cdot \delta = t\Psi\big((\xi \cdot \eta)\mathrm{I} - \eta \otimes \xi\big) \cdot \delta$$

This requires

$$\Psi\big((\xi \cdot \eta)\mathrm{I} - \eta \otimes \xi\big) \cdot \delta = 0$$

for every $\xi, \delta, \eta \in \mathbb{R}^3$ as above. Hence

$$\Psi((\eta \cdot \xi)\mathrm{I} - \eta \otimes \xi) = m(\xi)\eta$$

$\xi, \eta \in \mathbb{R}^3$ where $m(\xi) \in \mathbb{R}$. The linearity in ξ provides $m(\xi) = -c \cdot \xi$ for some $c \in \mathbb{R}^3$ and all ξ; hence

$$\Psi((\eta \cdot \xi)\mathrm{I} - \eta \otimes \xi) = -(c \cdot \xi)\eta = -(\eta \otimes \xi)c.$$

Setting $a := \Psi(\mathrm{I})$, we obtain

$$\Psi(\eta \otimes \xi) = (\eta \cdot \xi)a + (\eta \otimes \xi)c.$$

Since every $A \in \mathbb{M}^{3\times 3}$ is a sum of tensor products $\eta \otimes \xi$, the linearity of $\Psi(\cdot)$ yields

$$\Psi(A) = (\operatorname{tr} A)a + Ac$$

for each $A \in \mathbb{M}^{3\times 3}$. The consistency requires $a = \Psi(\mathrm{I}) = 3a + c$ and hence

$$\Psi(A) = -\frac{1}{2}(\operatorname{tr} A)c + Ac.$$

To complete the proof, let us show that $c = 0$. Let $\eta \in \mathbb{R}^3$ be any unit vector, and apply (8.3) with $F = \mathrm{I} - \eta \otimes \eta$, $d = 0$, $\delta \in \mathbb{R}^3$ satisfying $\delta \cdot \eta = 0$. Using

$$\operatorname{cof}(F + t\eta \otimes \eta) = tF + \eta \otimes \eta$$

one finds that Equation (8.3) reads

$$t\big(tFc - (1 + t/2)c\big) \cdot \delta = -tc \cdot \delta/2;$$

the arbitrariness of t then leads to the unique consequence

$$c \cdot \delta = 0$$

for any δ such that $\delta \cdot \eta = 0$ for some unit vector η. Taking η such that $\eta \cdot c = 0$, we can take $\delta = c$ to obtain $c = 0$. □

8.4 Lemma *Let $f : \mathbb{D}^n \to \mathbb{R}$ be given by*

$$f(F, d, b) = (\det F)c \cdot d$$

for every $(F, d, b) \in \mathbb{D}^n$ where $c \in \mathbb{R}^n$ is a constant. If f is Λ_{E}-affine then $f = 0$ identically.

Proof We apply (8.3) with $F = \mathrm{I}$, $d = 0$ and ξ, δ, $\eta \in \mathbb{R}^n$ as in (8.9). This gives

$$t(1 + t\xi \cdot \eta)(c \cdot d) = t(c \cdot d) + (1 - t)(1 + \xi \cdot \eta)(c \cdot d)$$

and clearly this can hold only if $c = 0$. □

8.5 Lemma *Let $n = 2$ and let $f : \mathbb{D}_2 \to \mathbb{R}$ be given by*

$$f(F, d, b) = m(F)(d \times b)$$

for every $(F, d, b) \in \mathbb{D}^n$ where $m : \mathbb{M}^{2\times 2} \to \mathbb{R}$ is a rank 1 affine function. Then f is Λ_{E}-affine if and only if

$$f(F, d, b) = c(d \times b) \tag{8.10}$$

for all $(F, d, b) \in \mathbb{D}_2$ and some $c \in \mathbb{R}$.

Proof By Lemma 13.2 we have $m(F) = A \cdot F + b \det F + c$ for each $F \in \mathbb{M}^{2\times 2}$ where $A \in \mathbb{M}^{2\times 2}$ and b, $c \in \mathbb{R}$ are constants. Hence

$$f(F, d, b) = (A \cdot F + b \det F + c)(d \times b).$$

Let $\eta \in \mathbb{R}^2$ be any unit vector, let $\beta = \eta^{\perp}$ and $\lambda > 0$. Let us write the equality (8.3) with $F = \lambda \mathrm{I}$, $d = \eta$, $b = 0$, $\xi \in \mathbb{R}^2$ arbitrary, $\delta = 0$. This gives

$$\begin{aligned} t\left[A \cdot (\lambda \mathrm{I} + t\xi \otimes \eta) + b\lambda^2\left(1 + t\lambda^{-1}(\xi \cdot \eta)\right) + c\right](\eta \times \eta^{\perp}) \\ = (1 - t)f(F, d, b) + tf(F + \xi \otimes \eta, d, b + \beta). \end{aligned}$$

The quadratic term (i.e., the coefficient of t^2) on the left–hand side is

$$A \cdot (\xi \otimes \eta) + b\lambda(\xi \cdot \eta).$$

This term must vanish. The arbitrariness of λ, ξ, η then gives $A = 0$, $b = 0$. Thus we have (8.10). Conversely, if f is given by (8.10), then clearly, f is Λ_{E}-affine. □

8.6 Lemma *A continuous function $f : \mathbb{D}^n \to \mathbb{R}$ is Λ_{E}-affine if and only if f is a linear combination, with constant coefficients, of the functions occurring in* (6.7).

Proof Let f be Λ_{E}-affine. By Lemma 8.1(ii) then

$$f(F, d, b) = c_0(F) + c_1(F) \cdot d + c_2(F) \cdot b + c_3(F)(d \times b)$$

for each $(F, d, b) \in \mathbb{D}^n$ where the last term must be omitted if $n = 3$. By Item (i) of the same lemma then $f(\cdot, d, b)$ is a rank 1 affine function for each d, $b \in \mathbb{R}^n$. The independence of d, b and $d \times b$ then implies that each of the coefficients c_0 to c_3 are rank 1 affine functions. Lemma 13.2 then asserts that c_0 and c_3 are exactly of the form described in (13.2). Since c_1 and c_2 are vector valued functions, a componentwise application of Lemma 13.2 gives

$$c_1(F) = c + \Omega(F) + \Psi(\operatorname{cof} F) + d \det F$$

for every $F \in \mathbb{M}^{n\times n}$, where $c \in \mathbb{R}^n$ and Ω, Ψ are linear transformations from $\mathbb{M}^{n\times n}$ to $\mathbb{R}^n$ with $\Psi = 0$ if $n = 2$. A similar form applies to c_2. Using the just described forms of $c_0(F)$ to $c_3(F)$ and collecting some of the terms of the same type into one, it is found that

$$\begin{aligned} f(F,d,b) &= m_0(F,d,b) + M_1 \cdot \operatorname{cof} F \\ &\quad + \Omega_1(F)\cdot d + \Omega_2(F)\cdot b + \Psi_1(\operatorname{cof} F)\cdot d + \Psi_2(\operatorname{cof} F)\cdot b \\ &\quad + (m_2\cdot d + m_3\cdot b)\det F + m_4(F)(d\times b) \end{aligned} \tag{8.11}$$

for every $(F,d,b) \in \mathbb{D}^n$ where m_0 is an affine function of F, d, b, the tensor M_1 is in $\mathbb{M}^{n\times n}$ with $M_1 = 0$ if $n = 2$, the objects Ω_i, Ψ_i $(i = 1,2)$ are linear transformations from $\mathbb{M}^{n\times n}$ into $\mathbb{R}^n$ with $\Psi_i = 0$ if $n = 2$, the vectors $m_i \in \mathbb{R}^n$ $(i = 1,2)$ are constants and m_4 is a rank 1 affine function.

We shall now make use of the full power of the Λ_{E}-affinity equality (8.3) (so far only various particular cases have been used). Inserting the form of f from (8.11) into (8.3) and noting that the affine function m_0 and the term $M_1 \cdot \operatorname{cof} F$ trivially satisfy that equality, we see that we have to require that the function f_1 given by

$$\begin{aligned} f_1(F,d,b) &= \Omega_1(F)\cdot d + \Omega_2(F)\cdot b + \Psi_1(\operatorname{cof} F)\cdot d + \Psi_2(\operatorname{cof} F)\cdot b \\ &\quad + (m_2\cdot d + m_3\cdot b)\det F + m_4(F)(d\times b) \end{aligned}$$

has to satisfy (8.3). Then of course the terms of different order in F have to satisfy the equality individually as well as the terms with d and b. Thus each of the functions

$$\Omega_1(F)\cdot d, \quad \Omega_2(F)\cdot b,$$
$$\Psi_1(\operatorname{cof} F)\cdot d, \quad \Psi_2(\operatorname{cof} F)\cdot b, \quad m_i\cdot d\det F, \quad m_4(F)(d\times b)$$

must be Λ_{E}-affine. By Lemma 8.2 then $\Omega_1(F)\cdot d = Fd\cdot c_1$, $\Omega_2(F)\cdot b = Fb\cdot c_2$ where $c_i \in \mathbb{R}^n$ are constants, by Lemma 8.3 then $\Psi_i = 0$, $i = 1,2$, by Lemma 8.4 $m_i = 0$ and by Lemma 8.5 m_4 is constant. The asserted form of f follows. The converse implication is immediate. □

We conclude this section with the following converse statement.

8.7 Lemma *Each function from the list* (6.7) *is* E*-quasiaffine.*

Proof We have to prove that any function f from the list (6.7) satisfies the equality (8.1) for each triplet $(\varphi,\delta,\beta) \in C^\infty_{\mathrm{per}}(\mathbb{R}^n,\mathbb{D}^n)$ satisfying (8.2).

Consider first the case $n = 3$. The system (8.2) implies that there are functions ω, π, $\rho \in C^\infty(\mathbb{R}^3,\mathbb{R}^3)$, such that $\varphi = \nabla\omega$, $\delta = \operatorname{curl}\pi$, $\beta = \operatorname{curl}\rho$; Equation (8.1) then reads

$$\int_Q f(F+\nabla\omega, d+\operatorname{curl}\pi, b+\operatorname{curl}\rho)\,dx = f(F,d,b). \tag{8.12}$$

An argument described in [2; Remark, p. 141] shows that it suffices to verify (8.12) only for ω, π, $\rho \in C^\infty_0(Q,\mathbb{R}^3)$. The verification of (8.12) for the functions $(F,d,b) \mapsto F$, $\operatorname{cof} F$, $\det F$ is standard, see, e.g., [1]. Consider now the functions $(F,d,b) \mapsto d$, b. Then (8.12) reads

$$\int_Q (d+\operatorname{curl}\pi)\,dx = d$$

and a similar equation for b, which is true since

$$\int_Q \operatorname{curl} \pi \, dx = \int_{\partial Q} \pi \times n \, dA(x) = 0$$

by Gauss theorem (where n is the normal to ∂Q) and π vanishes on ∂Q. Finally consider the functions $(F, d, b) \mapsto Fd, Fb$. We have

$$\begin{aligned}
\int_Q \big((F + \nabla\omega)(d + \delta)\big) \, dx &= \int_Q \operatorname{div}\big((Fx + \omega) \otimes (d + \delta)\big) \, dx \\
&= \int_{\partial Q} (Fx + \omega)\big((d + \delta) \cdot n\big) \, dA(x) \\
&= \int_{\partial Q} Fx \, (d \cdot n) \, dA(x) \quad (\text{since } \delta = 0 \text{ on } \partial Q) \\
&= \int_Q \operatorname{div}\big((Fx) \otimes d\big) \, dx = Fd.
\end{aligned}$$

This completes the proof for $n = 3$. The case $n = 2$ is similar; the details are omitted. □

9 Existence theorem

The present section deals with the existence of minimum energy states for the energy E of a E-polyconvex solid and the surrounding vacuum electromagnetic field. As is usual, the state space $\mathfrak{S}$ from Section 4 has to be enlarged as described in Definition 9.1 (below). Let us recall the definitions of $\mathbb{D}^n_+$ and $\mathbb{D}^n$ in (3.10) and (3.11).

The existence theory for a purely elastic material with a polyconvex energy is well understood [1, 17, 36, 18]. The corresponding part of the proof is based on the sequential weak continuity of the cofactor and determinant. The additional electromagnetic variables d and b interact with the mechanical variable in the nonlinear terms Fd and Fb and in dimension 2 we have also electrical–magnetic interactions $d \times b$. These terms are sequentially weakly continuous as well, but this time one has to use the div–curl lemma. We summarize the results on the weak convergence and weak continuity in Section 14, below.

9.1 Definition Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. We denote by $\mathfrak{S}$ the set of all triplets $(y, d, b) \in W^{1,1}(\Omega, \mathbb{R}^n) \times L^1(\Omega, \mathbb{R}^n) \times L^1(\Omega, \mathbb{R}^n)$ such that

$$\left.\begin{aligned} \operatorname{div} d = 0, \quad \operatorname{div} b = 0 \quad \text{in } \mathbb{R}^n, \\ y = \tilde{y} \quad \text{on} \quad \partial\Omega \end{aligned}\right\} \tag{9.1}$$

in the sense of distributions and in the sense of traces, respectively. Here $\tilde{y} : \partial\Omega \to \mathbb{R}^n$ is a prescribed function. As in § 3.1.2, we assume that $\tilde{y}$ can be extended to an equally denoted injective function on $\partial\Omega \cup \Omega^{\mathrm{c}}$; in the present section we assume that

$$c \cdot \mathrm{I} \geq J^{-1} F^{\mathrm{T}} F \geq c^{-1} \cdot \mathrm{I} \tag{9.2}$$

on Ω^{c} where F is as in (3.5) and c is a positive constant.

9.2 Definition The total energy of a state $\sigma = (y, d, b) \in \mathfrak{S}$ is defined by the original formula (4.1) where $\psi : \mathbb{D}^n_+ \to \mathbb{R}$ is the energy, which is assumed to be continuous and bounded from below and where we assume that the body force g is in $L^\infty(\Omega, \mathbb{R}^n)$ for notational simplicity.

9.3 Theorem *Let* (9.2) *hold and let* p, r, *and* s *be numbers satisfying*

$$2 \le p < \infty, \quad 1/r + 1/p \le 1, \quad 1/s + 1/p \le 1,$$

and additionally

$$\begin{cases} \textit{let } \ 1/r + 1/s \le 1 & \text{if} \quad n = 2, \\ \textit{let } \ q \ \textit{ be a number satisfying } \ 3/2 \le q < \infty & \text{if} \quad n = 3. \end{cases}$$

Extend the energy function $\psi : \mathbb{D}^n_+ \to \mathbb{R}$ *to* $\tilde{\psi} : \mathbb{D}^n \to \mathbb{R}$ *by setting* $\tilde{\psi}(F, d, b) = \infty$ *if* $\det F \le 0$ *and assume that the following conditions hold*:

(i) *there exists a continuous convex and bounded from below function* $\Phi : \tilde{\mathbb{D}}^n \to \mathbb{R} \cup \{\infty\}$ *such that*

$$\tilde{\psi}(F, d, b) = \Phi(\wedge(F, d, b)) \tag{9.3}$$

for every $(F, d, b) \in \mathbb{D}^n$;

(ii) *we have*

$$\psi(F, d, b) \ge \begin{cases} c(|F|^p + |d|^r + |b|^s) + d & \text{if} \quad n = 2, \\ c(|F|^p + |\operatorname{cof} F|^q + |d|^r + |b|^s) + d & \text{if} \quad n = 3 \end{cases}$$

for some $c > 0$, $d \in \mathbb{R}$ *and all* $(F, d, b) \in \mathbb{D}^n_+$.

If $\mathfrak{S}$ *contains an element of finite total energy then there exists a* $\sigma = (y, d, b) \in \mathfrak{S}$ *such that*

$$\mathsf{E}(\sigma) \le \mathsf{E}(\bar{\sigma})$$

for all $\bar{\sigma} \in \mathfrak{S}$; *each such a* σ *satisfies*

$$\det \nabla y > 0 \ \textit{ for almost every point of } \ \Omega.$$

Note that the continuity of Φ, the definition of $\tilde{\psi}$, and (9.3) imply that $\psi(F, d, b) \to \infty$ if $\det F \to 0$.

Proof Let $n = 3$ and assume that $q > 3/2$ rather than $q \ge 3/2$ to simplify the matters; the case $q = 3/2$ is similar but slightly more complicated (cf. [34; Proof of Theorem 5. 1, Case 2] for purely elastic bodies).

In the proof, we are going to apply Propositions 14.1 and 14.2, below. These propositions involve hypotheses on the exponents; we leave to the reader to verify that the hypotheses of the present theorem on p, q, r, and s are chosen exactly to satisfy the hypotheses of Propositions 14.1 and 14.2.

Let $\sigma_k = (y_k, d_k, b_k) \in \mathfrak{S}$ be a minimizing sequence and write $F_k = \nabla y_k$ for brevity. The coercivity condition (ii) implies that the sequence y_k is bounded in $W^{1,p}(\Omega, \mathbb{R}^3)$, the sequence $\operatorname{cof} F_k$ is bounded in $L^q(\Omega, \mathbb{M}^{3\times 3})$, the restrictions of d_k and b_k to Ω are bounded in $L^r(\Omega, \mathbb{R}^3)$ and $L^s(\Omega, \mathbb{R}^3)$, respectively, and the restrictions of d_k and b_k to Ω^c are bounded in $L^2(\Omega^c, \mathbb{R}^3)$. The reflexivity of these spaces implies that it is possible to extract a subsequence of the sequence $\sigma_k = (y_k, d_k, b_k)$, again denoted by σ_k, such that

$$
\begin{aligned}
y_k &\rightharpoonup y \quad \text{in} \quad W^{1,p}(\Omega,\mathbb{R}^3),\\
d_k &\rightharpoonup d \quad \text{in} \quad \begin{cases} L^r(\Omega,\mathbb{R}^3),\\ L^2(\Omega^c,\mathbb{R}^3),\end{cases}\\
b_k &\rightharpoonup b \quad \text{in} \quad \begin{cases} L^s(\Omega,\mathbb{R}^3),\\ L^2(\Omega^c,\mathbb{R}^3)\end{cases}
\end{aligned}
$$

for some (y,d,b) the indicated spaces. Proposition 14.1 then implies that

$$\operatorname{cof} F_k \rightharpoonup \operatorname{cof} F \qquad \text{in } L^q(\Omega,\mathbb{M}^{3\times 3}), \tag{9.4}$$

$$\det F_k \rightharpoonup \det F \qquad \text{in } L^{2q/3}(\Omega). \tag{9.5}$$

Furthermore, the components of the vector Fd with a general $F = \nabla y$ and a general d are $F_i \cdot d$ where $F_i = (F_{i1}, F_{i2}, F_{i3})$ and since

$$\operatorname{curl} F_i = 0, \quad \operatorname{div} d = 0,$$

the div–curl lemma Proposition 14.2 implies

$$F_k d_k \rightharpoonup Fd \quad \text{in } L^1(\Omega,\mathbb{R}^3) \quad \text{and similarly} \quad F_k b_k \rightharpoonup Fb \quad \text{in } L^1(\Omega,\mathbb{R}^3).$$

To summarize, we have

$$\wedge(F_k, d_k, b_k) \rightharpoonup \wedge(F,d,b) \qquad \text{in } L^1(\Omega) \tag{9.6}$$

and hence Proposition 14.3 gives

$$\liminf_{k\to\infty} \int_\Omega \Phi(\wedge(F_k,d_k,b_k))\,dx \ge \int_\Omega \Phi(\wedge(F,d,b))\,dx.$$

This can be rewritten as

$$\liminf_{k\to\infty} \int_\Omega \tilde\psi(F_k,d_k,b_k)\,dx \ge \int_\Omega \tilde\psi(F,d,b)\,dx. \tag{9.7}$$

The integral on the right–hand side is finite and the fact that $\tilde\psi = \infty$ if $\det F \le 0$, we see that $\det \nabla y > 0$ for almost every $x \in \Omega$. Furthermore, the weak form of $(9.1)_1$ reads

$$\int_{\mathbb{R}^3} d_k \cdot \nabla\phi\,dx = 0, \qquad \int_{\mathbb{R}^3} b_k \cdot \nabla\phi\,dx = 0$$

for each indefinitely integrable function $\phi : \mathbb{R}^3 \to \mathbb{R}$ with compact support. The convergence indicated in $(9.6)_{4,5}$ then yields

$$\int_{\mathbb{R}^3} d \cdot \nabla\phi\,dx = 0, \qquad \int_{\mathbb{R}^3} b \cdot \nabla\phi\,dx = 0,$$

i.e.,

$$\operatorname{div} d = 0, \quad \operatorname{div} b = 0 \quad \text{in } \mathbb{R}^3 \text{ in the sense of distributions.}$$

As one also finds that $y_k = \tilde y$ on $\partial\Omega$ implies $y = \tilde y$ on $\partial\Omega$, we see that the triplet $\sigma = (y,d,b)$ belongs to $\mathfrak{S}$. Also, trivially in view of (9.2),

$$\liminf_{k\to\infty} \int_{\Omega^c} J^{-1}(|Fd_k|^2 + |Fb_k|^2)\,dx \ge \int_{\Omega^c} J^{-1}(|Fd|^2 + |Fb|^2)\,dx, \tag{9.8}$$

$$\lim_{k\to\infty}\int_\Omega y_k\cdot g\,dx=\int_\Omega y\cdot g\,dx. \tag{9.9}$$

Inequalities (9.7)–(9.9) can be collected to show that

$$\liminf_{k\to\infty}\mathsf{E}(\sigma_k)\geq\mathsf{E}(\sigma)$$

which shows that σ is the required minimizer. This completes the proof in the case $n=3$.

The proof is similar if $n=2$. Instead of Proposition 14.1 one has to use the simpler result of Reshetnyak [45, 47] and Ball [1] that $y_k\rightharpoonup y$ in $W^{1,p}(\Omega,\mathbb{R}^2)$ with $p>2$ implies $\det F_k\rightharpoonup F$ in $L^{p/2}(\Omega)$; moreover, one more use of the div–curl lemma Proposition 14.2 is needed to show that $d_k\rightharpoonup d$ in $L^r(\Omega,\mathbb{R}^2)$ and $b_k\rightharpoonup b$ in $L^s(\Omega,\mathbb{R}^2)$ implies that $d_k\times b_k\rightharpoonup d\times b$ in $L^1(\Omega)$. For this, one has to identify the sequence g_k of Proposition 14.2 with $(d_{2,k},-d_{1,k})$ so that $\operatorname{div}d_k=0$ reads $\operatorname{curl}g_k=0$. □

10 Isotropic materials

This section deals with E-polyconvex energy functions of electro-magneto-elastic isotropic materials. Recall from §3.2.3 that a function $\psi:\mathbb{D}^n_+\to\bar{\mathbb{R}}$ is said to be isotropic (and objective) if

$$\psi(QFR^{\mathrm{T}},Rd,Rb)=\psi(F,d,b) \tag{10.1}$$

for each $Q,R\in\mathrm{SO}(n)$ and each $(F,d,b)\in\mathbb{D}^n_+$. In Theorem 10.1 we give a general sufficient condition for the E-polyconvexity of isotropic functions, which is then used in Examples A–C to describe various particular cases.

The following terminology is needed. Let $\phi:\operatorname{dom}\phi\to\bar{\mathbb{R}}$ be a function on a domain $\operatorname{dom}\phi\subset[0,\infty)^d$ where d is a positive integer. Denoting by $z=(z_1,\dots,z_d)$ the 'generic' variable of ϕ, and letting $K\subset\{1,\dots,d\}$ be a specified set of indices, we say that ϕ is non-decreasing in the variables z_k with $k\in K$ if for every $z=(z_1,\dots,z_d)\in\operatorname{dom}\phi$ and every $k\in K$ the function $t\mapsto\phi(z_1,\dots,z_k+t,\dots,z_d)$ is nondecreasing. Furthermore, following [50], we say that ϕ is pairwise nondecreasing in the variables z_k with $k\in K$ if for every integers $k,l\in K$ such that $k<l$ and every $z=(z_1,\dots,z_d)\in\operatorname{dom}\phi$ the function $t\mapsto\phi(z_1,\dots,z_k+t,\dots,z_l+t,\dots,z_d)$ is nondecreasing. Recall also that the singular values $v_1\geq\dots\geq v_n\geq 0$ of a transformation $A\in\mathbb{M}^{n\times n}$ are the eigenvalues of $\sqrt{A^{\mathrm{T}}A}$, i.e., the square roots of the eigenvalues of $A^{\mathrm{T}}A$.

10.1 Theorem *Let $\psi:\mathbb{D}^n_+\to\mathbb{R}$ be given by*

$$\psi(F,d,b)=\begin{cases}\Theta(v_1,v_2,d^{\parallel},b^{\parallel},d^{\times},b^{\times},v_1v_2,p) & \text{if } n=2,\\ \Theta(v_1,v_2,v_3,v_1v_2,v_1v_3,v_2v_3,d^{\parallel},b^{\parallel},d^{\times},b^{\times},v_1v_2v_3) & \text{if } n=3\end{cases} \tag{10.2}$$

for each $(F,d,b)\in\mathbb{D}^n_+$ where

$$v_1,\dots,v_n \text{ are the singular values of } F,$$

$$d^{\parallel}=|d|,\quad b^{\parallel}=|b|,\quad d^{\times}=|Fd|,\quad b^{\times}=|Fb| \quad\text{and}\quad p=d\times b \quad\text{if}\quad n=2,$$

and where Θ *is a convex function on* $[0,\infty)^7 \times (0,\infty)$ *if* $n = 2$ *and on* $[0,\infty)^{10} \times (0,\infty)$ *if* $n = 3$ *such that*

$$\left.\begin{array}{ll} \Theta(z_1,\dots,z_8) & \textit{is pairwise nondecreasing in } z_1, z_2, \textit{ nondecreasing in } z_3, \dots, z_6, \textit{ and symmetric under the exchange of } z_1 \textit{ and } z_2 \end{array}\right\} \quad \text{if } n = 2,$$

$$\left.\begin{array}{ll} \Theta(z_1,\dots,z_{11}) & \textit{is pairwise nondecreasing in } z_1, z_2, z_3, \textit{ pairwise nondecreasing in } z_4, z_5, z_6, \textit{ nondecreasing in the variables } z_7, \dots, z_{10}, \textit{ and symmetric under the permutations of } z_1, z_2, z_3 \textit{ and of } z_4, z_5, z_6 \end{array}\right\} \quad \text{if } n = 3.$$

Then ψ *is a* E*-polyconvex isotropic function.*

Theorem 10.1 is an extension of the well-known result of Ball [1; Theorem 5.2] for the purely mechanical case. However, even in this restricted context, Theorem 10.1 is more general than [1; Theorem 5.2], since the latter requires that $\Theta(z)$ be nondecreasing in $z_1,\dots,z_6$ while the present result requires less: the pairwise nondecreasing character as stated precisely above. Example C, below, shows the difference. This extension in the purely mechanical case is due to Rosakis [50] (see also [54] for $n = 2$). An as yet another extension in [31] will be discussed elsewhere.

Theorem 10.1 can be used to produce a wide variety of different isotropic E-polyconvex functions. It also covers the approach based on isotropic invariants. The following examples demonstrate these claims.

10.2 Example A The members of the following list are isotropic E-polyconvex functions:

$$\psi(F,d,b) = \begin{cases} \left.\begin{array}{l} v_1^\alpha + v_2^\alpha + v_3^\alpha \equiv \operatorname{tr}(F^{\mathrm{T}}F)^{\alpha/2} \\ v_1^\alpha v_2^\alpha + v_2^\alpha v_3^\alpha + v_3^\alpha v_1^\alpha \equiv \operatorname{tr}(\operatorname{cof}(F^{\mathrm{T}}F)^{\alpha/2}) \\ |d|^\alpha,\ |b|^\alpha,\ |Fd|^\alpha,\ |Fb|^\alpha \end{array}\right\} & \alpha \geq 1, \\ J^{-\gamma} & \gamma > 0, \\ |d|^\beta/J^\gamma,\ |b|^\beta/J^\gamma,\ |Fd|^\beta/J^\gamma,\ |Fb|^\beta/J^\gamma, & \beta - 1 \geq \gamma > 0, \end{cases} \tag{10.3}$$

$(F,d,b) \in \mathbb{D}^n_+$, where we abbreviate $J = \det F$. Indeed, we shall show that each of the above functions is of the format discussed in Theorem 10.1 with various choices of Θ. The first two members of (10.3) are the well-known forms proposed by Ogden [39–40]; the choice of Θ is obvious for them as well as for the members on the third and fourth lines of (10.3). To prove the E-polyconvexity of the functions on the last line of (10.3), we use the following elementary fact: if β and γ are positive numbers then the function $(a,b) \mapsto f(a,b) = a^\beta/b^\gamma$ is convex on $(0,\infty)^2$ if and only if $\beta \geq \gamma + 1$. This is verified by a direct check of the positive semidefinite character of the hessian of f. Employing the convexity of f, we prove the E-polyconvexity of, say, $\psi(F,d,b) = |Fd|^\beta/J^\gamma$ by observing that ψ is of the format of Theorem 10.1 with $\Theta(z) = z_9^\beta/z_{11}^\gamma$, and that Θ meets the requirements stated in that theorem. The E-polyconvexity of the remaining items in last line of (10.3) are proved similarly, which completes the proof of the E-polyconvexity of the list (10.3). Then also each

linear combination of the functions in (10.3) with positive coefficients (and of course with different values of the exponents) is E-polyconvex. This can be used to produce a big supply of isotropic E-polyconvex functions.

10.3 Example B (Isotropic invariants) An alternative approach to isotropic functions is based on isotropic invariants and on the representation theorems. That approach already occurred in the basic paper by Toupin [60] (who used a slightly different set of independent variables) and since then employed numerously in the literature.

The following complete list of isotropic invariantsis frequently used in the literature (see, e.g., [8]) for electro-elastic interactions

$$I_1, \quad I_2, \quad I_3, \quad K_1^{\mathrm{e}}, \quad K_2^{\mathrm{e}}, \quad K_3^{\mathrm{e}}$$

where

$$I_1 = \operatorname{tr}(F^{\mathrm{T}}F), \quad I_2 = \operatorname{tr}\big(\operatorname{cof}(F^{\mathrm{T}}F)\big), \quad I_3 = \det(F^{\mathrm{T}}F),$$

$$K_1^{\mathrm{e}} = |d|^2, \quad K_2^{\mathrm{e}} = |Fd|^2, \quad K_3^{\mathrm{e}} = |F^{\mathrm{T}}Fd|^2.$$

Similarly, for magneto-elastic interactions the following list is often employed (see, e.g., [7]):

$$I_1, \quad I_2, \quad I_3, \quad K_1^{\mathrm{m}}, \quad K_2^{\mathrm{m}}, \quad K_3^{\mathrm{m}}$$

where

$$K_1^{\mathrm{m}} = |b|^2, \quad K_2^{\mathrm{m}} = |Fb|^2, \quad K_3^{\mathrm{m}} = |F^{\mathrm{T}}Fb|^2.$$

For a general isotropic electro-magneto-elastic material, a complete list reads

$$I_1, \quad I_2, \quad I_3, \quad K_1^{\mathrm{e}}, \quad K_2^{\mathrm{e}}, \quad K_3^{\mathrm{e}}, \quad K_1^{\mathrm{m}}, \quad K_2^{\mathrm{m}}, \quad K_3^{\mathrm{m}}, \quad M^{\mathrm{em}},$$

where

$$M^{\mathrm{em}} = d \cdot b.$$

Interpreting the invariants as functions of (F, d, b), one readily sees that

$$I_1, \quad I_2, \quad I_3, \quad K_1^{\mathrm{e}}, \quad K_2^{\mathrm{e}}, \quad K_1^{\mathrm{m}}, \quad K_2^{\mathrm{m}},$$

are E-polyconvex, as they are of the format $(10.2)_2$ with the obvious choices of Θ (actually even their square roots are E-polyconvex), while the invariants

$$K_3^{\mathrm{e}}, \quad K_3^{\mathrm{m}} \quad \text{and} \quad M^{\mathrm{em}}$$

are not E-polyconvex, as will be proved in Subsection 12. Since any nondecreasing convex function of a family of convex functions is convex, see [49; Exercise 2.20(c)], one finds that if $\Psi : [0, \infty)^7 \to \mathbb{R}$ is a convex function nondecreasing in each argument, then

$$\psi(F, d, b) = \Psi(I_1, I_2, I_3, K_1^{\mathrm{e}}, K_2^{\mathrm{e}}, K_1^{\mathrm{m}}, K_2^{\mathrm{m}})$$

is an isotropic E-polyconvex function. This assertion is used in the literature in particular with Ψ representing convex powers.

10.4 Example C Let $\psi : \mathbb{M}^{3\times 3}_{+} \to \mathbb{R}$ be defined by

$$\psi(F) = a(v_1 + v_2 - v_3) + b(v_1 v_2 + v_1 v_3 - v_2 v_3) \tag{10.4}$$

for each $F \in \mathbb{M}^{3\times 3}_{+}$ where $v_1 \geq v_2 \geq v_3 > 0$ are the singular values of F and where a, b are positive constants. (Let us emphasize that the order $v_1 \geq v_2 \geq v_3 > 0$ is important for the definition (10.4).) We shall prove that ψ is polyconvex by applying Theorem 10.1 (neglecting the electromagnetic variables) at the same time noting that ψ is out of the scope of [1; Theorem 5.2]. This is based on the fact that the right-hand side of (10.4) is pairwise nondecreasing in the variables v_1, v_2, v_3 and in the variables $v_1 v_2$, $v_1 v_3$, $v_2 v_3$ but not nondecreasing in each of these variables. In more detail, let $\Theta : (0,\infty)^6 \to \mathbb{R}$ be defined by

$$\Theta(z_1, \dots, z_6) = a(\tilde{z}_1 + \tilde{z}_2 - \tilde{z}_3) + b(\tilde{z}_4 + \tilde{z}_5 - \tilde{z}_6)$$

for any $(z_1, \dots, z_6) \in (0,\infty)^6$, where $(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)$ is the unique permutation of (z_1, z_2, z_3) such that $\tilde{z}_1 \geq \tilde{z}_2 \geq \tilde{z}_3$ and $(\tilde{z}_4, \tilde{z}_5, \tilde{z}_6)$ is the unique permutation of (z_4, z_5, z_6) such that $\tilde{z}_4 \geq \tilde{z}_5 \geq \tilde{z}_6$. One has

$$\psi(F) = \Theta(v_1, v_2, v_3, v_1 v_2, v_1 v_3, v_2 v_3)$$

for any F and it is easily verified that Θ is convex, pairwise nondecreasing in z_1, z_2, z_3, pairwise nondecreasing in z_4, z_5, z_6, and symmetric under the permutations of z_1, z_2, z_3 and of z_4, z_5, z_6. Thus the polyconvexity of ψ follows from Theorem 10.1.

Proof of Theorem 10.1 Only an outline will be given, since the details are notationally complicated (too many variables). Let $n = 3$; the case $n = 2$ is similar (and simpler).

Clearly, the invariance of the variables v_1, v_2, v_3, $d^{\parallel}$, $b^{\parallel}$, $d^{\times}$, $b^{\times}$ under the passage $(F, d, b) \mapsto (QFR^{\mathrm{T}}, Rd, Rb)$ where Q, $R \in \mathrm{SO}(n)$ implies (10.1) and so ψ is an objective and isotropic function.

To prove the E-polyconvexity of ψ, we note that the result of Rosakis [50] implies that for each fixed $d^{\parallel}$, $b^{\parallel}$, $d^{\times}$, $b^{\times} \in [0,\infty)$ the function

$$F \mapsto \Theta(v_1, v_2, v_3, v_1 v_2, v_1 v_3, v_2 v_3, d^{\parallel}, b^{\parallel}, d^{\times}, b^{\times}, v_1 v_2 v_3)$$

is polyconvex in Ball's original sense. Next we note that by Theorem 6.3 the functions

$$(F, d, b) \mapsto |d|, \ |b|, \ |Fd|, \ |Fb|$$

are all E-polyconvex since the norm $|\cdot|$ is convex. Next, using the fact that a nondecreasing convex function of a family of convex functions is convex (see above), we deduce that for each fixed $z_1, \dots, z_6, z_{11}$ the function

$$(F, d, b) \mapsto \Theta(z_1, z_2, z_3, z_4, z_5, z_6, |d|, |b|, |Fd|, |Fb|, z_{11})$$

is E-polyconvex. The full E-polyconvexity is then essentially a combination of the two particular polyconvexity results stated above. □

11 Fluids

Recall from §3.2.3 that an electro-magneto-elastic material is a fluid if

$$\psi(QFH^{-1}, HRd, Hb) = \psi(F, d, b) \tag{11.1}$$

for each Q, $H \in \mathrm{SL}(n)$. Theorem 11.2 (below) shows that for fluids the main E-convexity conditions coincide and are easy to verify.

11.1 Proposition *A free energy function* $\psi : \mathbb{D}_+^n \to \mathbb{R}$ *of a electro-magnet-elastic fluid has the form*

$$\psi(F, d, b) = \sigma(\det F, Fd, Fb) \tag{11.2}$$

where $\sigma : (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *satisfies*

$$\sigma(v, Qd^\#, Qb^\#) = \sigma(v, d^\#, b^\#) \tag{11.3}$$

for every $(v, d^\#, b^\#) \in (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$ *and every* $Q \in \mathrm{SO}(n)$.

Proof We choose $H = F/(\det F)^{1/3}$ and $Q = \mathrm{I}$ in (11.1) to obtain

$$\psi(F, d, b) = \psi((\det F)^{1/3}\mathrm{I}, Fd/(\det F)^{1/3}, Fb/(\det F)^{1/3}).$$

If we define σ by

$$\sigma(v, d^\#, b^\#) = \psi(\sqrt[3]{v}\,\mathrm{I}, d^\#/\sqrt[3]{v}, b^\#/\sqrt[3]{v})$$

for every $(v, d^\#, b^\#) \in (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$, then we have (11.2). Equation (11.3) then follows from (11.1) with $Q \in \mathrm{SO}(n)$ and $H = \mathrm{I}$. □

11.2 Theorem *Consider a free energy function* $\psi : \mathbb{D}_+^n \to \mathbb{R}$ *of a electro-magnet-elastic fluid with the representation* (11.2). *Then the following conditions are equivalent*:

(i) ψ *is* E*-quasiconvex*;
(ii) ψ *is* Λ_{E}*-convex*;
(iii) ψ *is* E*-polyconvex*;
(iv) σ *is convex.*

Proof We shall prove

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iv)} \Rightarrow \text{(iii)} \Rightarrow \text{(i)}.$$

The implication (i) $\Rightarrow$ (ii) is a general assertion.

(ii) $\Rightarrow$ (iv): The Λ_{E}-convexity reqyuiures that the function

$$t \mapsto \psi(F + t\xi \otimes \eta, d + t\delta, b + t\beta) \tag{11.4}$$

is convex for every $(F, d, b) \in \mathbb{D}^n$ and every $\xi,\ \delta,\ \beta,\ \eta \in \mathbb{R}^n$ such that

$$\delta \cdot \eta = \beta \cdot \eta = 0, \quad \eta \neq 0.$$

In terms of the representation σ this requires that the function

$$t \mapsto \sigma(\det F + t \det F(F^{-1}\xi \cdot \eta), Fd + t\xi(\eta \cdot d) + tF\delta, Fb + t\xi(\eta \cdot b) + tF\beta)$$

is convex. Observing that the argument of σ is linear (affine) if t and using the arbitrariness of the elements occurring there one sees that the convexity of (11.4) implies the convexity of σ.

(iv) $\Rightarrow$ (iii) this is immediate in view of Theorem 6.5.

(iii) $\Rightarrow$ (i) is a general assertion. □

11.3 Theorem *Let* $\psi : \mathbb{D}_+^n \to \mathbb{R}$ *be defined by*

$$\psi(F, d, b) = \Theta(\det F, |Fd|, |Fb|)$$

for each $(F, d, b) \in \mathbb{D}_+^n$ *where* $\Theta : (0, \infty) \times [0, \infty)^2 \to \mathbb{R}$ *is a convex function that is nondecreasing in the last two variables. Then* ψ *is a* E*-polyconvex, objective function of an electro-magneto-elastic fluid.*

This is similar to Theorem 10.1. Details omitted.

12 Appendix A: E-quasiconvex envelopes of K_3^{e}, K_3^{m} and M^{em}

The E-quasiconvex envelope $\mathsf{Q}\psi : \mathbb{D}^n \to \bar{\mathbb{R}}$ of a function $\psi : \mathbb{D}^n \to \bar{\mathbb{R}}$ is defined by

$$\mathsf{Q}\psi(F,d,b) = \sup\{\omega(F,d,b) : \omega \text{ is E-quasiconvex and } \omega \le \psi \text{ on } \mathbb{D}^n\}, \tag{12.1}$$

$(F,d,b) \in \mathbb{D}^n$.

12.1 Proposition *The* E*-quasiconvex envelopes of the functions* $K_3^{\mathrm{e}}, K_3^{\mathrm{m}}$ *and* $M^{\mathrm{em}} : \mathbb{D}^n \to \mathbb{R}$*, defined by*

$$K_3^{\mathrm{e}} = |F^{\mathrm{T}}Fd|^2, \quad K_3^{\mathrm{m}} = |F^{\mathrm{T}}Fb|^2, \quad M^{\mathrm{em}} = d \cdot b,$$

are given by

$$\mathsf{Q}K_3^{\mathrm{e}} = |Fd|^4, \quad \mathsf{Q}K_3^{\mathrm{m}} = |Fb|^4, \quad \mathsf{Q}M^{\mathrm{em}} = -\infty \tag{12.2}$$

for every $(F,d,b) \in \mathbb{D}^3$.

Since the E-quasiconvexifications from (12.2) are different from the originals, those originals are not E-quasiconvex and hence not E-polyconvex.

For the following lemma, we refer to Definitions 13.1(i) and (iii) for the definitions of rank 1 convexity and rank 1 convex envelope $\mathsf{R}g$ of a function g. We shall employ Theorem 13.3 in the proof.

12.2 Lemma *Let* $g : \mathbb{M}^{n\times n} \to \mathbb{R}$ *be given by*

$$g(F) = |F^{\mathrm{T}}Fd|^2,$$

$F \in \mathbb{M}^{n\times n}$*, where* $d \in \mathbb{R}^n$ *is a fixed unit vector. Then*

$$\mathsf{R}g(F) = |Fd|^4 \tag{12.3}$$

for every $F \in \mathbb{M}^{n\times n}$.

Proof We have (13.3) with

$$m(C) = |Cd|^2,$$

$C \in \mathbb{S}_+^{n\times n}$. We observe that m is convex since the function $p : C \mapsto |Cd|$, being a seminorm, is convex and $m = p^2$ is then convex as well. To prove (12.3), we note that $g(F) \ge |Fd|^4$ and hence there is nothing to prove if $g(F) = |Fd|^4$; accordingly, assume that $g(F) > |Fd|^4$. We shall employ (13.4) with

$$Y = \tau^{-1}(Cd - (Cd\cdot d)d - \tau d) \otimes (Cd - (Cd\cdot d)d - \tau d)$$

where $\tau > 0$. One finds that

$$m(C+Y) = (|Fd|^2 + \tau)^2$$

and hence letting $\tau \to 0$, we obtain from (13.4) that

$$\mathsf{R}g(F) \le |Fd|^4.$$

However, the function $F \mapsto |Fd|^4$ is convex, hence rank 1 convex and thus we have (12.3). □

Proof of Proposition 12.1 Since for any function ψ on $\mathbb{D}^n$ we have

$$\begin{gathered}\text{the E-quasiconvexity of } \psi \\ \Downarrow \\ \Lambda_{\mathrm{E}}\text{-convexity of } \psi \\ \Downarrow \\ \text{rank 1 convexity of } \psi(\cdot, d, b) \text{ for every } d, b \in \mathbb{R}^n,\end{gathered}$$

we conclude that

$$(\mathsf{Q}\psi)(\cdot, d, b) \leq \mathsf{R}\psi(\cdot, d, b) \tag{12.4}$$

for every $d, b \in \mathbb{R}^3$. Fixing d and b and introducing $g_d, g_b : \mathbb{M}^{n\times n} \to \mathbb{R}$ by

$$g_d(F) := |F^{\mathrm{T}}Fd|^2, \quad g_b(F) := |F^{\mathrm{T}}Fb|^2,$$

$F \in \mathbb{M}^{n\times n}$ we obtain from Lemma 12.2 the formulas

$$\mathsf{R}g_d(F) = |Fd|^4, \quad \mathsf{R}g_b = |Fb|^4$$

and thus (12.4) yields

$$|Fd|^4 \geq \mathsf{Q}K_3^{\mathrm{e}}, \quad |Fb|^4 \geq \mathsf{Q}K_3^{\mathrm{m}}. \tag{12.5}$$

By a happy coincidence, the functions $(F, d, b) \mapsto |Fd|^4$, $(F, d, b) \mapsto |Fb|^4$ are E-polyconvex by Theorem 10.1; hence we have the equality signs throughout (12.5), which completes the proof of $(12.2)_{1,2}$.

Finally, $(12.2)_3$ is proved by an elementary observation that there is no Λ_{E}-convex function below M^{em}. Indeed, the existence of such function would imply that there exists a Λ_{E}-affine function below M^{em}, which by Theorem 6.3 would mean that

$$d \cdot b \geq c + A_1 \cdot F + A_2 \cdot \operatorname{cof} F + \alpha_7 \cdot d + \alpha_8 \cdot b + \alpha_9 \cdot Fd + \alpha_{10} \cdot Fb + \alpha_{11} \det F$$

for all $(F, d, b) \in \mathbb{D}^3$ and some consants c, $\alpha_7, \ldots, \alpha_{11}$. Appropriate choices of (F, d, b) show that such constants do not exist. □

13 Appendix B: rank 1 convex and rank 1 affine functions

The reader is referred to [1, 5] and [35] for the following notions.

13.1 Definitions Let $g : \mathbb{M}^{n\times n} \to \bar{\mathbb{R}}$.

(i) g is said to be rank 1 convex if

$$g(F + t\xi \otimes \eta) \leq (1 - t)g(F) + tg(F + \xi \otimes \eta) \tag{13.1}$$

for every $t \in (0, 1)$, every $F \in \mathbb{M}^{n\times n}$ and every $\xi, \eta \in \mathbb{R}^n$.

(ii) g is said to be rank 1 affine if it taken only finite values and (13.1) holds with the equality sign for every t, F, ξ, and η as in (i).

(iii) The rank 1 convex envelope $\mathsf{R}g : \mathbb{M}^{n\times n} \to \bar{\mathbb{R}}$ of g is defined by

$$\mathsf{R}g(F) = \sup\{h(F) : h \text{ is rank 1 convex and } h \leq g \text{ on } \mathbb{M}^{n\times n}\},$$

$F \in \mathbb{M}^{n\times n}$.

The following result is standard.

13.2 Lemma ([12], [10], [1; Theorem 4.1]) *A continuous function $g : \mathbb{M}^{n\times n} \to \mathbb{R}$ is rank 1 affine if and only if g is a linear combination, with constant coefficients, of the functions*

$$\left.\begin{array}{llll} 1, & F, & \det F, & \text{if} \quad n = 2, \\ 1, & F, \ \operatorname{cof} F, & \det F & \text{if} \quad n = 3. \end{array}\right\} \tag{13.2}$$

Finally, the following result was needed in Section 12.

13.3 Theorem ([28; Theorem 2]) *Let $g : \mathbb{M}^{n\times n} \to \mathbb{R}$ be of the form*

$$g(F) = m(F^{\mathrm{T}}F) \tag{13.3}$$

$F \in \mathbb{M}^{n\times n}$, where m is a convex function on the set $\mathbb{S}^{n\times n}_{+}$ of all positive semidefinite tensors. Then

$$\mathsf{R}g(F) = \inf\{m(F^{\mathrm{T}}F + Y) : Y \in \mathbb{S}^{n\times n}_{+}\}, \tag{13.4}$$

$F \in \mathbb{M}^{n\times n}$, and $\mathsf{R}g$ is convex on $\mathbb{M}^{n\times n}$.

14 Appendix C: weak convergence

We here gather some basic facts about maps that are continuous under the weak convergence.

Let $1 \le p \le \infty$ and let θ_k and θ be measurable functions on open subset Ω of $\mathbb{R}^n$. In this situation, we define the following three types of weak convergence:

- $\theta_k \rightharpoonup \theta$ in $L^p(\Omega, \mathbb{R}^m)$,
- $\theta_k \overset{*}{\rightharpoonup} \theta$ in $\mathscr{M}(\Omega, \mathbb{R}^m)$,
- $\theta_k \rightharpoonup \theta$ in $W^{1,p}(\Omega, \mathbb{R}^m)$

which mean, respectively,

- that $\theta, \theta_k \in L^p(\Omega, \mathbb{R}^m)$ and
$$\int_\Omega \theta_k \cdot \phi \, dx \to \int_\Omega \theta \cdot \phi \, dx \tag{14.4}$$
for each $\phi \in L^q(\Omega, \mathbb{R}^m)$ where $1/p + 1/q = 1$; we then say that the sequence θ_k converges weakly to θ in $L^p(\Omega, \mathbb{R}^m)$;
- that $\theta, \theta_k \in L^1(\Omega, \mathbb{R}^m)$ and (14.4) holds for each continuous function $\phi : \mathbb{R}^n \to \mathbb{R}^m$ which vanishes outside Ω; we then say that the sequence θ_k converges weak* to θ in the sense of measures;
- that $\theta, \theta_k \in W^{1,p}(\Omega, \mathbb{R}^m)$ and $\theta_k \rightharpoonup \theta$ in $L^p(\Omega, \mathbb{R}^m)$ and $\nabla\theta_k \rightharpoonup \nabla\theta$ in $L^p(\Omega, \mathbb{M}^{m\times n})$; we then say that θ_k converges weakly to θ in $W^{1,p}(\Omega, \mathbb{R}^m)$.

If $p = \infty$, we should actually write $\overset{*}{\rightharpoonup}$ instead of $\rightharpoonup$ and speak about the weak* convergence; however, this is consistently ignored here.

14.1 Proposition (Müller, Tang & Yan [36]) *Let Ω be a bounded open subset of $\mathbb{R}^n$ where n is arbitrary, let*

$$p \ge n - 1, \quad q > n/(n-1)$$

and let $y, y_k \in W^{1,p}(\Omega, \mathbb{R}^n)$ *satisfy*

$$y_k \rightharpoonup y \quad \text{in } W^{1,p}(\Omega, \mathbb{R}^n), \tag{14.5}$$

$$\operatorname{cof} F_k \quad \text{is bounded in } L^q(\Omega, \mathbb{M}^{n\times n}) \tag{14.6}$$

where $F = \nabla y$, $F_k = \nabla y_k$. *Then*

$$\operatorname{cof} F_k \rightharpoonup \operatorname{cof} F \quad \text{in } L^q(\Omega, \mathbb{M}^{n\times n}), \tag{14.7}$$

$$\det F_k \rightharpoonup \det F \quad \text{in } L^r(\Omega), \quad r = q(n-1)/n. \tag{14.8}$$

If $q = n/(n-1)$ *and* $\det F_k \geq 0$ *then instead of* (14.8) *we have*

$$\det F_k \rightharpoonup \det F \quad \text{in } L^1(K),$$

for all compact subsets $K \subset \Omega$.

14.2 Proposition (Murat [37], Tartar [59]) *Let* $\Omega \subset \mathbb{R}^n$ *be open bounded and let* $1 < p, q < \infty$ *satisfy* $1/p + 1/q = 1$. *Suppose* $d, d_k \in L^p(\Omega; \mathbb{R}^n)$, $g, g_k \in L^q(\Omega, \mathbb{R}^n)$ *are sequences such that*

$$\begin{aligned}
&d_k \rightharpoonup d && \text{in} \quad L^p(\Omega, \mathbb{R}^n)\\
&g_k \rightharpoonup g && \text{in} \quad L^q(\Omega, \mathbb{R}^n),\\
&\operatorname{div} d_k \to \operatorname{div} d && \text{in} \quad W^{-1,1}(\Omega),\\
&\operatorname{curl} g_k \to \operatorname{curl} g && \text{in} \quad W^{-1,1}(\Omega).
\end{aligned}$$

Then

$$d_k \cdot g_k \rightharpoonup d \cdot g \quad \text{in } \mathscr{M}(\Omega).$$

Here $W^{-1,1}(\Omega)$ is the dual of $W_0^{1,\infty}(\Omega)$.

14.3 Proposition (Reshetnyak [46], Ball & Murat [3]) *Let* $\Phi : \mathbb{R}^m \to \bar{\mathbb{R}}$ *be convex, lower semicontinuous and bounded below. Let* $\theta, \theta_k \in L^1(\Omega, \mathbb{R}^m)$ *with* $\theta_k \overset{*}{\rightharpoonup} \theta$ *in the sense of measures. Then*

$$\liminf_{j\to\infty} \int_\Omega \Phi(\theta_k)\, dx \geq \int_\Omega \Phi(\theta)\, dx.$$

15 Appendix D: notation, convexity

We use the direct notation with the same conventions as in [61, 53]. The following sets are used throughout:

$$\begin{aligned}
\bar{\mathbb{R}} &= \mathbb{R} \cup \{\infty\} = \text{ the extended real line},\\
\mathbb{R}^n &= \text{the } n\text{-dimensional euclidean space},\\
\mathbb{Z}^n &= \text{the set of all } n\text{-tuples of integers},\\
\mathbb{M}^{n\times n} &= \text{the space of all real } n\times n \text{ matrices},\\
\mathbb{M}^{n\times n}_+ &= \{F \in \mathbb{M}^{n\times n} : \det F > 0\},\\
\mathrm{O}(n) &= \{Q \in \mathbb{M}^{n\times n} : QQ^{\mathrm{T}} = \mathrm{I}\},\\
\mathrm{SO}(n) &= \{Q \in \mathbb{M}^{n\times n} : QQ^{\mathrm{T}} = \mathrm{I} \text{ and } \det Q = 1\},\\
\mathrm{SL}(n) &= \{H \in \mathbb{M}^{n\times n} : \det H = 1\}.
\end{aligned}$$

We interpret the matrices from $\mathbb{M}^{n\times n}$ as second–order tensors on $\mathbb{R}^n$. We denote by $\mathrm{I} \in \mathbb{M}^{n\times n}$ is the unit matrix, by $a \cdot b$ the usual scalar product of two vectors in $\mathbb{R}^n$ and by $A \cdot B := \mathrm{tr}(A^{\mathrm{T}}B)$ the scalar product of tensors. We recall that the tensor of cofactors of $F \in \mathbb{M}^{n\times n}_+$ is given by $\mathrm{cof}\, F = (\det F)F^{-\mathrm{T}}$.

If f is a map from an open subset of a finite-dimensional space X into a finite-dimensional space Y, we interpret the first- and second-order derivative $\mathrm{D}f(\xi) \equiv \mathrm{D}_{\xi} f(\xi) \equiv f_{,\xi}(\xi)$ and $\mathrm{D}^2 f(\xi) \equiv \mathrm{D}_{\xi\xi} f(\xi) \equiv f_{,\xi\xi}(\xi)$ of f at ξ as a linear and bilinear transformations from X into Y, respectively. In indices

$$\mathrm{D}f(\xi)[\mu] = \sum_{i=1}^{d} f_{,i}(\xi)\mu_i, \quad \mathrm{D}^2 f(\xi)[\eta,\lambda] = \sum_{i,j=1}^{d} f_{,ij}(\xi)\mu_i\lambda_j,$$

$\mu, \lambda \in \mathrm{X}$, where $d = \dim \mathrm{X}$, [6; Chapter VIII]. Alternatively, we use the round brackets: $\mathrm{D}f(\xi)(\mu)$ and $\mathrm{D}^2 f(\xi)(\eta,\lambda)$ if dictated by convenience.

If $n = 3$, we define the vector product and the curl in the usual way as vectors in $\mathbb{R}^3$ while if $n = 2$ then both the vector product and the curl are the numbers $a \times b = a_1 b_2 - a_2 b_1$, $\mathrm{curl}\, a = a_{2,1} - a_{1,2}$.

Although the main theme of the paper are various weakened notions of convexity, an essential use is made of the classical convexity. Recall that a function $f : \mathrm{X} \to \bar{\mathbb{R}}$ on a vector space X is said to be convex if

$$f\big((1-t)\xi_1 + t\xi_2\big) \le (1-t)f(\xi_1) + tf(\xi_2) \tag{15.1}$$

for every $\xi_1, \xi_2 \in \mathrm{X}$ and every $t \in (0,1)$. Further, f is said to be affine if we have the equality sign in (15.1) holding identically. f is affine if and only if there is a linear functional φ on X and a constant $c \in \mathbb{R}$ such that

$$f(\xi) = \langle \varphi, \xi \rangle + c \tag{15.2}$$

for every $\xi \in \mathrm{X}$ where $\langle \varphi, \xi \rangle$ is the value of φ on ξ. If $\mathrm{X} = \mathbb{R}^m$ then (15.2) reads $f(\xi) = \varphi \cdot \xi + c$ where $\varphi \in \mathbb{R}^m$. We refer to [48] and [11] for systematic expositions of the convexity theory.

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