Computing lower bounds of eigenvalues by the finite element method

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Abstract eigenvalue problem

Find $\lambda_i \in \mathbb{R}$ and $u_i \in V$, $u_i \neq 0$:

$$a(u_i, v) = \lambda_i b(u_i, v) \quad \forall v \in V$$

a . . . symmetric, continuous, V-elliptic
b . . . symmetric, continuous

Solution operator:
$S : V \mapsto V$
$Sz = y$, where $y \in V$:

$$a(y, v) = b(z, v) \quad \forall v \in V$$

Assumption: S is compact

Symmetric elliptic eigenvalue problem

Find $\lambda_i \in \mathbb{R}$ and $u_i \neq 0$:

$$\begin{aligned} -\operatorname{div}(\mathcal{A}\nabla u_i) + c u_i &= \lambda_i \beta_1 u_i && \text{in } \Omega \\ (\mathcal{A}\nabla u_i)\cdot \mathbf{n} + \alpha u_i &= \lambda_i \beta_2 u_i && \text{on } \Gamma_{\mathrm{N}} \\ u_i &= 0 && \text{on } \Gamma_{\mathrm{D}} \end{aligned}$$

Weak formulation:

- $V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_{\mathrm{D}}\}$
- $a(u,v) = (\mathcal{A}\nabla u, \nabla v) + (cu, v) + (\alpha u, v)_{\Gamma_{\mathrm{N}}}$
- $b(u,v) = (\beta_1 u, v) + (\beta_2 u, v)_{\Gamma_{\mathrm{N}}}$

Assumptions:

- $\mathcal{A} \in \mathbb{R}^{d\times d}$ uniformly positive definite, $c \geq 0$, $\alpha \geq 0$
- $\mathcal{A}$, c, α, β_1, β_2 piecewise constant

Finite element method

Find $\lambda_i^h \in \mathbb{R}$ and $u_i^h \in V^h$, $u_i^h \neq 0$:

$$a(u_i^h, v^h) = \lambda_i^h b(u_i^h, v^h) \quad \forall v^h \in V^h$$

where

- $V^h = \{v^h \in V : v^h|_K \in P^1(K) \quad \forall K \in \mathcal{T}_h\}$

Courant–Fischer–Weyl min-max principle:

$$\lambda_i \leq \lambda_i^h$$

Babuška, Osborn (1989):

$$|\lambda_i - \lambda_i^h| \leq Ch^2$$
$$\|u_i - u_i^h\| \leq Ch$$

Lower bound $\underline{\lambda}_i^h \leq \lambda_i$?

(A) Kuttler, Sigillito (1978):

If S compact, $\lambda_i^h \in \mathbb{R}$, $u_i^h \in V$, and $|u_i^h|_b = 1$.

$w_i \in V$: $a(w_i, v) = a(u_i^h, v) - \lambda_i^h b(u_i^h, v) \quad \forall v \in V$

Then

$$\min_j \left| \frac{\lambda_j - \lambda_i^h}{\lambda_j} \right| \leq |w_i|_b$$

(B) Friedrichs–Poincaré inequality:

$$|w_i|_b \leq C_{ab} \|w_i\|_a, \quad C_{ab} = \frac{1}{\sqrt{\lambda_1}}$$

(C) Complementarity estimate:

$$\|w_i\|_a \leq A_i + C_{ab} B_i$$

Hence

$$\min_j \left| \frac{\lambda_j - \lambda_i^h}{\lambda_j} \right| \leq \frac{1}{\sqrt{\lambda_1}} A_i + \frac{1}{\lambda_1} B_i$$

Lower bound $\underline{\lambda}_i^h \le \lambda_i$

Relative closeness assumption:

Let the (relatively) closest eigenvalue to λ_i^h be λ_i.

Case 1: $i = 1$

$$\frac{\lambda_1^h - \lambda_1}{\lambda_1} = \min_j \left| \frac{\lambda_j - \lambda_1^h}{\lambda_j} \right| \le \frac{1}{\sqrt{\lambda_1}} A_1 + \frac{1}{\lambda_1} B_1$$

$\Rightarrow$

$$\underline{\lambda}_1^h \le \lambda_1, \quad \underline{\lambda}_1^h = \frac{1}{4}\left(-A_1 + \sqrt{A_1^2 + 4\left(\lambda_1^h - B_1\right)} \right)^2$$

Case 2: $i > 1$

$$\frac{\lambda_i^h - \lambda_i}{\lambda_i} = \min_j \left| \frac{\lambda_j - \lambda_i^h}{\lambda_j} \right| \le \frac{1}{\sqrt{\underline{\lambda}_1^h}} A_i + \frac{1}{\underline{\lambda}_1^h} B_i$$

$\Rightarrow$

$$\underline{\lambda}_i^h \le \lambda_i, \quad \underline{\lambda}_i^h = \left(1 + \frac{A_i}{\sqrt{\underline{\lambda}_1^h}} + \frac{B_i}{\underline{\lambda}_1^h} \right)^{-1} \lambda_i^h$$

Complementarity estimate:

Flux reconstruction $\mathbf{q}_i^h$:

- $\mathbf{q}_i^h \in \mathbf{H}(\mathrm{div}, \Omega)$,
- $\mathbf{q}_i^h|_K \in \mathrm{RT}_1(K) \quad \forall K \in \mathcal{T}_h$
- $\mathbf{q}_i^h$ solves local problems on patches $\omega_{\mathbf{a}}$
- $\mathrm{div}\,\mathbf{q}_i^h = c u_i^h - \lambda_i^h \beta_1 u_i^h \quad$ in Ω
- $\mathbf{q}_i^h \cdot \mathbf{n} = -\alpha u_i^h + \lambda_i^h \beta_2 u_i^h \quad$ on Γ_{N}

Theorem

$w_i \in V\colon\ a(w_i, v) = a(u_i^h, v) - \lambda_i^h b(u_i^h, v) \quad \forall v \in V$

Then

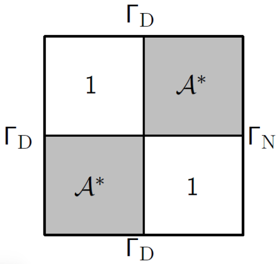
$$\|w_i\|_a \leq A_i + C_{ab} B_i,$$

where

- $A_i = \|\nabla u_i^h - \mathcal{A}^{-1}\mathbf{q}_i^h\|_{\mathcal{A}}$
- $B_i = 0$

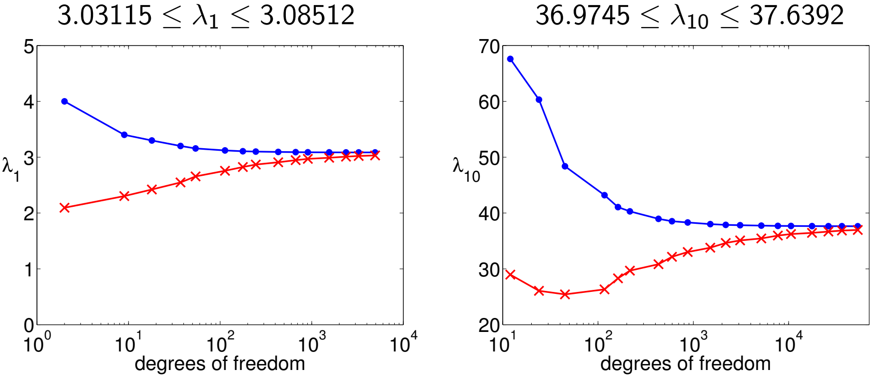
Numerical example

$$
\begin{aligned}
-\operatorname{div}(\mathcal{A}\nabla u_i) &= \lambda_i u_i && \text{in } \Omega \\
(\mathcal{A}\nabla u_i)\cdot \mathbf{n} &= 0 && \text{on } \Gamma_{\mathrm{N}} \\
u_i &= 0 && \text{on } \Gamma_{\mathrm{D}}
\end{aligned}
$$



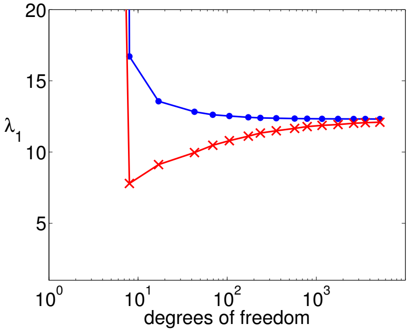
- $\Omega = (-1,1)^2$
- $\mathcal{A} = \begin{cases} 1 & \text{for } xy \leq 0 \\ \mathcal{A}^* & \text{for } xy > 0 \end{cases}$
- Adaptive algorithm driven by $\eta_K = \|\nabla u_i^h - \mathcal{A}^{-1}\mathbf{q}_i^h\|_{\mathcal{A},K}$

Smooth case ($\mathcal{A}^* = 1$)

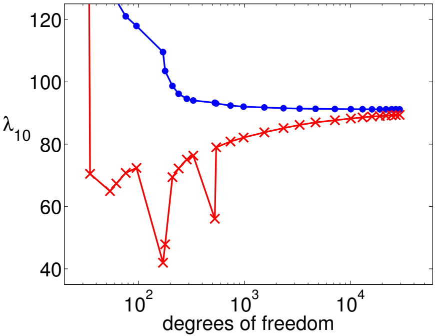


Singularity ($\mathcal{A}^* = 1000$)

$$12.0944 \leq \lambda_1 \leq 12.3214$$



$$89.3918 \leq \lambda_{10} \leq 91.1678$$



Conclusions

Highlights

- Upper bound by the (conforming) FEM
- Lower bound by postprocessing
- Mixed boundary conditions (includes Steklov eigenvalue problem)
- Efficiency of the error indicators
- Convergence of the adaptive algorithm

Future work

- Indicator for the relative closeness assumption
- Estimates of the error in eigenvectors

Thank you for your attention

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