Inviscid and viscous fluid flows

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based on joint work with several coauthors mentioned in the text

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Compressible Euler system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0$$

Boundary conditions

Periodic:

$$\Omega = (\mathcal{T})^N - \text{ flat torus in } R^N, \ N = 2, 3$$

Whole space - far field conditions:

$$\varrho \to \varrho_\infty, \ \mathbf{u} \to \mathbf{u}_\infty \text{ as } |x| \to \infty$$

Initial data

$$\varrho(0, \cdot) = \varrho_0, \ \mathbf{u}(0, \cdot) = \mathbf{u}_0$$

Admissibility criteria

Pressure potential

$$P(\varrho) = \varrho \int_1^{\varrho} \frac{p(z)}{z^2} \, \mathrm{d}z, \;\; p' > 0, \;\; p(\varrho) = \varrho^{\gamma}$$

Energy

$$E = \frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)$$

Dissipative solutions - global energy inequality

$$\int_{\Omega} E(t) \, \mathrm{d}x \boxed{\leq} \int_{\Omega} E_0 \, \mathrm{d}x, \;\; \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} E(t) \, \mathrm{d}x \leq 0$$

Entropy solutions - local energy balance

$$\partial_t E + \mathrm{div}_x(E\mathbf{u}) + \mathrm{div}_x(p(\varrho)\mathbf{u}) \boxed{\leq} 0$$

Existence of weak solutions

Weak solution - E.Chiodaroli, EF

For *any* (sufficiently smooth) initial data ϱ_0, $\mathbf{u}_0$ the compressible Euler system admits infinitely many global-in-time *weak* solutions

Dissipative solutions - E.Chiodaroli, EF

For any (smooth) initial density ϱ_0 and any given $T > 0$, there is $u_0 \in L^\infty$ such that the compressible Euler system admits infinitely many *dissipative weak solutions* in $(0, T)$.

Entropy solutions - E.Chiodaroli, EF

For a large class of initial densities, there is $u_0 \in L^\infty$ such that the compressible Euler system admits infinitely many *entropy weak solutions* in $(0, T)$.

Measure-valued solutions

Parameterized (Young) measure

$$\nu_{t,x} \in L^\infty_{\rm weak}((0,T)\times\Omega; \mathcal{P}([0,\infty)\times R^N)),\ \ [s,\mathbf{m}] \in [0,\infty)\times R^N$$

$$\varrho(t,x) = \langle \nu_{t,x}; s\rangle\ ,\ \ \varrho\mathbf{u} = \langle \nu_{t,x}; \mathbf{m}\rangle$$

Equations

$$\int_0^T \int_\Omega \left[\langle \nu_{t,x}; \varrho\rangle \partial_t\varphi + \langle \nu_{t,x}; \mathbf{m}\rangle \cdot \nabla_x\varphi \right]\ {\rm d}x + \langle S_1, \nabla_x\varphi\rangle\,{\rm d}t = 0,$$

$$\int_0^T \int_\Omega \left[\langle \nu_{t,x}; \mathbf{m}\rangle \cdot \partial_t\varphi + \left\langle \nu_{t,x}; \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} \right\rangle : \nabla_x\varphi \right]\ {\rm d}x{\rm d}t$$

$$+ \int_0^T \int_\Omega \left[\langle \nu_{t,x}; p(\varrho)\rangle\, {\rm div}_x\varphi + S_2 : \nabla_x\varphi \right]\ {\rm d}x{\rm d}t = 0$$

Energy balance - dissipation defect

Energy inequality

$$\int_\Omega \left\langle \nu_{\tau,x}; \frac{1}{2}\left(\frac{|\mathbf{m}|^2}{\varrho}\right) + P(\varrho) \right\rangle \, \mathrm{d}x + D(\tau)$$
$$\leq \int_\Omega \left\langle \nu_{0,x}; \frac{1}{2}\left(\frac{|\mathbf{m}|^2}{\varrho}\right) + P(\varrho) \right\rangle \, \mathrm{d}x$$

Dissipation defect

$$\int_0^\tau \int_\Omega \sum_{j=1,2} |S_j| \, \mathrm{d}x \, \mathrm{d}t \leq c \int_0^\tau D(t) \, \mathrm{d}t \text{ for a.a. } \tau \in (0,T).$$

Truly measure-valued solutions

Truly measure-valued solutions for the Euler system (E.Chiodaroli, EF, O.Kreml, E. Wiedemann)

There is a measure-valued solution to the compressible Euler system (without viscosity) that *is not* a limit of bounded L^p weak solutions to the Euler system.

Weak (mv) - strong uniqueness

Theorem - EF, P.Gwiazda, A.Świerczewska-Gwiazda, E. Wiedemann

A measure valued and a strong solution emanating from the same initial data coincide as long as the latter exists. The result holds true for both Euler and (compressible) Navier-Stokes system

“Onsager’s conjecture”

Energy balance equation

$$\partial_t \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right) + \mathrm{div}_x \left(\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right) \mathbf{u} \right) + \mathrm{div}_x (p(\varrho)\mathbf{u}) \boxed{=} 0$$

First criterion - EF, P.Gwiazda, A.Swierczewska-Gwiazda, E.Wiedemann

If ϱ, $\mathbf{u} \in L^\infty$,

$$\varrho(t) \in BV \cap C(\mathcal{T}^3),\ \mathbf{u}(t) \in BV \cap C(\mathcal{T}^3) \text{ for a.a.} t,$$

$$\varrho, \mathbf{u} \in L^\infty(0,T;C),\ \nabla_x \varrho,\ \nabla_x \mathbf{u} \in L^\infty(0,T;\mathcal{M}),$$

then the *energy balance equation* holds.

“Onsager’s conjecture” in Besov spaces

Besov spaces

$$\|w\|_{B_p^{\alpha,\infty}(\Omega)} = \|w\|_{L^p(\Omega)} + \sup_{\xi\in\Omega} \frac{\|w(\cdot+\xi) - w\|_{L^p(\Omega\cap(\Omega-\xi))}}{|\xi|^\alpha}$$

Second criterion - EF, P.Gwiazda, A.Swierczewska-Gwiazda, E.Wiedemann

Suppose

$$\varrho,\ \varrho u \in B_3^{\beta,\infty}((0,T)\times\mathcal{T}^3),\ \mathbf{u} \in B_3^{\alpha,\infty}((0,T)\mathcal{T}^3),$$

$$0 \le \underline{\varrho} \le \varrho \le \overline{\varrho},\ p \in C^2[\underline{\varrho},\overline{\varrho}],$$

$$\beta > \max\left\{1-2\alpha, \frac{1-\alpha}{2}\right\},\ 0 \le \alpha,\beta \le 1.$$

Then the *energy balance equation* holds.
Note that

$$BV \cap L^\infty \subset B_3^{1/3,\infty}$$

Robustness of 1D viscosity solutions

Navier-Stokes system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}),$$

Pressure, viscous stress

$$p(\varrho) = a\varrho^\gamma, \ a > 0, \ \gamma > 1,$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{N} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0.$$

1D problem

1D Navier-Stokes system

$$\partial_t R + \partial_y (RV) = 0,$$

$$\partial_t (RV) + \partial_y (RV^2) + \partial_y p(R) = \left[2\mu \left(1 - \frac{1}{N} \right) + \eta \right] \partial^2_{y,y} V.$$

Stability of 1D solutions - hypotheses

Theorem EF, Y.Sun

$$\gamma > \frac{N}{2},\ q > \max\left\{2, \gamma'\right\},\ \frac{1}{\gamma} + \frac{1}{\gamma'} = 1 \text{ if } N = 2$$

$$q > \max\left\{3, \frac{6\gamma}{5\gamma - 6}\right\} \text{ if } N = 3$$

Let $[R, V]$ be a (strong) solution of the one-dimensional problem, with the initial data belonging to the class

$$R_0 \in W^{1,q}(0,1),\ R_0 > 0,\ V_0 \in W_0^{1,q}(0,1)$$

Let $[\varrho, \mathbf{u}]$ be a finite energy weak solution to the Navier-Stokes system in

$$(0,T) \times \Omega,\ \Omega = (0,1) \times \mathcal{T}^{N-1},$$

with the initial data

$$\varrho_0 \in L^\infty(\Omega),\ \varrho_0 > 0,\ \mathbf{u}_0 \in L^2(\Omega; R^3).$$

Stability of 1D solutions - conclusion

Conclusion

Then

$$\int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{V}|^2 + P(\varrho) - P'(R)(\varrho - R) - P(R) \right] (\tau, \cdot) \, \mathrm{d}x$$

$$\leq c(T) \int_\Omega \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0 - \mathbf{V}_0|^2 + P(\varrho_0) - P'(R_0)(\varrho_0 - R_0) - P(R_0) \right] \, \mathrm{d}x$$

for a.a. $\tau \in (0, T)$,

$$P(\varrho) = \frac{a}{\gamma - 1} \varrho^\gamma .$$