INSTABILITY OF TURING TYPE FOR A REACTION-DIFFUSION SYSTEM WITH UNILATERAL OBSTACLES MODELED BY VARIATIONAL INEQUALITIES

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Abstract. We consider a reaction-diffusion system of activator-inhibitor type which is subject to Turing's diffusion-driven instability. It is shown that unilateral obstacles of various type for the inhibitor, modeled by variational inequalities, lead to instability of the trivial solution in a parameter domain where it would be stable otherwise. The result is based on a previous joint work with I.-S. Kim, but a refinement of the underlying theoretical tool is developed. Moreover, a different regime of parameters is considered for which instability is shown also when there are simultaneously obstacles for the activator and inhibitor, obstacles of opposite direction for the inhibitor, or in the presence of Dirichlet conditions.

Keywords: reaction-diffusion system; Signorini condition; unilateral obstacle; instability; asymptotic stability; parabolic obstacle equation

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary, and $\Gamma_D \subseteq \partial\Omega$. Given $f_1, f_2 \colon \mathbb{R}^2 \to \mathbb{R}$, we consider the reaction-diffusion system

$$(1.1) \qquad \begin{aligned} u_t &= d_1 \Delta u + f_1(u, v) \\ v_t &= d_2 \Delta v + f_2(u, v) \end{aligned}$$

with diffusion coefficients $d_1, d_2 > 0$ and Neumann-Dirichlet boundary conditions

$$(1.2) \qquad \begin{cases} u = v = 0 & \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \setminus \Gamma_D. \end{cases}$$

We will supplement (1.1), (1.2) with unilateral obstacle conditions and show that these obstacles will change the stability for certain diffusion coefficients (d_1, d_2). The result is based on a refinement of a theoretical tool developed in [4]. In a sense, the result presented in this paper complements the application from [4], since we consider a different range of values (d_1, d_2) than in [4], which leads to a result under somewhat different assumptions on the obstacle/boundary conditions (and allows us also to treat $\Gamma_D \neq \emptyset$).

2. Auxiliary and main results

We assume that $f_i(0, 0) = 0$ with f_i being differentiable at $(0, 0)$, and we are interested in the stability of the trivial solution $u = v = 0$ of (1.1), (1.2). We suppose that system (1.1) is subject to Turing's diffusion-driven instability [7], that is, we assume that without diffusion $(d_1 = d_2 = 0)$ the solution $u = v = 0$ is linearly stable. In other words, we assume that the Jacobi-matrix (b_{ij}) of (f_1, f_2) at $(0, 0)$ has its spectrum in the left half-plane. For definiteness, and to avoid a certain Hopf-type instability later on, we assume $b_{11} > 0$. It follows from Vieta's theorem that these assumptions are equivalent to the sign conditions

(2.1) $\qquad b_{11} > 0 > b_{22}, \quad b_{12}b_{21} < 0, \quad b_{11} + b_{22} < 0, \quad b_{11}b_{22} - b_{12}b_{21} > 0.$

We rewrite (1.1) in the form

(2.2)
$$u_t = d_1 \Delta u + b_{11}u + b_{12}v + g_1(u, v),$$
$$v_t = d_2 \Delta v + b_{21}u + b_{22}v + g_2(u, v),$$

where

(2.3)
$$\lim_{(u,v) \to (0,0)} \frac{g_i(u, v)}{|u| + |v|} = 0 \quad \text{for } i = 1, 2.$$

We assume that f_i (and thus g_i) satisfy the Lipschitz type condition

(2.4) $|f_i(u_1, v_1) - f_i(u_2, v_2)| \leqslant C(1 + |u_1| + |u_2| + |v_1| + |v_2|)^{\alpha_N}(|u_1 - u_2| + |v_1 - v_2|)$

for all $u_j, v_j \in \mathbb{R}$, where $C \in [0, \infty)$ and $\alpha_N := 2/(N - 2)$ if $N > 2$ or $0 < \alpha_N < \infty$ if $N = 2$. In case of space dimension $N = 1$, we assume instead of (2.4) only that f_i satisfy a local Lipschitz condition in some neighborhood of $(u, v) = (0, 0)$.

It is well-known (see [6] for $N = 1$ or [1] for the general case) that $(0, 0)$ is linearly stable if and only if $(d_1, d_2) \in \mathbb{R}_+^2$ lies in the open set D_S to the right/under each of

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the countably many hyperbolas

$$(2.5) \qquad C_n = \{(d_1, d_2) \in \mathbb{R}_+^2 \colon (\kappa_n d_1 - b_{11})(\kappa_n d_2 - b_{22}) = b_{12} b_{21}\}$$
$$= \Big\{(d_1, d_2) \in \mathbb{R}_+^2 \colon d_2 = \frac{b_{12} b_{21}/\kappa_n^2}{d_1 - b_{11}/\kappa_n} + \frac{b_{22}}{\kappa_n}\Big\},$$

see Figure 1. Here, $0 < \kappa_1 < \kappa_2 < \ldots \to \infty$ denote the strictly positive eigenvalues of $-\Delta$ with Dirichlet-Neumann boundary condition (1.2) (in case $\Gamma_D = \emptyset$ the trivial eigenvalue $\kappa_0 = 0$ is omitted from the sequence).
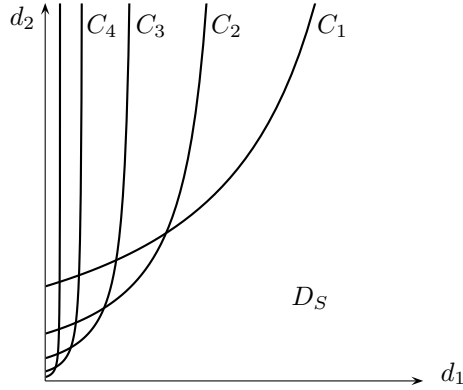

Figure 1. Hyperbolas (2.5) determining D_S.

Although the above mentioned result is mathematical folklore, we will give a new simple proof of this observation based on spectral calculus in Section 5. In order to compare this observation with our subsequent main result concerning obstacles, we formulate now precisely what the linear stability implies in the terminology of dynamical systems.

To this end, we consider the weak form of (1.1), (1.2), that is, we consider the Hilbert space

$$\mathbb{V} := \{u \in W^{1,2}(\Omega) \colon u|_{\Gamma_D} = 0\}$$

(here and in the following, restrictions $u|_{\Gamma_D}$ are understood in the sense of traces, of course), and then understand the solutions of (1.1), (1.2) as absolutely continuous functions $u, v \colon I \to L_2(\Omega)$ (I denoting some interval) satisfying the variational equations

$$(2.6) \qquad u \in \mathbb{V}, \quad \int_\Omega (u' - f_1(u,v))\varphi \, dx + d_1 \int_\Omega \nabla u \nabla \varphi \, dx = 0 \quad \text{for all } \varphi \in \mathbb{V},$$
$$v \in \mathbb{V}, \quad \int_\Omega (v' - f_2(u,v))\psi \, dx + d_2 \int_\Omega \nabla v \nabla \psi \, dx = 0 \quad \text{for all } \psi \in \mathbb{V}.$$

Note that, since u and v are absolutely continuous and $L_2(\Omega)$ is separable and reflexive, the derivatives u' and v' exist almost everywhere, and the corresponding fundamental theorem of calculus holds with the Lebebesgue-Bochner integral, and so it follows from e.g. [8], Theorem 4.4.4, that the partial derivatives of $u(t,s) = u(t)(s)$ with respect to t exist for almost all (t,s) and satisfy $u_t(t,s) = u'(t)(s)$ for almost all (t,s).

We call the trivial solution $u = v = 0$ of system (2.6) *asymptotically stable* in the $\mathbb{V}^2$ topology if each neighborhood $U \subseteq \mathbb{V}^2$ of $(0,0)$ contains a neighborhood $V \subseteq \mathbb{V}^2$ of $(0,0)$ such that all solutions (u,v) of (2.11) with $(u(0), v(0)) \in V$ satisfy $(u(t), v(t)) \in U$ for all $t > 0$ and $u(t) \to 0$ and $v(t) \to 0$ as $t \to \infty$ in the topology of $\mathbb{V}$.

Proposition 1. *Assume (2.1), (2.3), and (2.4). Then for $(d_1, d_2) \in D_S$ the trivial solution $u = v = 0$ of system (2.6) is asymptotically stable in the $\mathbb{V}^2$ topology, the convergence $u(t), v(t) \to 0$ as $t \to \infty$ being exponentially fast.*

The associated linearization has its spectrum in the left half-plane, the real part of the spectral values being even uniformly bounded by a negative number.

P r o o f.   See Section 5. $\square$

Our aim in this paper is to show that the (asymptotic) stability of $(0,0)$ in (1.1), (1.2) is lost even for certain $(d_1, d_2) \in D_S$ if on some (nontrivial) parts of $\overline{\Omega} \setminus \Gamma_D$ appropriate unilateral conditions are prescribed.

More precisely, we fix measurable (with respect to Lebesgue measure) subsets $\Omega_i^{\pm} \subseteq \Omega$ and measurable (with respect to Hausdorff measure of dimension $N-1$) sets $\Gamma_i^{\pm} \subseteq \Gamma := \partial\Omega \setminus \Gamma_D$ such that

$$(2.7) \qquad (\overline{\Omega_i^+ \cup \Gamma_i^+}) \cap (\overline{\Omega_i^- \cup \Gamma_i^-}) = \emptyset \quad (i = 1, 2).$$

In physical terms, these eight sets are the locations in the interior or on the boundary of Ω where we will describe a certain regulating system or unilateral membrane which provides some unilateral flow in or out for u or v, respectively. Condition (2.7) means that the locations for opposite flux should not touch each other. In applications, most of these eight sets are empty, that is, not all possible sorts of obstacles are described simultaneously in the same problem (and we need not have flux in- and outside simultaneously). However, we will *not exclude* the possibility that they are all nonempty and thus write a complete system which *can* contain all possible cases simulteanously.

In order to simplify notation, we set

$$\Omega_i := \Omega_i^+ \cup \Omega_i^-, \quad \Gamma_i := \Gamma_i^+ \cup \Gamma_i^- \quad (i = 1, 2).$$

We will require that there is at least some obstacle for the inhibitor v, that is:

$$(2.8) \qquad \operatorname{mes}_N \Omega_2 > 0 \quad \text{or} \quad \operatorname{mes}_{N-1} \Gamma_2 > 0 \quad \text{(or both)}.$$

Actually, we will allow that Ω_i and Γ_i consist of subregions where we will describe "integral" obstacles (in the remaining regions, "pointwise" obstacles will be described). To fix the locations of these "integral" obstacles, we assume that finitely many measurable subsets $\Omega_{i,j}^{\pm} \subseteq \Omega_i^{\pm}$ $(j \in J_{1,i}^{\pm})$ and $\Gamma_{i,j}^{\pm} \subseteq \Gamma_i^{\pm}$ $(j \in J_{2,i}^{\pm})$ are given such that

$$\overline{\Omega}_{i,j}^{\pm} \cap \overline{\Omega}_{i,k}^{\pm} = \emptyset \quad \text{for all } j, k \in J_{1,i}^{\pm}, \ j \neq k,$$

and

$$\overline{\Gamma}_{i,j}^{\pm} \cap \overline{\Gamma}_{i,k}^{\pm} = \emptyset \quad \text{for all } j, k \in J_{2,i}^{\pm}, \ j \neq k,$$

and we define the remainder regions (where we will describe "pointwise" obstacles) as

$$\widehat{\Omega}_i^{\pm} := \Omega_i^{\pm} \setminus \bigcup_{j \in J_{1,i}^{\pm}} \Omega_{i,j}^{\pm} \quad \text{and} \quad \widehat{\Gamma}_i^{\pm} := \Gamma_i^{\pm} \setminus \bigcup_{j \in J_{2,i}^{\pm}} \Gamma_{i,j}^{\pm}.$$

It is explicitly admissible that some (or all) of the finite index sets $J_{1,i}^{\pm}, J_{2,i}^{\pm}$ $(i = 1, 2)$ are empty, that is, we do *not require* integral obstacles but we do *not exclude* their presence. Similarly, also the remainder regions $\widehat{\Omega}_i^{\pm}$ or $\widehat{\Gamma}_i^{\pm}$ are allowed to be empty. Now our general unilateral problem can be formulated as follows:

$$(2.9) \qquad u_t = d_1 \Delta u + f_1(u, v) \quad \text{on } \Omega \setminus \Omega_1,$$
$$\pm u_t \mp (d_1 \Delta u + f_1(u, v)) \geqslant 0, \quad \pm u \geqslant 0 \quad \text{on } \widehat{\Omega}_1^{\pm},$$
$$(-u_t + d_1 \Delta u + f_1(u, v)) u = 0 \quad \text{on } \widehat{\Omega}_1^{\pm},$$
$$\pm u_t \mp (d_1 \Delta u + f_1(u, v)) = \text{const} \geqslant 0, \quad \pm \int_{\Omega_{1,j}^{\pm}} u \, dx \geqslant 0 \quad \text{on } \Omega_{1,j}^{\pm},$$
$$(-u_t + d_1 \Delta u + f_1(u, v)) \int_{\Omega_{1,j}^{\pm}} u \, dx = 0 \quad \text{on } \Omega_{1,j}^{\pm},$$
$$v_t = d_2 \Delta v + f_2(u, v) \quad \text{on } \Omega \setminus \Omega_2,$$
$$\pm v_t \mp (d_2 \Delta v + f_2(u, v)) \geqslant 0, \quad \pm v \geqslant 0 \quad \text{on } \widehat{\Omega}_2^{\pm},$$
$$(-v_t + d_2 \Delta v + f_2(u, v)) v = 0 \quad \text{on } \widehat{\Omega}_2^{\pm},$$
$$\pm v_t \mp (d_2 \Delta v + f_2(u, v)) = \text{const} \geqslant 0, \quad \pm \int_{\Omega_{1,j}^{\pm}} v \, dx \geqslant 0 \quad \text{on } \Omega_{1,j}^{\pm},$$
$$(-v_t + d_2 \Delta v + f_2(u, v)) \int_{\Omega_{1,j}^{\pm}} v \, dx = 0 \quad \text{on } \Omega_{1,j}^{\pm},$$

with Dirichlet/Neumann/Signorini/integral-Signorini type boundary conditions:

$$(2.10) \qquad u = v = 0 \quad \text{on } \Gamma_D,$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma \setminus \Gamma_1,$$

$$\pm \frac{\partial u}{\partial n} \geqslant 0, \quad \pm u \geqslant 0, \quad \frac{\partial u}{\partial n} u = 0 \quad \text{on } \widehat{\Gamma}_1^{\pm},$$

$$\pm \frac{\partial u}{\partial n} = \text{const} \geqslant 0, \quad \pm \int_{\Gamma_{1,j}^{\pm}} u \, dx \geqslant 0, \quad \frac{\partial u}{\partial n} \int_{\Gamma_{1,j}^{\pm}} u \, dx = 0 \quad \text{on } \Gamma_{1,j}^{\pm},$$

$$\frac{\partial v}{\partial n} = 0 \quad \text{on } \Gamma \setminus \Gamma_2,$$

$$\pm \frac{\partial v}{\partial n} \geqslant 0, \quad \pm v \geqslant 0, \quad \frac{\partial v}{\partial n} v = 0 \quad \text{on } \widehat{\Gamma}_2^{\pm},$$

$$\pm \frac{\partial v}{\partial n} = \text{const} \geqslant 0, \quad \pm \int_{\Gamma_{2,j}^{\pm}} v \, dx \geqslant 0, \quad \frac{\partial v}{\partial n} \int_{\Gamma_{2,j}^{\pm}} v \, dx = 0 \quad \text{on } \Gamma_{2,j}^{\pm}.$$

Here, the shortcuts $\pm$ and $\mp$ are understood such that in each line either the upper or lower sign is chosen simultaneously. Roughly speaking, the unilateral conditions mean the following: At each obstacle point (or integral region) "normally" the original system (1.1)/(1.2) is satisfied unless u or v lies under/over the threshold 0. In the latter case, the obstacle (e.g. a unilateral membrane) becomes active, forcing u or v to be 0 by activating a corresponding source/sink. On each integral part, the "measurement" of the membrane is only "in the mean", and the activated source/sink is uniform (that is, constant on the whole integral part).

Actually, we will consider only the weak form of (2.9), (2.10) which is much easier to formulate: We consider the following two cones of obstacles for u ($i = 1$) or v ($i = 2$), respectively:

$$K_i := \left\{ u \in \mathbb{V} : \pm u|_{\widehat{\Omega}_i^{\pm}} \geqslant 0 \ \text{ and } \ \pm \int_{\Omega_{i,j}^{\pm}} u \, dx \geqslant 0 \ \text{ for all } j \in J_{1,i}^{\pm} \right.$$

$$\left. \text{and } \ \pm u|_{\widehat{\Gamma}_i^{\pm}} \geqslant 0 \ \text{ and } \ \pm \int_{\Gamma_{i,j}^{\pm}} u \, dx \geqslant 0 \ \text{ for all } j \in J_{2,i}^{\pm} \right\}.$$

Then the weak form of (2.9), (2.10) becomes

$$(2.11) \quad u \in K_1, \quad \int_{\Omega} (u' - f_1(u, v))(\varphi - u) \, dx + d_1 \int_{\Omega} \nabla u (\nabla \varphi - \nabla u) \, dx \geqslant 0$$
$$\text{for all } \varphi \in K_1,$$

$$v \in K_2, \quad \int_{\Omega} (v' - f_2(u, v))(\psi - v) \, dx + d_2 \int_{\Omega} \nabla v (\nabla \psi - \nabla v) \, dx \geqslant 0$$
$$\text{for all } \psi \in K_2.$$

In order to formulate our main result, we have to consider the eigenspaces E_n of $-\Delta$ (with (1.2)) to the eigenvalues κ_n, that is,

$$E_n := \left\{ u \in \mathbb{V} : \int_\Omega \nabla u \nabla \varphi \, dx = \kappa_n \int_\Omega u\varphi \, dx \ \text{ for all } \varphi \in \mathbb{V} \right\}.$$

Moreover, we define the sets

$$K_i^\circ := \left\{ u \in \mathbb{V} : \operatorname*{ess\,inf}_{\widehat{\Omega}_i^\pm}(\pm u) > 0 \ \text{ and } \ \pm \int_{\Omega_{i,j}^\pm} u \, dx > 0 \ \text{ for all } j \in J_{1,i}^\pm \right.$$

$$\left. \text{and } \operatorname*{ess\,inf}_{\widehat{\Gamma}_i^\pm}(\pm u) > 0 \ \text{ and } \ \pm \int_{\Gamma_{i,j}^\pm} u \, dx > 0 \ \text{ for all } j \in J_{2,i}^\pm \right\},$$

which in a certain sense might be considered as an "interior" of K_i. (However, K_i° is not the interior of K_i in the topological sense for $N > 1$, since the latter is empty in case $N > 1$.)

R e m a r k 1. For the case when $\partial\Omega$, Γ_D, and $\widehat{\Gamma}_i^\pm$ are smooth manifolds with a smooth boundary, we can relax the definition of K_i° by replacing the requirement $\operatorname*{ess\,inf}_{\widehat{\Gamma}_i^\pm}(\pm u) > 0$ by the weaker requirement that $\pm u|_{\widehat{\Gamma}_i^\pm} > 0$ almost everywhere.

Let $d_0 = (d_{1,0}, d_{2,0}) \in \mathbb{R}_+^2 \cap \partial D_S$, that is, d_0 lies either on a single hyperbola $C_n \setminus \bigcup_{m \neq n} C_m$, or d_0 is the intersection point of exactly two hyperbolas $C_n \cap C_m$ $(n \neq m)$. In the former case, we put $\sigma := \operatorname{sgn} b_{12}$ and require

$$(2.12) \qquad E_n \cap K_2^\circ \cap \sigma K_1^\circ \neq \emptyset \quad \text{and} \quad E_n \cap K_2 \cap (-\sigma K_1) \subseteq \sigma K_1,$$

and in the latter case, we put for $i = n, m$

$$\alpha_i(d_0) := \frac{-b_{12}}{b_{11} - d_{1,0}\kappa_i} = \frac{d_{2,0}\kappa_i - b_{22}}{b_{21}},$$

$$\alpha_i^*(d_0) := \frac{d_{2,0}^{-1}(-b_{21})}{d_{1,0}^{-1}b_{11} - \kappa_i} = \frac{\kappa_i - d_{2,0}^{-1}b_{22}}{d_{1,0}^{-1}b_{12}},$$

and require the following:

(2.13) There are $e_i \in E_i$ $(i = n, m)$

         with $e_n + e_m \in K_2^\circ$, $\alpha_n^*(d_0)e_n + \alpha_m^*(d_0) \in K_1^\circ$.

       For every $e_i \in E_i$ $(i = n, m)$ there holds

       $(e_n + e_m \in K_2 \text{ and } \alpha_n(d)e_n + \alpha_m(d)e_m \in K_1)$

           $\Longrightarrow \alpha_n^*(d)e_n + \alpha_m^*(d)e_m \in K_1.$

Noting that $\operatorname{sgn} \alpha_i(d_0) = \sigma = -\operatorname{sgn} \alpha_i^*(d_0)$, we have actually a consistency of the requirements for the two cases: The first hypothesis of (2.13) is weaker than the first hypothesis from (2.12) while the second hypothesis is stricter.

Since these hypotheses are rather technical, let us first discuss them: First of all, the requirements $K_i^\circ \neq \emptyset$ imply that the regions of "pointwise" obstacles should be strictly separated from Γ_D. Hence, the following condition is necessary for (2.12) or (2.13):

$$(2.14) \qquad \overline{\widehat{\Gamma}_i^\pm} \cap \overline{\Gamma}_D = \emptyset \quad \text{and} \quad \overline{\widehat{\Omega}_i^\pm} \cap \overline{\Gamma}_D = \emptyset \quad \text{for } i = 1, 2.$$

R e m a r k 2. In the setting of Remark 1, we can relax the first condition in (2.14) slightly by allowing the boundary manifolds of $\widehat{\Gamma}_i^\pm$ and Γ_D to intersect.

The second hypothesis in (2.12) and (2.13) restricts the possible obstacles for u: If we have no obstacles for u ($K_1 = \mathbb{V}$), then the second hypothesis in (2.12) and (2.13) is automatically satisfied. However, the less obstacles we have for v, the more restrictions are imposed by this hypothesis on our choice of obstacles for u. In the extreme case, if there were no obstacle for v ($K_2 = \mathbb{V}$), then the second condition would imply that K_1 is a linear subspace and thus would not allow any reasonable obstacle for u: This is the reason why our whole discussion only makes sense under hypothesis (2.8).

In some important situations, the necessary requirements (2.14) are already sufficient for (2.12)/(2.13): Recall that if we have a Dirichlet condition, that is, if

$$(2.15) \qquad \Gamma_D \text{ has positive } (N-1)\text{-dimensional Hausdorff measure,}$$

then the first eigenspace E_1 is one-dimensional and generated by a strictly positive function. Thus, under hypothesis (2.14) and (2.15) one can usually apply the following result at least with $n = 1$:

Proposition 2. *Suppose* (2.1). *Assume that the obstacles for u and v act only into the same (if $b_{12} > 0$) or opposite (if $b_{12} < 0$) direction, that is, we assume that one of the following four cases holds:*

$$(2.16) \qquad \begin{aligned} &b_{12} > 0 \quad and \quad \Gamma_1^+ = \Omega_1^+ = \Gamma_2^+ = \Omega_2^+ = \emptyset, \\ &b_{12} > 0 \quad and \quad \Gamma_1^- = \Omega_1^- = \Gamma_2^- = \Omega_2^- = \emptyset, \\ &b_{12} < 0 \quad and \quad \Gamma_1^+ = \Omega_1^+ = \Gamma_2^- = \Omega_2^- = \emptyset, \\ &b_{12} < 0 \quad and \quad \Gamma_1^- = \Omega_1^- = \Gamma_2^+ = \Omega_2^+ = \emptyset. \end{aligned}$$

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Then $d_0 \in C_n \cap \partial D_S$ satisfies (2.12) or (2.13), respectively, under the following assumptions: E_n contains a function which is uniformly positive (bounded away from 0) on $\Omega_1 \cup \Omega_2 \cup \Gamma_1 \cup \Gamma_2$, and at least one of the following holds:

(1) $\Omega_1 = \Gamma_1 = \emptyset$.

(2) $d_0 \notin \bigcup\limits_{m \neq n} C_m$ and there is no nonzero function in E_n which is nonnegative on $\Omega_1 \cup \Gamma_1$ and nonpositive on $\Omega_2 \cup \Gamma_2$.

P r o o f. The positive function from the hypothesis (or its negative) proves that the first assumption of (2.12) or (2.13) holds, respectively. In case $\Omega_1 = \Gamma_1 = \emptyset$ we have $K_1 = \mathbb{V}$, and otherwise the hypothesis implies $E_n \cap K_1 \cap (-\sigma K_2) = \{0\}$. In both cases, the second assumption in (2.13) or (2.12), respectively, is trivially satisfied. $\qquad \square$

In order to formulate a slightly more powerful version of our main result, we need yet another notion.

We call a point $(\hat{d}_1, \hat{d}_2) \in \mathbb{R}_+^2$ a *non-bifurcation point* of stationary solutions of (2.11) if there is a neighborhood of $(\hat{d}_1, \hat{d}_2, 0, 0)$ in $\mathbb{R}^2 \times \mathbb{V}^2$ such that every (d_1, d_2, u, v) in this neighborhood satisfying (2.11) (with $u' = v' := 0$) satisfies $u = v = 0$.

Now our main result can be formulated as follows:

Theorem 1. *Suppose* (2.1), (2.3), (2.4), (2.8), *and* (2.15). *Let* $d_0 \in \mathbb{R}_+^2 \cap \partial D_S$ *be as above, satisfying* (2.12) *or* (2.13). *Then the set D of non-bifurcation points of stationary solutions of* (2.11) *in D_S is open and contains $D_S \cap B_r(d_0)$ for some open ball $B_r(d_0) \subseteq \mathbb{R}_+^2$ around d_0; $r > 0$ can even be chosen independent of g_i. Let D_0 denote the connected component of D which contains $D_S \cap B_r(d_0)$. Then for each $(d_1, d_2) \in D_0$ the trivial solution of system* (2.11) *fails to be asymptotically stable in the $\mathbb{V}^2$ topology.*

We require the condition (2.15) only to simplify the proof and to use a technical result from [2] where this condition was imposed. Actually, Theorem 1 can be shown also without the hypothesis (2.15) if one uses a corresponding variant of the mentioned technical result which will be given in a forthcoming paper.

Our proof of Theorem 1 does not show that the trivial solution is unstable; only the failure of *asymptotic* stability is claimed. However, our proof will show that the trivial solution fails to be "spherically stable" (which will be defined in Remark 3).

The main difference of Theorem 1 to the corresponding result in [4] is that in the latter, points (d_1, d_2) with large values $d_1, d_2 > 0$ play a crucial role instead of points (d_1, d_2) close to a certain hyperbola C_n (or to $C_n \cap C_m$). This makes a fundamental difference in the hypotheses of the two results: While in [4] there is apparently no

hypothesis similar to (2.12)/(2.13), this hypothesis is implicitly in an requirement about E_0 (corresponding to the eigenvalue $\kappa_0 = 0$). In fact, the corresponding result in [4] applies only to the non-Dirichlet case $\Gamma_D = \emptyset$ and only if there is no obstacle for u ($\Omega_1 = \Gamma_1 = \emptyset$) and the obstacle for v acts in a uniform direction ($\Gamma_2^- = \Omega_2^- = \emptyset$ or $\Gamma_1^- = \Omega_1^- = \emptyset$). It seems likely that none of these additional requirements can be dropped for the result from [4] (since they are crucial for a major tool provided by [5]). In contrast, Theorem 1 needs none of these requirements, but instead (2.12)/(2.13) is needed.

3. The Krasnosel'skii-Quittner formula and instability

In this section, we show a variant of the main result from [4], which in contrast to [4] does not require us to consider artificial scalar products in the spaces for the application.

Let $(V, \langle \cdot, \cdot \rangle, \|\cdot\|)$ and $(H, (\cdot, \cdot), |\cdot|)$ be Hilbert spaces such that there is a compact dense embedding $V \subseteq H$ which we treat notationally as the identity. As is common practice in this setting, we identify notationally also the dual space $H' \cong H$ and work with the adjoint embedding $H' \subseteq V'$ (which just restricts functionals), assuming that the corresponding duality map $(\cdot, \cdot) \colon V' \times V \to \mathbb{R}$ is compatible with the scalar product of H and thus using the same symbol. We denote the functional norm in V' induced by this duality map by $\|\cdot\|_{V'}$.

Let $\mathfrak{A} \colon V \to V'$ be a linear isomorphism onto V'. Putting $D(A) := \mathfrak{A}^{-1}(H) \subseteq V \subseteq H$, we let $A := \mathfrak{A}|_{D(A)} \colon D(A) \to H$ denote the H-realization of $\mathfrak{A}$. Suppose that there exist $c_0, c_1 \in [0, \infty)$ with

$$(3.1) \qquad |\langle A^{-1}u, v\rangle| \leqslant c_0 |u||v| \quad \text{for all } u \in H,\ v \in V \subseteq H,$$

$$(3.2) \qquad |u|^2 \leqslant c_1 \langle A^{-1}u, u\rangle \quad \text{for all } u \in V \subseteq H,$$

$$(3.3) \qquad \langle A^{-1}u, v\rangle = \langle u, A^{-1}v\rangle \quad \text{for all } u, v \in V \subseteq H.$$

We consider a closed convex set $K \subseteq V$ with the following property:

$$(3.4) \quad ((u, \varphi - v) \geqslant 0 \quad \text{for all } \varphi \in K) \iff (\langle A^{-1}u, \varphi - v\rangle \geqslant 0 \quad \text{for all } \varphi \in K)$$
$$\text{for all } v \in K,\ u \in V.$$

Given an open set $\mathscr{U} \subseteq V$ and a map $F \colon \mathscr{U} \to H$, we are interested in the variational inequality

$$(3.5) \qquad u(t) \in K, \quad \left(\frac{du(t)}{dt} + \mathfrak{A}(u(t)) - F(u(t)), \varphi - u(t)\right) \geqslant 0 \quad \text{for all } \varphi \in K.$$

More precisely, we assume that $\mathscr{U}$ is a neighborhood of a given stationary solution u_0 of (3.5), that is,

$$(3.6) \qquad u_0 \in K, \quad (\mathfrak{A}u_0 - F(u_0), \varphi - u_0) \geqslant 0 \quad \text{for all} \quad \varphi \in K,$$

and we require that F satisfies the Lipschitz/Hölder type conditions

$$(3.7) \quad |F(u) - F(v)| \leqslant c_2\|u - v\|, \quad \|F(u) - F(v)\|_{V'} \leqslant c_3|u - v|^\alpha \quad \text{for all } u, v \in \mathscr{U},$$

where $\alpha > 0$ and $c_2, c_3 \in [0, \infty)$.

For every compact and continuous map $G\colon \mathscr{U} \to V$ which has u_0 as an isolated fixed point, we denote by $\text{ind}(G, u_0)$ the local fixed point index of G at u_0, that is, $\text{ind}(G, u_0)$ is defined as the Leray-Schauder degree $\deg(\text{id} - G, \mathscr{U}_0, 0)$ on an open bounded set $\mathscr{U}_0 \subseteq \mathscr{U}$ containing u_0 as the only fixed point. The excision property of the Leray-Schauder degree implies that this index is independent of the particular choice of $\mathscr{U}_0$.

Similarly, if $G\colon \mathscr{U} \cap K \to K$ is compact and continuous and has u_0 as an isolated fixed point, we define $\text{ind}_K(G, u_0)$ as the local fixed point index of G at u_0 relative to K, that is, as the local fixed point index of the map $G \circ \varrho$ where ϱ is a retraction of an open neighborhood $\mathscr{U}_0 \subseteq \mathscr{U}$ onto $K \cap \mathscr{U}_1$ where $\mathscr{U}_1 \subseteq V$ is an open neighborhood of u_0. The commutativity and excision property of the fixed point index imply that this value is independent of the choice of ϱ and $\mathscr{U}_i$. In particular, one can choose $\varrho = P_K$ as the metric projection, that is, $P_K(u)$ is the unique element of K of the closest distance (with respect to $\|\cdot\|$) to u.

Theorem 2. *Under the above hypotheses, we find for each initial value $u_1 \in \mathscr{U}$ a unique weak solution $\Phi(t)u_1 = u(t)$ of (3.5) satisfying $u(0) = u_1$ which exists until $u(t)$ meets the boundary of $\mathscr{U}$ (or tends to ∞). This solution satisfies (3.5) for every such $t > 0$ if we interpret $\mathrm{d}u(t)/\mathrm{d}t$ as the right-sided derivative. For every time $t > 0$ the translation-in-time operator $\Phi(t)$ is compact and continuous on its domain of definition; it is at least defined for small $t > 0$. Moreover, if $\text{ind}(P_K \circ A^{-1} \circ F, u_0)$ is defined then so is $\text{ind}_K(\Phi(t), u_0)$ for small $t > 0$ and has the same value. If this value is defined and differs from 1 then at least one of following two assertions holds:*

(1) *For sequences $0 < r_n \to 0$ and $t_n > 0$ there is some n such that there are $u \in K$ and $t > t_n$ such that $\|u - u_0\| = r_n$ and $\Phi(t)(u)$ is undefined or satisfies $\|\Phi(t)(u) - u_0\| > r_n$.*

(2) *There is a sequence $u_n \in K$ with $0 < \|u_n - u_0\| \to 0$ such that u_n is the initial value of a periodic (non-stationary) solution. In both of these cases, u_0 fails to be asymptotically stable in the topology of V.*

R e m a r k 3. We do not claim in Theorem 2 that u_0 fails to be (Ljapunov) stable. However, Theorem 2 implies that u_0 lacks the following stability property.

We call u_0 *spherically stable* if u_0 has a neighborhood (in V) which is free of (initial values of) non-stationary periodic solutions, and if for every $\varepsilon > 0$ there is $t_\varepsilon > 0$ such that every $u \in K$ with $\|u - u_0\| = \varepsilon$ satisfies $\|\Phi(t)(u) - u_0\| \leqslant \varepsilon$ for all $t > t_\varepsilon$.

The latter is neither stronger nor weaker than the classical notion of (Ljapunov) stability: It is stronger in the sense that the same $\varepsilon > 0$ occurs twice, but it is weaker in the sense that there is no requirement for $t \leqslant t_\varepsilon$, i.e., the trajectories may go arbitrarily far away from u_0 if they only return close enough to u_0 in time.

P r o o f. We replace first $(\cdot, \cdot)|_{V' \times V}$ by

$$(u, v)^* := \langle \mathfrak{A}^{-1} u, v \rangle \quad \text{for all } u \in V' \text{ and } v \in V.$$

The associated functional norm $\|\cdot\|^*$ in V' satisfies

$$\|u\|^* := \sup_{\|v\| \leqslant 1} |(u, v)^*| = \|\mathfrak{A}^{-1} u\|$$

and thus is equivalent to $\|\cdot\|_{V'}$ since $\mathfrak{A}\colon V \to V'$ is an isomorphism. For $u \in H$, we have by (3.1) that

(3.8)
$$(u, v)^* \leqslant c_0 |u| |v| \quad \text{for all } v \in V,$$

and so $(u, \cdot)^*$ is Lipschitz on $V \subseteq H$ with respect to $|\cdot|$ with Lipschitz constant $c_0 |u|$ and thus has a unique continuous extension to H with the same Lipschitz constant, which we denote again by $(u, \cdot)^*$. It follows that such extended bilinear form $(\cdot, \cdot)^*$ is continuous on $H \times H$. From (3.3) we obtain that $(\cdot, \cdot)^*$ is symmetric on $V \times V$ and thus by continuity and density also on $H \times H$. By (3.2), we have

$$c_1 (u, u)^* = c_1 \langle \mathfrak{A}^{-1} u, u \rangle \geqslant (u, u)$$

for all $u \in V$, and from density and continuity, we obtain that $(\cdot, \cdot)^*$ is positive definite and thus an inner product on H inducing a norm $|\cdot|^*$ satisfying $\sqrt{c_1} |u|^* \geqslant |u|$. In view of (3.8), we conclude that the two scalar products on H are actually equivalent. Note now that none of the assertions or hypotheses of the theorem changes when we replace $(\cdot, \cdot)$ by $(\cdot, \cdot)^*$: Indeed, (3.4) implies

$$((u, \varphi - v) \geqslant 0 \quad \text{for all } \varphi \in K) \iff ((u, \varphi - v)^* \geqslant 0 \quad \text{for all } \varphi \in K)$$

also for all $u \in H$ by the density of V and by continuity of the scalar product. Hence, even (3.5) does not change when we replace $(\cdot, \cdot)$ by $(\cdot, \cdot)^*$. Thus, we can assume without loss of generality that $(\cdot, \cdot) = (\cdot, \cdot)^*$. In this case, $(\mathfrak{A}u, v) = (\mathfrak{A}u, v)^* = \langle u, v \rangle$, that is, the operator $\mathfrak{A}$ has the particular form required in [4], and so the assertion follows from the main theorem of [4] and the remarks given there. $\qquad\square$

Theorem 2 becomes in case $(u, v) = \langle A^{-1}u, v\rangle$ exactly a reformulation of [4]. However, the above formulation is more handy since it allows us to put parameters into the operator $\mathfrak{A}$ instead of the scalar products which caused technical inconveniences in [4]. The apparently restrictive hypothesis (3.4) is actually trivially satisfied for convex sets of the form $K = K_1 \times K_2$ in the setting which we describe now.

4. Proof of Theorem 1

The first step consists of rewriting (2.11) equivalently in the form (3.5) by defining H, V, $\mathfrak{A}$, and F appropriately.

It will simplify our considerations to equip $\mathbb{V}$ with the scalar product

$$\langle u, \varphi \rangle := \int_\Omega \nabla u \cdot \nabla \varphi \, dx \quad \text{for all} \ \ u, \varphi \in \mathbb{V}$$

which in view of (2.15) generates the topology inherited from $W^{1,2}(\Omega)$, see e.g. [10]. We denote the usual scalar product in $L_2(\Omega)$ by $(\cdot, \cdot)$. Our aim is to apply Theorem 2 in the Hilbert spaces $V = \mathbb{V} \times \mathbb{V}$ and $U = L_2(\Omega) \times L_2(\Omega)$ with the sum scalar products, that is (using in a slight misuse of notation the symbols for the scalar product in a duplicate meaning), we put

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle := \langle u, \varphi \rangle + \langle v, \psi \rangle = \int_\Omega (\nabla u \cdot \nabla \varphi + \nabla v \cdot \nabla \psi) \, dx,$$

$$\left(\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right) := (u, \varphi) + (v, \psi) = \int_\Omega (u\varphi + v\psi) \, dx.$$

We define a linear $A_0 \colon L_2(\Omega) \to \mathbb{V}$ by the duality

$$\langle A_0 u, \varphi \rangle = (u, \varphi) = \int_\Omega u\varphi \, dx \quad \text{for all} \ \ u \in L_2(\Omega), \ \varphi \in \mathbb{V}.$$

For fixed $d_1, d_2 > 0$, we define $\mathfrak{A} \colon V \to V'$ by

$$\left(\mathfrak{A} \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right) = \int_\Omega (d_1 \nabla u \cdot \nabla \varphi + d_2 \nabla v \cdot \nabla \psi) \, dx.$$

Since the right-hand side is (up to an equivalence) the scalar product in V, it is clear that $\mathfrak{A} \colon V \to V'$ is an isomorphism. Letting $A := \mathfrak{A}|_{\mathfrak{A}^{-1}(H)}$ we have $A^{-1} = \mathfrak{A}^{-1}|_H$,

$$(4.1) \qquad\qquad A^{-1} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} d_1^{-1} A_0 u \\ d_2^{-1} A_0 v \end{pmatrix} \quad \text{for all} \ \begin{pmatrix} u \\ v \end{pmatrix} \in H,$$

or in other words

$$\left\langle A^{-1} \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle = d_1^{-1}(u, \varphi) + d_2^{-1}(v, \psi) \quad \text{for all } \begin{pmatrix} u \\ v \end{pmatrix} \in H, \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in V.$$

From this representation it follows immediately that (3.1), (3.2), and (3.3) are satisfied for every $d_1, d_2 > 0$.

Moreover, putting $K := K_1 \times K_2$, we have (3.4). Indeed, with the notation $u = (u_1, u_2)$, $v = (v_1, v_2)$ the particular choices $\varphi \in K_1 \times \{v_2\}$ and $\varphi \in \{v_1\} \times K_2$ show that actually both statements in (3.4) are equivalent to

$$(u_i, \varphi_i - v_i) \geqslant 0 \quad \text{for all } \varphi_i \in K_i \text{ and } i = 1, 2.$$

For $i = 1, 2$, we define $F_i, G_i \colon V \to L_2(\Omega)$ as the superposition operators

$$F_i(u, v)(x) = f_i(u(x), v(x)), \quad G_i(u, v)(x) = g_i(u(x), v(x)).$$

In view of Sobolev's embedding theorems it follows straightforwardly from our hypothesis (2.4) that these operators are indeed well-defined and satisfy a Lipschitz condition on some neighborhood $\mathscr{U} \subseteq V$ of $u_0 := 0$; see e.g. [4] for an analogous calculation. Hence, putting $F := (F_1, F_2) \colon \mathscr{U} \to H$, we obtain that the Lipschitz condition in (3.7) is satisfied, and a calculation analogous to that given in [4] shows that also the Hölder condition in (3.7) is satisfied if $\mathscr{U} \subseteq V$ is bounded (and small enough in case $N = 1$).

Now we prove the assertions of Theorem 1: The fact that the set D of non-bifurcation points is open, follows immediately from the definition. In [2], Lemma 3.4, it has been proved that for every $d_0 \in \mathbb{R}_+^2 \cap \partial D_S$ satisfying (2.12) or (2.13), there is $r > 0$ such that the open ball $B_r(d_0) \subseteq \mathbb{R}_+^2$ has the property that $B_r(d_0) \cap D_S$ is free of so-called critical points, in particular, it consists only of non-bifurcation points (independent of the particular higher-order term g_i). Observe that (4.1) implies

$$(A^{-1} \circ F) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} d_1^{-1} b_{11} A_0 u + d_1^{-1} b_{12} A_0 v \\ d_2^{-1} b_{21} A_0 u + d_2^{-1} b_{22} A_0 v \end{pmatrix} + \begin{pmatrix} d_1^{-1} A_0 G_1(u, v) \\ d_2^{-1} A_0 G_2(u, v) \end{pmatrix}.$$

This is exactly the operator which occurs in [2] (the second summand is called F in [2]), and [2], Theorem 3.1, thus implies that

(4.2) $$\operatorname{ind}(P_K \circ A^{-1} \circ F, 0) = 0$$

for $(d_1, d_2) \in D_S \cap B_r(d_0)$. Note that the homotopy invariance of the degree implies that the local fixed point index $\operatorname{ind}(P_K \circ A^{-1} \circ F, 0)$ is a locally constant function of $(d_1, d_2) \in D$. Hence, this index is constant on the connected components of D, and so (4.2) holds for all $(d_1, d_2) \in D_S$. Now the assertion about the instability follows from Theorem 2. $\qquad \square$

We consider almost the same setting as in Section 4, but in order to include $\Gamma_D = \emptyset$, we have to equip $\mathbb{V}$ with the usual scalar product

$$\langle u, v \rangle := (u, v) + (\nabla u, \nabla v).$$

We use the same formulas as in Section 4 to define A_0, $\mathfrak{A}$, A, F, and G, observing that the definitions of $\mathfrak{A}$, A, F, and G do not involve the scalar product of $\mathbb{V}$ at all. In particular, $F \colon V \to H$ is locally Lipschitz near 0 by the same arguments as before. Note that (2.4) implies for $N \geqslant 2$ that

(5.1) $$|f_i(u, v)| \leqslant C(1 + |u| + |v|)^{\alpha_N + 1}.$$

By the Sobolev embedding theorems (see e.g. [10]), the space $W^{1,2}(\Omega)$ is continuously embedded into $C(\Omega)$ (in case $N = 1$) or into $L_q(\Omega)$ with $q/2 = \alpha_N + 1$ (in case $N \geqslant 2$). (Indeed, in case $N \geqslant 3$ the critical Sobolev exponent is $q := 2N/(N-2) = 2(\alpha_N + 1)$.) Thus, we obtain from (2.3) and (5.1) (using e.g. [9], Theorem 4.16, in case $N \geqslant 2$) that

$$\lim_{\|(u,v)\| \to 0} \frac{|G(u, v)|}{\|(u, v)\|} = 0.$$

Hence, everything is prepared to apply [3], Theorem 5.1.1, in the space $X := H$ with our operator $A \colon D(A) \to X$, the corresponding fractional power space $X^{1/2} = V$, and with our nonlinearity F whose linearization is

$$B_0(u, v) := F(u, v) - G(u, v) = \begin{pmatrix} b_{11} u & b_{12} v \\ b_{21} u & b_{22} v \end{pmatrix}$$

(in [3], Theorem 5.1.1, this map is called B). It remains to show that for $(d_1, d_2) \in D_S$ the real parts of the spectrum of $A - B_0$ are bounded by some $\gamma < 0$, that is, that for all $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geqslant \gamma$ the operator $C_\lambda := A - B_0 - \lambda \operatorname{id}$ has a bounded inverse in H. Putting $A_1 := A_0^{-1} - \operatorname{id} \colon \mathbb{V} \to L_2(\Omega)$, we can rewrite $C_\lambda(u, v) = (f, g)$ as the system

(5.2) $$d_1 A_1 u - (b_{11} - \lambda)u + b_{12} v = f,$$
$$d_2 A_1 v - b_{21} u - (b_{22} - \lambda)v = g.$$

Note that A_1 is a self-adjoint operator in $L_2(\Omega)$ with spectrum $\sigma(A_1)$ consisting exactly of the values $\kappa_n > 0$ $(n = 1, 2, \ldots)$ and in case $\operatorname{mes} \Gamma_D = 0$ also of $\kappa_0 := 0$. For $\lambda \notin (-\infty, b_{22}]$, we can thus calculate C_λ^{-1} "explicitly" by first solving the second

equation in (5.2) for v and then inserting into the first equation. We do this in terms of spectral calculus: Defining $h\colon \sigma(A_1) \to \mathbb{C}$ by

$$h(\kappa) := d_1\kappa - (b_{11} - \lambda) - b_{12}\frac{-b_{21}}{d_2\kappa - (b_{22} - \lambda)},$$

we have to show that $h(A_1)$ has a bounded inverse (for $\operatorname{Re}\lambda \geqslant \gamma$), that is, by the spectral mapping theorem, that $0 \notin \overline{h(\sigma(A))} = h(\sigma(A))$ (the closure can be omitted since $|h(\kappa_n)| \to \infty$ as $n \to \infty$.)

Let Λ denote the set of all $\lambda \in \mathbb{C} \setminus (-\infty, b_{22}]$ with $0 \in h(\sigma(A))$. Multiplying by the nominator (for which we already verified that it is nonzero), we find that $\lambda \in \Lambda$ implies that there is some $n = 0, 1, \dots$ with

$$(5.3) \qquad (d_1\kappa_n - b_{11} + \lambda)(d_2\kappa_n - b_{22} + \lambda) = b_{12}b_{21}.$$

For each n, we thus have at most two values $\lambda_{n,1}, \lambda_{n,2}$ in Λ which are exactly the eigenvalues of the matrix

$$B + \kappa_n \begin{pmatrix} -d_1 & 0 \\ 0 & -d_2 \end{pmatrix}.$$

Dividing this matrix by $\kappa_n \to \infty$, we can assume that (after an appropriate ordering) $\kappa_n^{-1}\lambda_{n,j} \to -d_j < 0$ so that the set of all real parts of elements of Λ has a maximum γ. We fix some n with $\max\limits_{j=1,2} \operatorname{Re}\lambda_{n,j} = \gamma$. Using the shortcuts $\alpha_n := d_1\kappa_n - b_{11}$ and $\beta_n := d_2\kappa_n - b_{22}$, we find by Vieta's theorem from (5.3) that

$$(5.4) \qquad \lambda_{n,1} + \lambda_{n,2} = -(\alpha_n + \beta_n) \leqslant b_{11} + b_{22} < 0,$$
$$\lambda_{n,1}\lambda_{n,2} = \alpha_n\beta_n - b_{12}b_{21} > 0,$$

where the last inequality follows in case $n = 0$ from (2.1) and in case $n \geqslant 1$ from the assumption $(d_1, d_2) \in D_S$. Note that $\lambda_{n,1}, \lambda_{n,2}$ are either both real or conjugate complex. In both cases, we obtain from (5.4) that $\operatorname{Re}\lambda_{n,j} < 0$, and so $\gamma < 0$. Hence, the real parts of the elements of Λ are all bounded from above by $\gamma < 0$. By construction, $\Lambda \cup (-\infty, b_{22}]$ contains the spectrum of $A - B_0$, and so we are done. $\qquad \square$

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