HOMOMORPHISMS BETWEEN ALGEBRAS OF HOLOMORPHIC FUNCTIONS IN INFINITE DIMENSIONAL SPACES

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Abstract. It is shown that a homomorphism between certain topological algebras of holomorphic functions is continuous if and only if it is a composition operator.

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1. Introduction

Let E and F be complex locally convex spaces. Let $H(U)$ denote the algebra of all holomorphic functions on an open subset U of E. Let τ_w denote the compact-ported topology introduced by Nachbin [7] on the space $H(U)$. Let V be an open subset in F. In [4] Isidro has characterized the spectrum of the topological algebra $(H(U),\tau_w)$, when E is a complete locally convex space with the approximation property and U is a balanced convex open subset of E. Using this result, in this note we prove that if E is complete and has the approximation property then a homomorphism $A\colon (H(U),\tau_w)\to (H(V),\tau_w)$ is continuous if and only if A is a composition operator. As a consequence we prove that if E is the Tsirelson space each continuous homomorphism between topological algebras of germs of holomorphic functions is a composition operator.

We refer to the books of Dineen [2] or Mujica [6] for background information from infinite dimensional complex analysis.

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Before stating our results, let us fix some notation and terminology. By a homomorphism between algebras we mean an algebra homomorphism which is not identically zero. A topological algebra is an algebra and a topological vector space such that ring multiplication is separately continuous.

Let U and V be open subsets of complex locally convex spaces E and F respectively. We say that a homomorphism $A\colon H(U) \to H(V)$ is a composition operator if there exists a holomorphic function $g\colon V \to E$ such that $g(V) \subset U$ and for each $f \in H(U)$ we have $A(f) = f \circ g$.

A seminorm p on $H(U)$ is ported by a compact set $K \subset U$ if for each open set W with $K \subset W \subset U$, there exists a constant $c(W) > 0$ such that $p(f) \leqslant c(W)\|f\| = \sup_{x\in W} |f(x)|$ for all $f \in H(U)$. The Nachbin topology on $H(U)$, denoted by τ_w, is the locally convex topology defined by all such seminorms. It is known that for any open set U of E, $(H(U), \tau_w)$ is a locally m-convex algebra. We denote by τ_0 the topology on $H(U)$ of the uniform convergence on the compact sets $K \subset U$.

We recall that a complete locally convex space E has the approximation property, if for each neighbourhood of zero V in E and each compact set $K \subset E$ there exists a continuous linear mapping $T\colon E \to E$ with $\dim(T(E)) < \infty$ such that $T(x) - x \in V$, for every $x \in K$.

In [4] Isidro has proved that every complex homomorphism on $(H(U), \tau_w)$ is an evalution at a point of U, where U is a balanced convex open set of E. Using this result we can prove the next proposition.

Proposition 2.1. *Let E and F be complex locally convex spaces such that E is complete and has the appoximation property. Let $U \subset E$ be a convex balanced open subset, and let V be an open subset of F. Then for each homomorphism $A\colon H(U) \to H(V)$ the following statements are equivalent.*

(a) $A\colon (H(U), \tau_w) \to (H(V), \tau_w)$ *is continuous.*

(b) $A\colon (H(U), \tau_w) \to (H(V), \tau_0)$ *is continuous.*

(c) A *is a composition operator.*

Proof. (a) $\Rightarrow$ (b). Let $A\colon (H(U), \tau_w) \to (H(V), \tau_w)$ be a continuous homomorphism. Since the natural inclusion $(H(V), \tau_w) \hookrightarrow (H(V), \tau_0)$ is continuous we have that $A\colon (H(U), \tau_w) \to (H(V), \tau_0)$ is a continuous homomorphism.

(b) $\Rightarrow$ (c). Let $A\colon (H(U), \tau_w) \to (H(V), \tau_0)$ be a continuous homomorphism. For each $y \in V$ we consider the evaluation function at $y, \delta_y\colon (H(V), \tau_0) \to \mathbb{C}$ given by $\delta_y(f) = f(y)$, for every $f \in H(V)$. Thus $\delta_y \circ A\colon (H(U), \tau_w) \to \mathbb{C}$ is a continuous homomorphism and by [4, Corollary 2], there exists a unique $x(y) \in U$ such that $\delta_y \circ A(f) = f(x(y))$, for all $f \in H(U)$, $y \in V$.

Therefore, we can define a mapping $\Phi\colon V \to U$ by $\Phi(y) = x(y)$, for all $y \in V$ and consequently $A(f) = f \circ \Phi$, for all $f \in H(U)$.

(c) $\Rightarrow$ (a). Let $A\colon (H(U), \tau_w) \to (H(V), \tau_w)$ be a composition operator. This means, there exists a holomorphic function $\Phi\colon V \to U$ such that $A(f) = f \circ \Phi$, for all $f \in H(U)$.

Let $q\colon H(V) \to \mathbb{R}$ be a seminorm on $H(V)$ ported by a compact subset L of V. We consider the mapping $p\colon H(U) \to \mathbb{R}$ given by $p(f) = q(A(f))$, for $f \in H(U)$. Since A is a linear mapping we have that p is a seminorm on $H(U)$. We claim that p is ported by the compact subset $\Phi(L)$ of U. Since q is ported by L we obtain a constant $C_{U_1} > 0$ such that $q(g) \leqslant C_{U_1} \|g\|_{\Phi^{-1}(U_1)}$, for all $g \in H(V)$, thus $p(f) = q(A(f)) \leqslant C_{U_1} \|f \circ \Phi\|_{\Phi^{-1}(U_1)} \leqslant C_{U_1} \|f\|_{U_1}$, for all $f \in H(U)$. It follows from [3, Proposition 2, pg. 97] that A is continuous. $\square$

Our next proposition shows that in the case E to be the Tsirelson space (defined by B. Tsirelson in [9]), every continuous homomorphism between algebras of holomorphic germs is a composition operator. Before proving the proposition 2.2 we need some preparation. Let E be a Banach space. Let $\mathscr{H}(K)$ denote the space of all germs of holomorphic functions on a compact subset K of E and let us also denote by τ_w the locally convex inductive limit topology on $\mathscr{H}(K)$ which is defined by $(\mathscr{H}(K), \tau_w) = \varinjlim_{U \supset K} (H(U), \tau_w)$. It is known that $(H(K), \tau_w)$ is an m-locally convex algebra.

Proposition 2.2. *Let E be a Tsirelson space and F be a Banach space. Let $K \subset E$ be an absolutely convex and compact subset and $L \subset F$ a compact subset. Let $A\colon (\mathscr{H}(K), \tau_w) \to (\mathscr{H}(L), \tau_w)$ a homomorphism. Then A is continuous if and only if A is a composition operator.*

Proof. Let $A\colon (\mathscr{H}(K), \tau_w) \to (\mathscr{H}(L), \tau_w)$ be a continuous homomorphism. By [1, Corollary 3.3] we have that A is a composition operator.

Conversely, if A is a composition operator then there exist an open subset $V_0 \supset L$ and a holomorphic function $\Phi\colon V_0 \to E$ such that $\Phi(L) \subset K$ and $A([f]) = [f \circ \Phi]$ for each holomorphic function f defined on a neighbourhood of K.

Thus, for each open subset $U \supset K$, by Theorem 3.2 in [1] there exists an open subset V such that $L \subset V \subset V_0$ with $\Phi(V) \subset U$ and a composition operator $\tilde{A}_U\colon (H(U), \tau_w) \to (H(V), \tau_w)$ given by $\tilde{A}_U(f) = f \circ \Phi$, for $f \in H(U)$. Therefore,

$A \circ \mathscr{I}_U^K = \mathscr{I}_V^L \circ \tilde{A}_U$. That is, we obtain the commutative diagram

$$\begin{array}{ccc} (\mathscr{H}(K), \tau_w) & \xrightarrow{\quad A \quad} & (\mathscr{H}(L), \tau_w) \\ \mathscr{I}_U^K \Big\uparrow & & \Big\uparrow \mathscr{I}_V^L \\ (H(U), \tau_w) & \xrightarrow{\quad \tilde{A}_U \quad} & (H(V), \tau_w) \end{array}$$

So, by the proposition 2.1 (c) $\longrightarrow$ (a) we have that $\tilde{A}_U$ is continuous. Then A is continuous by a result of Nachbin [8, Proposition 45]. This completes the proof. □

Now, we need some additional notation and terminology. Let E be a complex Banach space. For each $m \in \mathbb{N}$ let $\mathscr{P}(^m E)$ denote the space of all continuous m-homogeneous polynomials on E. As usual the space $\sum_{n=0}^{\infty} \mathscr{P}(^n E)$ is denoted by $\mathscr{P}(E)$. We denote by $\mathscr{P}_f(^m E)$ the space generated by all m-homogeneous polynomials of the form $P(x) = \psi(x)^m$, with $\psi \in E'$.

Given a compact set $K \subset E$ we define its *polynomially convex hull* $\widehat{K}_{\mathscr{P}(E)}$ by

$$\widehat{K}_{\mathscr{P}(E)} = \bigl\{x \in E\colon\ |P(x)| \leqslant \sup_{y \in K} |P(y)|\ = \|P\|_K,\ \forall\, P \in \mathscr{P}(E)\bigr\}.$$

The compact set K is said to be *polynomially convex* if $\widehat{K}_{\mathscr{P}(E)} = K$. Let U be an open set in E. We say that U is *polynomially convex* if for each compact set $K \subset U$, the set $\widehat{K}_{\mathscr{P}(E)} \cap U$ is compact.

Corollary 2.3. *Let E be a reflexive Banach space such that $\mathscr{P}_f(^n E)$ is dense in $\mathscr{P}(^n E)$ for each $n \in \mathbb{N}$. Let $K \subset E$ be an absolutely convex and compact subset of E. Let F be a Banach space and $L \subset F$ be a compact subset. Let A: $(\mathscr{H}(K), \tau_w) \to (\mathscr{H}(L), \tau_w)$ be a homomorphism. Then, A is continous if and only if A is a composition operator.*

Proof. The result follows arguing as in Proposition 2.2 and using a result of the authors [1, Corollary 3.4]. □

In [5] Mujica has extended the Corollary 2 of Isidro [4] for polynomially convex open set in locally convex space quasi-complete with the approximation property. As a consequence of Mujica's results we get the next proposition.

Proposition 2.4. *Let E be a quasi-complete space with the approximation property and let F be a locally convex space. Let $U \subset E$ be a polynomially convex open subset and $V \subset F$ an open subset. Let $A\colon H(U) \to H(V)$ a homomorphisms. The following statements are equivalent.*

(a) *$A\colon (H(U), \tau_w) \to (H(V), \tau_w)$ is continuous.*

(b) *$A\colon (H(U), \tau_w) \to (H(V), \tau_0)$ is continuous.*

(c) *A is a composition operator.*

Proof. The proof here is similar to the proof of the proposition 2.1. □

References

[1] *L. O. Condori, M. L. Lourenço*: Continuous homomorphism between topological algebras of holomorphic germs. Rocky Moutain J. Math. *36(5)* (2006), 1457–1469.

[2] *S. Dineen*: Complex Analysis on Infinite Dimensional Spaces. Springer, 1999. zbl

[3] *J. Horvath*: Topological Vector Spaces and Distributions Vol. 1. Addison-Wesley, 1966. zbl

[4] *J. M. Isidro*: Characterization of the spectrum of some topological algebras of holomorphic functions. Advances in Holomorphy, (ed. J.A. Barroso), North-Holland, 1979, pp. 407–416. zbl

[5] *J. Mujica*: The Oka-Weil theorem in locally convex spaces with the approximation property. Séminaire Paul Krée 1977/1978. Institute Henri Poincaré, Paris, 1979. zbl

[6] *J. Mujica*: Complex Analysis in Banach Spaces. Math. Stud. Vol. 120, North-Holland, 1986. zbl

[7] *L. Nachbin*: On the topology of the space of all holomorphic functions on a given open subset. Indag. Math. *29* (1967), 366–368. zbl

[8] *L. Nachbin*: Topics on Topological Vector Spaces. Textos de Métodos Matemáticos da Universidade Federal do Rio de Janeiro, 1974.

[9] *B. Tsirelson*: Not every Banach space contains an imbedding of ℓ_p or c_0. Funct. Anal. Appl. *8* (1974), 138–144. zbl

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