ON SOLUTIONS OF THE DIFFERENCE EQUATION $x_{n+1} = x_{n-3}/(-1 + x_n x_{n-1} x_{n-2} x_{n-3})$

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Abstract. We study the solutions and attractivity of the difference equation $x_{n+1} = x_{n-3}/(-1 + x_n x_{n-1} x_{n-2} x_{n-3})$ for $n = 0, 1, 2, \ldots$ where x_{-3}, x_{-2}, x_{-1} and x_0 are real numbers such that $x_0 x_{-1} x_{-2} x_{-3} \neq 1$.

Keywords: difference equation, recursive sequence, solutions, equilibrium point

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1. Introduction

A lot of work has been done concerning the attractivity and solutions of the rational difference equations, for example in [1]–[9]. In [3] Cinar studied the positive solutions of the difference equation $x_{n+1} = x_{n-1}/(1 + x_n x_{n-1})$ for $n = 0, 1, 2, \ldots$ and proved by induction the formula

$$x_n = \begin{cases} x_{-1} \dfrac{\prod\limits_{i=0}^{[(n+1)/2]-1} (2x_{-1}x_0 i + 1)}{\prod\limits_{i=0}^{[(n+1)/2]-1} ((2i+1)x_{-1}x_0 + 1)} & \text{for } n \text{ odd,} \\[2ex] x_0 \dfrac{\prod\limits_{i=1}^{n/2} ((2i-1)x_{-1}x_0 + 1)}{\prod\limits_{i=1}^{n/2} (2i x_{-1}x_0 + 1)} & \text{for } n \text{ is even.} \end{cases}$$

In [6] Stevic studied the stability properties of the solutions of Cinar's equation. Also in [7] Stevic investigated the solutions of the difference equation $x_{n+1} =$

$Bx_{n-1}/B + x_n$ and gave the formulas

$$x_{2n} = x_0\left(1 - x_1 \sum_{j=1}^{n} \prod_{i=1}^{2j-1} \frac{1}{1+x_i}\right),$$

$$x_{2n+1} = x_{-1}\left(1 - \frac{x_0}{1+x_0} \sum_{j=0}^{n} \prod_{i=1}^{2j} \frac{1}{1+x_i}\right).$$

Moreover, in [1] Aloqeili generalized the results from [3], [6] to the kth order case and investigated the solutions, stability character and semicycle behavior of the difference equation $x_{n+1} = x_{n-k}/(A + x_{n-k}x_n)$ where $x_{-k}, \ldots, x_0 > 0$ and $A > 0$, k being any positive integer.

Our aim in this paper is to investigate the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{-1 + x_n x_{n-1} x_{n-2} x_{n-3}} \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{1.1}$$

where x_{-3}, x_{-2}, x_{-1} and x_0 are real numbers such that $x_0 x_{-1} x_{-2} x_{-3} \neq 1$.

First, we give two definitions which will be useful in our investigation of the behavior of solutions of Eq. (1.1).

Definition 1. Let I be an interval of real numbers and let $f\colon I^4 \to I$ be a continuously differentiable function. Then for every $x_{-i} \in I$, $i = 0, 1, 2, 3$, the difference equation $x_{n+1} = f(x_n, x_{n-1}, x_{n-2}, x_{n-3})$, $n = 0, 1, 2, \ldots$, has a unique solution $\{x_n\}_{n=-3}^{\infty}$.

Definition 2. The equilibrium point $\bar{x}$ of the equation $x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k})$, $n = 0, 1, 2, \ldots$, is the point that satisfies the condition $\bar{x} = f(\bar{x}, \ldots, \bar{x})$.

2. Main results

Theorem 1. *Assume that $x_0 x_{-1} x_{-2} x_{-3} \neq 1$ and let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq. (1.1). Then for $n = 0, 1, 2, \ldots$ all solutions of Eq. (1.1) are of the form*

$$x_{4n+1} = x_{-3}/\left(-1 + x_0 x_{-1} x_{-2} x_{-3}\right)^{n+1}, \tag{2.1}$$

$$x_{4n+2} = x_{-2}\left(-1 + x_0 x_{-1} x_{-2} x_{-3}\right)^{n+1}, \tag{2.2}$$

$$x_{4n+3} = x_{-1}/(-1 + x_0 x_{-1} x_{-2} x_{-3})^{n+1}, \tag{2.3}$$

$$x_{4n+4} = x_0\left(-1 + x_0 x_{-1} x_{-2} x_{-3}\right)^{n+1}. \tag{2.4}$$

Proof. x_1, x_2, x_3 and x_4 are clear from Eq. (1.1). Also, for $n = 1$ the result holds. Now suppose that $n > 1$ and our assumption holds for $(n-1)$. We shall show that the result holds for n. From our assumption for $(n-1)$ we have

$$\begin{aligned}
x_{4n-3} &= x_{-3}/\left(-1+x_0x_{-1}x_{-2}x_{-3}\right)^n,\\
x_{4n-2} &= x_{-2}(-1+x_0x_{-1}x_{-2}x_{-3})^n,\\
x_{4n-1} &= x_{-1}/\left(-1+x_0x_{-1}x_{-2}x_{-3}\right)^n,\\
x_{4n} &= x_0\left(-1+x_0x_{-1}x_{-2}x_{-3}\right)^n.
\end{aligned}$$

Then, from Eq. (1.1) and the above equality, we have

$$\begin{aligned}
x_{4n+1} &= x_{4n-3}/(-1+x_{4n}x_{4n-1}x_{4n-2}x_{4n-3})\\
&= \frac{x_{-3}/\left(-1+x_0x_{-1}x_{-2}x_{-3}\right)^n}{-1+x_0x_{-1}x_{-2}x_{-3}} = \frac{x_{-3}}{\left(-1+x_0x_{-1}x_{-2}x_{-3}\right)^{n+1}}.
\end{aligned}$$

That is,

$$x_{4n+1} = \frac{x_{-3}}{\left(-1+x_0x_{-1}x_{-2}x_{-3}\right)^{n+1}}.$$

Also,

$$\begin{aligned}
x_{4n+2} &= \frac{x_{4n-2}}{-1+x_{4n+1}x_{4n}x_{4n-1}x_{4n-2}}\\
&= \frac{x_{-2}\left(-1+x_0x_{-1}x_{-2}x_{-3}\right)^n}{-1+x_0x_{-1}x_{-2}x_{-3}/(-1+x_0x_{-1}x_{-2}x_{-3})}\\
&= x_{-2}\left(-1+x_0x_{-1}x_{-2}x_{-3}\right)^{n+1}.
\end{aligned}$$

Hence, we have

$$x_{4n+2} = x_{-2}\left(-1+x_0x_{-1}x_{-2}x_{-3}\right)^{n+1}.$$

Similarly,

$$\begin{aligned}
x_{4n+3} = \frac{x_{4n-1}}{-1+x_{4n+2}x_{4n+1}x_{4n}x_{4n-1}} &= \frac{x_{-1}/\left(-1+x_0x_{-1}x_{-2}x_{-3}\right)^n}{-1+x_0x_{-1}x_{-2}x_{-3}}\\
&= \frac{x_{-1}}{\left(-1+x_0x_{-1}x_{-2}x_{-3}\right)^{n+1}}.
\end{aligned}$$

Consequently, we have

$$x_{4n+3} = \frac{x_{-1}}{\left(-1+x_0x_{-1}x_{-2}x_{-3}\right)^{n+1}}.$$

Now we prove the last formula. Since

$$\begin{aligned}
x_{4n+4} &= \frac{x_{4n}}{-1+x_{4n+3}x_{4n+2}x_{4n+1}x_{4n}}\\
&= \frac{x_0\left(-1+x_0x_{-1}x_{-2}x_{-3}\right)^n}{-1+x_0x_{-1}x_{-2}x_{-3}/(-1+x_0x_{-1}x_{-2}x_{-3})}\\
&= x_0\left(-1+x_0x_{-1}x_{-2}x_{-3}\right)^{n+1},
\end{aligned}$$

we have

$$x_{4n+4} = x_0 \left(-1 + x_0 x_{-1} x_{-2} x_{-3}\right)^{n+1}.$$

Thus, we have proved (2.1), (2.2), (2.3) and (2.4). □

Theorem 2. *Eq.* (1.1) *has three equilibrium points which are* 0, $\sqrt[4]{2}$ *and* $-\sqrt[4]{2}$.

Proof. For the equilibrium points of Eq. (1.1) we write

$$\bar{x} = \bar{x}/(-1 + \bar{x}\bar{x}\bar{x}\bar{x}).$$

Then we have

$$\bar{x}^5 - 2\bar{x} = 0.$$

Thus, the equilibrium points of Eq. (1.1) are 0, $\sqrt[4]{2}$ and $-\sqrt[4]{2}$. □

Corollary 1. *Let* $\{x_n\}$ *be a solution of Eq.* (1.1). *Assume that* $x_{-3}, x_{-2}, x_{-1}, x_0 > 0$ *and* $x_{-3}x_{-2}x_{-1}x_0 > 1$. *Then all solutions of Eq.* (1.1) *are positive.*

Proof. This is clear from Eqs. (2.1), (2.2), (2.3) and (2.4). □

Corollary 2. *Let* $\{x_n\}$ *be a solution of Eq.* (1.1). *Assume that* $x_{-3}, x_{-2}, x_{-1}, x_0 < 0$ *and* $x_{-3}x_{-2}x_{-1}x_0 > 1$. *Then all solutions of Eq.* (1.1) *are negative.*

Proof. This is clear from Eqs. (2.1), (2.2), (2.3) and (2.4). □

Corollary 3. *Let* $\{x_n\}$ *be a solution of Eq.* (1.1). *Assume that* $x_{-3}, x_{-2}, x_{-1}, x_0 > 0$ *and* $x_{-3}x_{-2}x_{-1}x_0 > 2$. *Then*

$$\lim_{n\to\infty} x_{4n+1} = 0,\ \lim_{n\to\infty} x_{4n+2} = \infty,\ \lim_{n\to\infty} x_{4n+3} = 0 \text{ and } \lim_{n\to\infty} x_{4n+4} = \infty.$$

Proof. Let $x_{-3}, x_{-2}, x_{-1}, x_0 > 0$ and $x_{-3}x_{-2}x_{-1}x_0 > 2$.
Then $x_{-3}x_{-2}x_{-1}x_0 - 1 > 1$ and Eq. (2.1), (2.2), (2.3) and (2.4) imply

$$\begin{aligned}
\lim_{n\to\infty} x_{4n+1} &= \lim_{n\to\infty} \frac{x_{-3}}{(-1 + x_0 x_{-1} x_{-2} x_{-3})^{n+1}} = 0,\\
\lim_{n\to\infty} x_{4n+2} &= \lim_{n\to\infty} x_{-2}(-1 + x_0 x_{-1} x_{-2} x_{-3})^{n+1} = \infty,\\
\lim_{n\to\infty} x_{4n+3} &= \lim_{n\to\infty} \frac{x_{-1}}{(-1 + x_0 x_{-1} x_{-2} x_{-3})^{n+1}} = 0,\\
\lim_{n\to\infty} x_{4n+4} &= \lim_{n\to\infty} x_0(-1 + x_0 x_{-1} x_{-2} x_{-3})^{n+1} = \infty.
\end{aligned}$$

□

Corollary 4. *Let* $\{x_n\}$ *be a solution of Eq.* (1.1). *Assume that* $x_{-3}, x_{-2}, x_{-1}, x_0 < 0$ *and* $x_{-3}x_{-2}x_{-1}x_0 > 2$. *Then*

$$\lim_{n\to\infty} x_{4n+1} = 0, \lim_{n\to\infty} x_{4n+2} = -\infty, \lim_{n\to\infty} x_{4n+3} = 0 \text{ and } \lim_{n\to\infty} x_{4n+4} = -\infty.$$

The proof is similar to that of Corollary 3. Thus it is omitted.

Now, we give the following result about the product of solutions of Eq. (1.1).

Corollary 5. $\prod_{n=0}^{s} x_{4n+1}x_{4n+2}x_{4n+3}x_{4n+4} = (x_0x_{-1}x_{-2}x_{-3})^{s+1}$ *where* $s \in \mathbb{Z}^+$.

Proof. From Eqs. (2.1), (2.2), (2.3) and (2.4) we obtain

$$x_{4n+1}x_{4n+2}x_{4n+3}x_{4n+4} = \frac{x_{-3}}{(-1+x_0x_{-1}x_{-2}x_{-3})^{n+1}}\, x_{-2}\,(-1+x_0x_{-1}x_{-2}x_{-3})^{n+1}$$
$$\times \frac{x_{-1}}{(-1+x_0x_{-1}x_{-2}x_{-3})^{n+1}}\, x_0\,(-1+x_0x_{-1}x_{-2}x_{-3})^{n+1} = x_0x_{-1}x_{-2}x_{-3}$$

and the above equality yields

$$\prod_{n=0}^{s} x_{4n+1}x_{4n+2}x_{4n+3}x_{4n+4} = (x_0x_{-1}x_{-2}x_{-3})^{s+1}.$$

Thus, the proof is complete. □

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