Entropy methods in compressible fluid modeling

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Navier-Stokes-Fourier system

Field equations

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

$$\partial_t (\varrho e(\varrho, \vartheta)) + \mathrm{div}_x (\varrho e(\varrho, \vartheta) \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u}$$

Newton's rheological law, Fourier's law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \ \mu > 0, \ \ \eta \geq 0$$

$$\mathbf{q} = -\kappa \nabla_x \vartheta, \ \ \kappa > 0$$

No flux conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \ \ \nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Thermodynamics

Gibbs' relation - entropy

$$\vartheta Ds(\varrho,\vartheta) = De(\varrho,\vartheta) + p(\varrho,\vartheta) D\left(\frac{1}{\varrho}\right)$$

Thermodynamics stability

$$\frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0, \quad \frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} > 0$$

Entropy equation

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma$$

Entropy production rate

$$\sigma = \frac{1}{\vartheta}\left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right) \geq 0$$

Navier-Stokes-Fourier system - weak formulation

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Entropy production

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma$$

$$\sigma \boxed{\geq} \frac{1}{\vartheta} \left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] \, \mathrm{d}x = 0$$

Relative energy (entropy)

Ballistic free energy

$$H_\Theta(\varrho,\vartheta) = \varrho e(\varrho,\vartheta) - \Theta \varrho s(\varrho,\vartheta)$$

Relative energy

$$\mathcal{E}\left(\varrho,\vartheta,\mathbf{u} \Big| r,\Theta,\mathbf{U}\right)$$

$$= \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial\varrho}(\varrho - r) - H_\Theta(r,\Theta) \right] \ \mathrm{d}x$$

$$= \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right] \ \mathrm{d}x - \int_\Omega \varrho\mathbf{u}\cdot\mathbf{U} \ \mathrm{d}x + \int_\Omega \frac{1}{2}\varrho|\mathbf{U}|^2 \ \mathrm{d}x$$

$$- \int_\Omega \varrho s(\varrho,\vartheta)\Theta \ \mathrm{d}x - \int_\Omega \frac{\partial H_\Theta(r,\Theta)}{\partial\varrho}\varrho \ \mathrm{d}x + \int_\Omega p(r,\Theta) \ \mathrm{d}x$$

Dissipative solutions

Relative energy inequality

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{t=\tau}$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R} \left(\varrho, \vartheta, \mathbf{u} \ \middle| \ r, \Theta, \mathbf{U} \right) \mathrm{d}t$$

Test functions

$$r, \Theta > 0, \ \ \mathbf{U}|_{\partial\Omega} = 0 \text{ (or other relevant b.c.)}$$

Dissipative solutions - remainder

Remainder

$$\mathcal{R}\left(\varrho, \vartheta, \mathbf{u} \,\middle|\, r, \Theta, \mathbf{U}\right) = \int_\Omega \varrho(\mathbf{u} - \mathbf{U}) \cdot \nabla_x \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x$$

$$+ \int_\Omega \left[\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right] \, \mathrm{d}x$$

$$+ \int_\Omega \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \Big(\mathbf{U} - \mathbf{u} \Big) \cdot \nabla_x \Theta \, \mathrm{d}x$$

$$+ \int_\Omega \varrho \Big(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x - \int_\Omega p(\varrho, \vartheta) \mathrm{div}_x \mathbf{U} \, \mathrm{d}x$$

$$- \int_\Omega \left(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{U} \cdot \nabla_x \Theta \right) \, \mathrm{d}x$$

$$+ \int_\Omega \left(\left(1 - \frac{\varrho}{r} \right) \partial_t p(r, \Theta) - \frac{\varrho}{r} \mathbf{u} \cdot \nabla_x p(r, \Theta) \right) \, \mathrm{d}x$$

Weak solutions - the-state-of-the-art

Existence [EF, A.Novotný [2009]]

Weak solutions exists for any finite energy initial data *under certain restrictions* on the constitutive relations

Weak-strong uniqueness [EF, A.Novotný [2012]]

Weak and strong solution emanating from the same initial data coincide as long as the latter exists

Weak-strong uniqueness [EF, A.Novotný [2012]]

Weak and strong solution emanating from the same initial data coincide as long as the latter exists

Conditional regularity [EF, A.Novotný, Y.Sun [2013]]

Weak solution emanating from regular initial data and having bounded velocity gradient $\nabla_x \mathbf{u}$ is regular

Long-time behavior

Driven system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho \mathbf{f}$$

$$\partial_t(\varrho e(\varrho, \vartheta)) + \mathrm{div}_x(\varrho e(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u}$$

Driving force

$$\mathbf{f} = \mathbf{f}(x)$$

Long-time behavior

Pressure hypothesis

$$\frac{\partial p}{\partial \varrho}(0, \vartheta) > 0 \text{ for all } \vartheta > 0$$

Dichotomy - EF, H.Petzeltová [2006]

$\mathbf{f}(x) = \nabla_x F(x) \ \Rightarrow$ convergence to an equilibriun solution for $t \to \infty$

$$\mathbf{f}(x) \neq \nabla_x F(x) \ \Rightarrow \ E(t) \to \infty \ as \ t \to \infty$$

Incompressible vanishing dissipation limit

Scaled system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) = \varepsilon^\alpha \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \frac{1}{\varepsilon} \varrho \nabla_x F$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta) \mathbf{u}) + \varepsilon^\beta \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

$$\sigma = \frac{1}{\vartheta} \left(\varepsilon^{2+\alpha} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \varepsilon^\beta \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{\varepsilon^2} \varrho e(\varrho, \vartheta) \right] + \frac{1}{\varepsilon} \varrho F \ \mathrm{d}x = 0$$

Geometry, (ill prepared) data

Spatial domain, boundary conditions

$$\Omega \subset R^3 \quad \text{– exterior domain}$$

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = \nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

$$\varrho \to \overline{\varrho} > 0, \ \vartheta \to \overline{\vartheta} > 0, \ \mathbf{u} \to 0 \text{ as } |x| \to \infty$$

Initial data

$$\mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}, \ \varrho(0,\cdot) = \varrho_0 = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \ \vartheta(0,\cdot) = \vartheta_0 = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon}$$

$$\{\varrho^{(1)}_{0,\varepsilon}\}_{\varepsilon>0}, \ \{\vartheta^{(1)}_{0,\varepsilon}\}_{\varepsilon>0}, \ \{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ precompact in } L^1 \cap L^2(\Omega)$$

Target system

Euler-Boussinesq system

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = -a(\overline{\varrho}, \overline{\vartheta}) \nabla_x F$$

$$c_p(\overline{\varrho}, \overline{\vartheta}) \left(\partial_t \theta + \mathbf{v} \cdot \nabla_x \theta \right) - \overline{\vartheta} a(\overline{\varrho}, \overline{\vartheta}) \mathbf{v} \cdot \nabla_x F = 0$$

Insviscid limit EF, A.Novotný [2014]

$$\beta > 0, \ 0 < \alpha < \frac{5}{3}$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \theta \text{ as } \varepsilon \to 0$$

$$\mathbf{u}_\varepsilon \to \mathbf{v} \text{ as } \varepsilon \to 0$$

Boundary layer at $t = 0$

Vanishing dissipation limit

Navier-Stokes-Fourier system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) - \lambda \mathbf{u}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

$$\sigma = \frac{1}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right),$$

Slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Constitutive relations - scaling

Pressure

$$p(\varrho,\vartheta) = p_M(\varrho,\vartheta) + p_R(\varrho,\vartheta),\ p_M = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right),\ p_R(\varrho,\vartheta) = \frac{\boxed{a}}{3}\vartheta^4$$

Viscous stress

$$\mathbb{S}(\vartheta,\nabla_x \mathbf{u}) = \boxed{\nu}\left[\mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}\right]$$

Heat flux

$$\mathbf{q} = -\boxed{\omega}\kappa(\vartheta)\nabla_x \vartheta$$

Brinkman type "damping"

$$D = -\boxed{\lambda}\mathbf{u}$$

Target system

Full Euler system

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p_M(\varrho, \vartheta) = 0$$

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e_M(\varrho, \vartheta) \right)$$

$$+\mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e_M(\varrho, \vartheta) \right) \mathbf{u} + p_M(\varrho, \vartheta) \mathbf{u} \right] = 0$$

Slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0$$

Vanishing dissipation limit

Theorem EF [2015]

Let $[\varrho_E, \vartheta_E, \mathbf{u}_E]$ be the classical solution of the Euler system in a time interval $(0, T)$, with the initial data $[\varrho_{0,E}, \vartheta_{0,E}, \mathbf{u}_{0,E}]$. Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak (dissipative) solution of the Navier-Stokes-Fourier system, with the initial data $[\varrho_0, \vartheta_0, \mathbf{u}_0]$.
Then

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \Big| \varrho_E, \vartheta_E, \mathbf{u}_E \right)(\tau)$$

$$\leq c_1(T, \mathrm{data}) \mathcal{E}\left(\varrho_0, \vartheta_0, \mathbf{u}_0 \Big| \varrho_{0,E}, \vartheta_{0,E}, \mathbf{u}_{0,E} \right)$$

$$+ c_2(T, \mathrm{data}) \max \left\{ a, \nu, \omega, \lambda, \frac{\nu}{\sqrt{a}}, \frac{\omega}{a}, \left(\frac{a}{\sqrt{\nu^3 \lambda}} \right)^{1/3} \right\}$$

for a.a. $\tau \in (0, T)$.

Full Euler system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \varrho \vartheta = 0$$

Energy balance

$$\partial_t \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + c_v \varrho \vartheta \right] + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + c_v \varrho \vartheta + \varrho \vartheta \right) \mathbf{u} \right] = 0$$

Entropy inequality

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) \geq 0, \ \ s = s(\varrho, \vartheta) \equiv \log \left(\frac{\vartheta^{c_v}}{\varrho} \right)$$

Riemann problem

Geometry

$$\Omega = R^1 \times \mathcal{T}^1, \text{ where } \mathcal{T}^1 \equiv [0,1]_{\{0,1\}} \text{ is the ``flat'' sphere}$$

Initial data

$$\varrho(0, x_1, x_2) = R_0(x_1), \ R_0 = \begin{cases} R_L \text{ for } x_1 \leq 0 \\ R_R \text{ for } x_1 > 0 \end{cases}$$

$$\vartheta(0, x_1, x_2) = \Theta_0(x_1), \ \Theta_0 = \begin{cases} \Theta_L \text{ for } x_1 \leq 0 \\ \Theta_R \text{ for } x_1 > 0 \end{cases}$$

$$u^1(0, x_1, x_2) = U_0(x_1), \ U_0 = \begin{cases} U_L \text{ for } x_1 \leq 0, \\ U_R \text{ for } x_1 > 0 \end{cases} \quad u^2(0, x_1, x_2) = 0.$$

Shock free Riemann solutions

Solution class

$$0 < \varrho \leq \overline{\varrho},\ 0 < \vartheta \leq \overline{\vartheta},\ |s(\varrho, \vartheta)| < \overline{s},\ |\mathbf{u}| < \overline{u}$$

Isentropic solutions

- the entropy S is *constant* in $[0, T] \times \Omega$
- $\Theta = R^{\frac{1}{c_v}} \exp\left(\frac{1}{c_v} S\right)$
- $R = R(t, x_1)$ and $U = U(t, x_1)$ represent a rarefaction wave

solution of the 1-D *isentropic* system

$$\partial_t R + \partial_{x_1}(RU) = 0,\ R\left[\partial_t U + U \partial_{x_1} U\right] + \exp\left(\frac{1}{c_v} S\right) \partial_{x_1} R^{\frac{c_v+1}{c_v}} = 0$$

Uniqueness

Theorem, EF, O.Kreml, A.Vasseur [2014]

Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Euler system in $(0, T) \times \Omega$ originating from the Riemann data. Suppose in addition that the Riemann data give rise to the shock-free solution $[R, \Theta, U]$ of the 1-D Riemann problem.

Then

$$\varrho = R, \ \vartheta = \Theta, \ \mathbf{u} = [U, 0] \text{ a.a. in } (0, T) \times \Omega$$