On measure-valued solutions to the Euler and Navier-Stokes system

Eduard Feireisl
based on joint work with P. Gwiazda, A. Świerczewska-Gwiazda (Warsaw), E.Wiedemann (Hannover)

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

AIMS 2016 conference, Orlando, 2 July 2016

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078

Compressible Navier-Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Newton's rheological law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \ \mu > 0, \ \ \eta \geq 0$$

No-slip boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0$$

Thermodynamics stability hypothesis

Pressure potential

$$P(\varrho) = \varrho \int_1^{\varrho} \frac{p(z)}{z^2}\, \mathrm{d}z$$

Pressure-density state equation

$$p \in C[0,\infty) \cap C^2(0,\infty),\ \ p(0) = 0$$

$$p'(\varrho) > 0 \text{ for } \varrho > 0,\ \liminf_{\varrho\to\infty} p'(\varrho) > 0$$

$$\liminf_{\varrho\to\infty} \frac{P(\varrho)}{p(\varrho)} > 0$$

Isentropic pressure-density state equation

$$p(\varrho) = a\varrho^{\gamma},\ \ a > 0,\ \ \gamma \geq 1$$

What is the “right” solution

Several definitions of solutions - no uniqueness

- weak solution
- limit of suitable approximations
- measure-valued solution
- (higher) viscosity solution
- limits of numerical schemes

Weak-strong uniqueness principle

Generalized and strong solutions emanating from the same initial data should coincide as long as the latter exists

Hierarchy of solutions

Classical solutions

Solutions are (sufficiently) smooth satisfying the equations point-wise, determined uniquely by the data. Requires strong *a priori* bounds usually not available

Weak solutions

Equations satisfied in the sense of distributions. Requires *a priori* bounds to ensure equi-integrability of nonlinearities + compactness

Measure-valued solutions

Equations satisfied in the sense of distributions, nonlinearities replaced by Young measures (weak limits) $f(u)(t,x) \approx \langle \nu_{t,x}; f(\mathbf{v}) \rangle$. Requires *a priori* bounds to ensure equi-integrability of nonlinearities.

Measure-valued solutions with concentration measure

Measure-valued solutions + concentration defects. Requires *a priori* bounds to ensure integrability of nonlinearities.

Dissipative solutions

Energy (entropy) inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right)\ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\ \mathrm{d}x \boxed{\leq} 0$$

$$P(\varrho) = \varrho\int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z$$

Known results

- **Local strong solution for any data and global strong solutions for small data.** Matsumura and Nishida [1983], Valli and Zajaczkowski [1986], among others
- **Global-in-time weak solutions.** $p(\varrho) = \varrho^\gamma$, $\gamma \geq 9/5, N = 3$, $\gamma \geq 3/2, N = 2$ P.L. Lions [1998], $\gamma > 3/2$, $N = 3$, $\gamma > 1$, $N = 2$ EF, Novotný, Petzeltová [2000], $\gamma = 1$, $N = 2$ Plotnikov and Vaigant [2014]
- **Measure-valued solutions.** Neustupa [1993], related results Málek, Nečas, Rokyta, Růžička, Nečasová - Novotný

Bounded sequences of integrable functions

Boundedness

$$\mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^1(Q; R^M)$$

$$\|F(\mathbf{v}_n)\|_{L^1(Q)} \leq C \;\Rightarrow\; F(\mathbf{v}_n) \to \overline{F(\mathbf{v})} \not\equiv F(\mathbf{v}) \text{ weakly-(*) in } \mathcal{M}(\overline{Q})$$

Biting limit - parameterized Young measure

$$\langle \nu_{t,x}; F_k(\mathbf{v})\rangle = \overline{F_k(\mathbf{v})}(t,x),\; F_k \in BC(R^M)$$

$$\langle \nu_{t,x}; F(\mathbf{v})\rangle = \lim_{k\to\infty} \overline{F_k(\mathbf{v})}(t,x),\; F_k \nearrow F,\; \|F(\mathbf{v}_n)\|_{L^1(Q)} \leq C$$

Concentration part - defect measure

$$\overline{F(\mathbf{v})}(t,x) = \underbrace{\langle \nu_{t,x}; F(\mathbf{v})\rangle}_{integrable} + \underbrace{\left[\overline{F(\mathbf{v})}(t,x) - \langle \nu_{t,x}; F(\mathbf{v})\rangle\right]}_{concentration\;\; defect}$$

Measure-valued solutions

Parameterized (Young) measure

$$\nu_{t,x} \in L^\infty_{\rm weak}((0,T)\times\Omega; \mathcal{P}([0,\infty)\times R^N)),\ [s,\mathbf{v}] \in [0,\infty)\times R^N$$

$$\varrho(t,x) = \langle \nu_{t,x}; s \rangle,\ \mathbf{u} = \langle \nu_{t,x}; \mathbf{v}\rangle \in L^2(0,T; W^{1,2}_0(\Omega; R^N))$$

Field equations revisited

$$\int_0^T \int_\Omega \langle \nu_{t,x}; s\rangle \partial_t \varphi + \langle \nu_{t,x}; s\mathbf{v}\rangle \cdot \nabla_x \varphi \ {\rm d}x\ {\rm d}t = \langle R_1; \nabla_x \varphi \rangle$$

$$\int_0^T \int_\Omega \langle \nu_{t,x}; s\mathbf{v}\rangle \cdot \partial_t \varphi + \langle \nu_{t,x}; s\mathbf{v}\otimes \mathbf{v}\rangle \cdot \nabla_x \varphi + \langle \nu_{t,x}; p(s)\rangle {\rm div}_x \varphi \ {\rm d}x\ {\rm d}t$$

$$= \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi \ {\rm d}x\ {\rm d}t + \langle R_2; \nabla_x \varphi \rangle$$

Dissipativity

Energy inequality

$$\int_\Omega \left\langle \nu_{\tau,x}; \left(\frac{1}{2} s|\mathbf{v}|^2 + P(s) \right) \right\rangle \, \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t + \boxed{\mathcal{D}(\tau)}$$

$$\leq \int_\Omega \left\langle \nu_0; \left(\frac{1}{2} s|\mathbf{v}|^2 + P(s) \right) \right\rangle \, \mathrm{d}x$$

Compatibility

$$\left| R_1[0,\tau] \times \overline{\Omega} \right| + \left| R_2[0,\tau] \times \overline{\Omega} \right| \leq \xi(\tau)\mathcal{D}(\tau), \ \xi \in L^1(0,T)$$

$$\int_0^\tau \int_\Omega \left\langle \nu_{t,x}; |\mathbf{v} - \mathbf{u}|^2 \right\rangle \, \mathrm{d}x \, \mathrm{d}t \leq c_P \mathcal{D}(\tau)$$

Truly measure-valued solutions

Truly measure-valued solutions for the Euler system (with E.Chiodaroli, O.Kreml, E. Wiedemann)

There is a measure-valued solution to the compressible Euler system (without viscosity) that *is not* a limit of bounded L^p weak solutions to the Euler system.

Do we need measure valued solutions?

Limits of problems with higher order viscosities

Multipolar fluids with complex rheologies (Nečas - Šilhavý)

$$\mathbb{T}(\mathbf{u}, \nabla_x \mathbf{u},\ \nabla_x^2 \mathbf{u}, \dots)$$

$$= \mathbb{S}(\nabla_x \mathbf{u}) + \delta \sum_{j=1}^{k-1} \left((-1)^j \mu_j \Delta^j (\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}) + \lambda_j \Delta^j \mathrm{div}_x \mathbf{u}\ \mathbb{I} \right)$$

$$+ \text{ non-linear terms}$$

Limit for $\delta \to 0$

Limits of numerical solutions

Numerical solutions resulting from Karlsen-Karper and other schemes

Sub-critical parameters

$p(\varrho) = a\varrho^\gamma$, $\gamma < \gamma_{\rm critical}$

Weak (mv) - strong uniqueness

Theorem - EF, P.Gwiazda, A.Świerczewska-Gwiazda, E. Wiedemann [2015]

A measure valued and a strong solution emanating from the same initial data coincide as long as the latter exists

Relative energy (entropy)

Relative energy functional

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\right)(\tau)$$

$$= \int_\Omega \left\langle \nu_{\tau,x}; \frac{1}{2} s|\mathbf{v} - \mathbf{U}|^2 + P(s) - P'(r)(s-r) - P(r) \right\rangle \, \mathrm{d}x$$

$$= \int_\Omega \left\langle \nu_{\tau,x}; \frac{1}{2} s|\mathbf{v}|^2 + P(s) \right\rangle \, \mathrm{d}x - \int_\Omega \langle \nu_{\tau,x}; s\mathbf{v} \rangle \cdot \mathbf{U} \, \mathrm{d}x$$

$$+ \int_\Omega \frac{1}{2} \langle \nu_{\tau,x}; s \rangle \, |\mathbf{U}|^2 \, \mathrm{d}x$$

$$- \int_\Omega \langle \nu_{\tau,x}; s \rangle \, P'(r) \, \mathrm{d}x + \int_\Omega p(r) \, \mathrm{d}x$$

Relative energy (entropy) inequality

Relative energy inequality

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) + \int_0^\tau \mathbb{S}(\nabla_x \mathbf{u}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t + \mathcal{D}(\tau)$$

$$\leq \int_\Omega \left\langle \nu_{0,x}; \frac{1}{2} s |\mathbf{v} - \mathbf{U}_0|^2 + P(s) - P'(r_0)(s - r_0) - P(r_0) \right\rangle \ \mathrm{d}x$$

$$+ \int_0^\tau \mathcal{R}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) \ \mathrm{d}t$$

Remainder

$$\mathcal{R}\left(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\right)$$

$$= - \int_0^\tau \int_\Omega \langle \nu_{t,x}, s\mathbf{v} \rangle \cdot \partial_t \mathbf{U} \, \mathrm{d}x \, \mathrm{d}t$$

$$- \int_0^\tau \int_{\overline{\Omega}} [\langle \nu_{t,x}; s\mathbf{v} \otimes \mathbf{v} \rangle : \nabla_x \mathbf{U} + \langle \nu_{t,x}; p(s) \rangle \, \mathrm{div}_x \mathbf{U}] \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega [\langle \nu_{t,x}; s \rangle \, \mathbf{U} \cdot \partial_t \mathbf{U} + \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \mathbf{U} \cdot \nabla_x \mathbf{U}] \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \left[\left\langle \nu_{t,x}; \left(1 - \frac{s}{r}\right) \right\rangle p'(r) \partial_t r - \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \frac{p'(r)}{r} \nabla_x r \right] \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_0^\tau \left\langle R_1; \frac{1}{2} \nabla_x \left(|\mathbf{U}|^2 - P'(r) \right) \right\rangle \, \mathrm{d}t - \int_0^\tau \langle R_2; \nabla_x \mathbf{U} \rangle \, \mathrm{d}t$$

Regularity

Theorem - EF, P.Gwiazda, A. Świerczewska-Gwiazda, E. Wiedemann

Suppose that the initial data are smooth and satisfy the relevant compatibility conditions. Let $\nu_{t,x}$ be a measure-valued solution to the compressible Navier-Stokes system with a dissipation defect $\mathcal{D}$ such that

$$\mathrm{supp}\ \nu_{t,x} \subset \left\{ (s, \mathbf{v}) \ \Big| \ 0 \leq s \leq \overline{\varrho},\ \mathbf{v} \in R^N \right\}$$

for a.a. $(t, x) \in (0, T) \times \Omega$.
Then $\mathcal{D} = 0$ and

$$\nu_{t,x} = \delta_{\varrho(t,x),\mathbf{u}(t,x)}$$

where ϱ, $\mathbf{u}$ is a smooth solution.

Sketch of the proof

- The Navier-Stokes system admits a local-in-time smooth solution
- The measure-valued solution coincides with the smooth solution on its life-span
- The smooth solution density component remains bounded by $\overline{\varrho}$ as long as the solution exists
- Y. Sun, C. Wang, and Z. Zhang [2011]: The strong solution can be extended as long as the density component remains bounded

Corollary

Convergence of numerical solutions

Bounded numerical solutions emanating from smooth data that converge to a measure-valued solution converge, in fact, unconditionally to the unique strong solution