ON THE EIGENVALUES OF A ROBIN PROBLEM WITH A LARGE PARAMETER

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Abstract. We consider the Robin eigenvalue problem $\Delta u+\lambda u=0$ in Ω, $\partial u/\partial\nu+\alpha u=0$ on $\partial\Omega$ where $\Omega\subset\mathbb{R}^n$, $n\geqslant 2$ is a bounded domain and α is a real parameter. We investigate the behavior of the eigenvalues $\lambda_k(\alpha)$ of this problem as functions of the parameter α. We analyze the monotonicity and convexity properties of the eigenvalues and give a variational proof of the formula for the derivative $\lambda_1'(\alpha)$. Assuming that the boundary $\partial\Omega$ is of class C^2 we obtain estimates to the difference $\lambda_k^D-\lambda_k(\alpha)$ between the k-th eigenvalue of the Laplace operator with Dirichlet boundary condition in Ω and the corresponding Robin eigenvalue for positive values of α for every $k=1,2,\ldots$.

Keywords: Laplace operator; Robin boundary condition; eigenvalue; large parameter

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1. Introduction

Let us consider the eigenvalue problem

$$\Delta u+\lambda u=0 \quad \text{in } \Omega, \tag{1}$$

$$\frac{\partial u}{\partial\nu}+\alpha u=0 \quad \text{on } \Gamma, \tag{2}$$

where $\Omega\subset\mathbb{R}^n$, $n\geqslant 2$ is a bounded domain with C^2 class boundary surface $\Gamma=\partial\Omega$. By ν we mean the outward unit normal vector to Γ, α is a real parameter.

The problem (1), (2) is usually referred to as the Robin problem for $\alpha>0$ (see [6], Chapter 7, Paragraph 7.2) and as the generalized Robin problem for all α ([5]).

We have the sequence of eigenvalues $\lambda_1(\alpha)<\lambda_2(\alpha)\leqslant\ldots\to\infty$ enumerated according to their multiplicities where $\lambda_1(\alpha)$ is simple with a positive eigenfunction.

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By the variational principle ([11], Chapter 4, Paragraph 1, no. 4) we have

$$(3)\qquad \lambda_k(\alpha)=\sup_{v_1,\dots,v_{k-1}\in L_2(\Omega)}\ \inf_{\substack{v\in H^1(\Omega)\\ (v,v_j)_{L_2(\Omega)}=0\\ j=1,\dots,k-1}}\frac{\int_\Omega|\nabla v|^2\,\mathrm{d}x+\alpha\int_\Gamma v^2\,\mathrm{d}s}{\int_\Omega v^2\,\mathrm{d}x},\qquad k=1,2,\dots.$$

Let $0<\lambda_1^D<\lambda_2^D\leqslant\dots\to\infty$ be the sequence of eigenvalues of the Dirichlet eigenvalue problem

$$(4)\qquad \Delta u+\lambda u=0\quad\text{in }\Omega,$$

$$(5)\qquad u=0\quad\text{on }\Gamma.$$

Also, by the variational principle we have

$$(6)\qquad \lambda_k^D=\sup_{v_1,\dots,v_{k-1}\in L_2(\Omega)}\ \inf_{\substack{v\in \mathring{H}^1(\Omega)\\ (v,v_j)_{L_2(\Omega)}=0\\ j=1,\dots,k-1}}\frac{\int_\Omega|\nabla v|^2\,\mathrm{d}x}{\int_\Omega v^2\,\mathrm{d}x},\qquad k=1,2,\dots.$$

It is easy to show the inequality $\lambda_1(\alpha)\leqslant\lambda_1^D$ which gives an upper bound of $\lambda_1(\alpha)$ for all values of α. It was noticed in ([2], Chapter 6, Paragraph 2, No. 1) that for $n=2$ and smooth boundary $\lim_{\alpha\to\infty}\lambda_1(\alpha)=\lambda_1^D$. Later in [12] for $n=2$ the two-side estimates

$$\lambda_1^D\Big(1+\frac{\lambda_1^D}{\alpha q_1}\Big)^{-1}\leqslant\lambda_1(\alpha)\leqslant\lambda_1^D\Big(1+\frac{4\pi}{\alpha|\Gamma|}\Big)^{-1},\qquad \alpha>0,$$

were obtained where q_1 is the first eigenvalue of the Steklov problem

$$\Delta^2u=0\quad\text{in }\Omega,$$

$$u=0,\quad \Delta u-q\frac{\partial u}{\partial\nu}=0\quad\text{on }\Gamma.$$

In [4] for any $n\geqslant 2$ we established the asymptotic expansion

$$\lambda_1(\alpha)=\lambda_1^D-\frac{\int_\Gamma\big(\partial u_1^D/\partial\nu\big)^2\,\mathrm{d}s}{\int_\Omega(u_1^D)^2\,\mathrm{d}x}\alpha^{-1}+o(\alpha^{-1}),\qquad \alpha\to\infty,$$

where u_1^D is the first eigenfunction of the Dirichlet problem (4), (5).

The case $\alpha<0$ has received attention in the last years after [9]. It was shown in [9] that for piecewise-C^1 boundary $\liminf_{\alpha\to-\infty}\lambda_1(\alpha)/-\alpha^2\geqslant 1$. Later for C^1-class boundaries it was proved ([10], [5]) that $\lim_{\alpha\to-\infty}\lambda_1(\alpha)/-\alpha^2=1$. Here the condition of C^1-class is optimal, in [9] plane triangle domains were prepared for which $\lim_{\alpha\to-\infty}\lambda_1(\alpha)/-\alpha^2>1$. In [3] authors proved that for C^1 boundaries $\lim_{\alpha\to-\infty}\lambda_k(\alpha)/-\alpha^2=1$ for all $k=1,2,\dots$.

2. Main results

Theorem 1. *The eigenvalues $\lambda_k(\alpha)$ have the following properties:*

(i) $\lambda_k(\alpha_1) \leqslant \lambda_k(\alpha_2) \leqslant \lambda_k^D$ *for* $\alpha_1 < \alpha_2$, $k = 1, 2, \ldots$;

(ii) $\lambda_1(\alpha)$ *is differentiable and*

$$\lambda_1'(\alpha) = \frac{\int_\Gamma u_{1,\alpha}^2 \, \mathrm{d}s}{\int_\Omega u_{1,\alpha}^2 \, \mathrm{d}x} > 0, \tag{7}$$

where $u_{1,\alpha}(x)$ *is the corresponding eigenfunction;*

(iii) $\lambda_1(\alpha)$ *is a concave function of* α*:*

$$\lambda_1(\beta\alpha_1 + (1-\beta)\alpha_2) \geqslant \beta\lambda_1(\alpha_1) + (1-\beta)\lambda_1(\alpha_2), \quad 0 < \beta < 1. \tag{8}$$

Theorem 1 establishes some known properties of eigenvalues of the problem (1), and (2) (see [2], Chapter 6 for (i) and [9], [1] for (ii) and (iii) (in [1] planar domains with piecewise analytic boundaries were considered)).

Hence the behavior of eigenvalues can be illustrated by Figure 1:

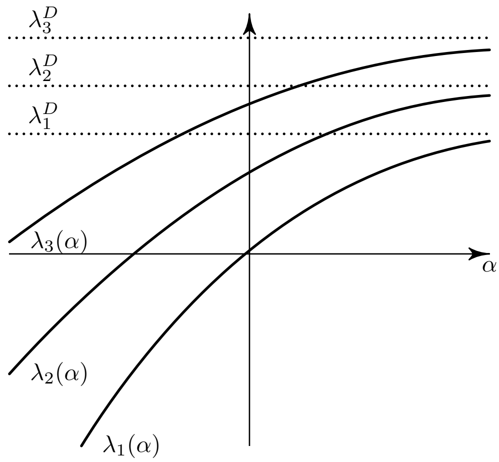

Figure 1.

Theorem 2. *The eigenvalues* $\lambda_k(\alpha)$, $k = 1, 2, \ldots$, *satisfy the estimates*

$$0 \leqslant \lambda_k^D - \lambda_k(\alpha) \leqslant C_1\alpha^{-1/2}(\lambda_k^D)^2, \quad \alpha > 0, \tag{9}$$

where the constant C_1 *does not depend on* k.

Proof of Theorem 1. The increasing of $\lambda_k(\alpha)$ follows from (3). Using (6) and the inclusion $\mathring{H}^1(\Omega) \subset H^1(\Omega)$, we have

$$\begin{aligned}\lambda_k(\alpha) &= \sup_{v_1,\dots,v_{k-1}\in L_2(\Omega)} \inf_{\substack{v\in H^1(\Omega)\\ (v,v_j)_{L_2(\Omega)}=0\\ j=1,\dots,k-1}} \frac{\int_\Omega |\nabla v|^2\,\mathrm{d}x + \alpha\int_\Gamma v^2\,\mathrm{d}s}{\int_\Omega v^2\,\mathrm{d}x}\\ &\leqslant \sup_{v_1,\dots,v_{k-1}\in L_2(\Omega)} \inf_{\substack{v\in \mathring{H}^1(\Omega)\\ (v,v_j)_{L_2(\Omega)}=0\\ j=1,\dots,k-1}} \frac{\int_\Omega |\nabla v|^2\,\mathrm{d}x + \alpha\int_\Gamma v^2\,\mathrm{d}s}{\int_\Omega v^2\mathrm{d}x}\\ &= \sup_{v_1,\dots,v_{k-1}\in L_2(\Omega)} \inf_{\substack{v\in \mathring{H}^1(\Omega)\\ (v,v_j)_{L_2(\Omega)}=0\\ j=1,\dots,k-1}} \frac{\int_\Omega |\nabla v|^2\,\mathrm{d}x}{\int_\Omega v^2\,\mathrm{d}x} = \lambda_k^D.\end{aligned}$$

To obtain (7) we use the inequalities

$$\begin{aligned}\lambda_1(\alpha_1)-\lambda_1(\alpha) &= \lambda_1(\alpha_1) - \inf_{v\in H^1(\Omega)} \frac{\int_\Omega |\nabla v|^2\,\mathrm{d}x + \alpha\int_\Gamma v^2\,\mathrm{d}s}{\int_\Omega v^2\,\mathrm{d}x}\\ &\geqslant \lambda_1(\alpha_1) - \frac{\int_\Omega |\nabla u_{1,\alpha_1}|^2\,\mathrm{d}x + \alpha\int_\Gamma u_{1,\alpha_1}^2\,\mathrm{d}s}{\int_\Omega u_{1,\alpha_1}^2\,\mathrm{d}x}\\ &= (\alpha_1-\alpha)\frac{\int_\Gamma u_{1,\alpha_1}^2\,\mathrm{d}s}{\int_\Omega u_{1,\alpha_1}^2\,\mathrm{d}x},\\ \lambda_1(\alpha_1)-\lambda_1(\alpha) &= \inf_{v\in H^1(\Omega)} \frac{\int_\Omega |\nabla v|^2\,\mathrm{d}x + \alpha_1\int_\Gamma v^2\,\mathrm{d}s}{\int_\Omega v^2\,\mathrm{d}x} - \lambda_1(\alpha)\\ &\leqslant \frac{\int_\Omega |\nabla u_{1,\alpha}|^2\,\mathrm{d}x + \alpha_1\int_\Gamma u_{1,\alpha}^2\,\mathrm{d}s}{\int_\Omega u_{1,\alpha}^2\,\mathrm{d}x} - \lambda_1(\alpha)\\ &= (\alpha_1-\alpha)\frac{\int_\Gamma u_{1,\alpha}^2\,\mathrm{d}s}{\int_\Omega u_{1,\alpha}^2\,\mathrm{d}x}.\end{aligned}$$

Therefore

$$\frac{\int_\Gamma u_{1,\alpha_1}^2\,\mathrm{d}s}{\int_\Omega u_{1,\alpha_1}^2\,\mathrm{d}x} \leqslant \frac{\lambda_1(\alpha_1)-\lambda_1(\alpha)}{\alpha_1-\alpha} \leqslant \frac{\int_\Gamma u_{1,\alpha}^2\,\mathrm{d}s}{\int_\Omega u_{1,\alpha}^2\,\mathrm{d}x}. \tag{10}$$

Considering the problem (1), (2) in the space $H^1(\Omega)$ we search the values of λ for which there exists a nonzero function $u \in H^1(\Omega)$ satisfying the integral identity

$$\int_\Omega (\nabla u, \nabla v)\,\mathrm{d}x + \alpha\int_\Gamma uv\,\mathrm{d}s = \lambda\int_\Omega uv\,\mathrm{d}x \tag{11}$$

for any $v \in H^1(\Omega)$. The relation (11) can be rewritten as

$$\int_\Omega ((\nabla u, \nabla v) + Muv)\,\mathrm{d}x + \alpha \int_\Gamma uv\,\mathrm{d}s = (\lambda + M)\int_\Omega uv\,\mathrm{d}x \tag{12}$$

with an arbitrary $M > 0$. Let us define an equivalent scalar product in the space $H^1(\Omega)$ by the formula

$$[u,v]_M = \int_\Omega ((\nabla u, \nabla v) + Muv)\,\mathrm{d}x, \quad \|u\|_M^2 = [u,u]_M.$$

Now (12) transforms to

$$[u,v]_M + \alpha[Tu,v]_M = (\lambda + M)[Bu,v]_M,$$

where self-adjoint nonnegative operators $T\colon H^1(\Omega) \to H^1(\Omega)$ and $B\colon H^1(\Omega) \to H^1(\Omega)$ were determined by bilinear forms

$$[Tu,v]_M = \int_\Gamma uv\,\mathrm{d}s, \quad [Bu,v]_M = \int_\Omega uv\,\mathrm{d}x, \quad u,v \in H^1(\Omega). \tag{13}$$

So we have the following equation in the space $H^1(\Omega)$ with the norm $\|\cdot\|_M$:

$$(I + \alpha T)u = (\lambda + M)Bu. \tag{14}$$

Now we use the inequality ([11], Chapter 3, Paragraph 5, Formula 19)

$$\|v\|^2_{L_2(\Gamma)} \leqslant \varepsilon \|\nabla v\|^2_{L_2(\Omega)} + C_\varepsilon \|v\|^2_{L_2(\Omega)}, \tag{15}$$

valid for $v(x) \in H^1(\Omega)$ with an arbitrary $\varepsilon > 0$. Using (13), (15), we obtain

$$\begin{aligned} \|Tu\|_M^2 = [Tu,Tu]_M &= \int_\Gamma uTu\,\mathrm{d}s \leqslant \|u\|_{L_2(\Gamma)}\|Tu\|_{L_2(\Gamma)} \\ &\leqslant \varepsilon\biggl(\int_\Omega \Bigl(|\nabla Tu|^2 + \frac{C_\varepsilon}{\varepsilon}(Tu)^2\Bigr)\,\mathrm{d}x\biggr)^{1/2}\biggl(\int_\Omega \Bigl(|\nabla u|^2 + \frac{C_\varepsilon}{\varepsilon}u^2\Bigr)\,\mathrm{d}x\biggr)^{1/2} \\ &\leqslant \varepsilon \|Tu\|_M \|u\|_M, \end{aligned} \tag{16}$$

where $\varepsilon > 0$, $M = M_\varepsilon = C_\varepsilon/\varepsilon$. It follows from (16) that

$$\|Tu\|_{M_\varepsilon} \leqslant \varepsilon \|u\|_{M_\varepsilon},$$

so for any $\varepsilon > 0$ we have $\|\alpha T\|_{H^1(\Omega)\to H^1(\Omega)} < 1$ for $|\alpha| < 1/\varepsilon$. Hence, the inverse operator $(I + \alpha T)^{-1}$ is bounded and $\|(I + \alpha T)^{-1}\| \leqslant (1 - |\alpha|\|T\|)^{-1}$. Therefore the equation (14) is equivalent to

$$(I - (\lambda + M)(I + \alpha T)^{-1}B)u = 0.$$

The operator B is compact ([11], Chapter 3, Paragraph 4, Theorem 3) and the operator $(I+\alpha T)^{-1}B\colon H^1(\Omega)\to H^1(\Omega)$ is compact too. So the spectrum of the problem (14) consists of eigenvalues $\lambda_j(\alpha)\in\mathbb{R}$, $j=1,2,\dots$, of finite multiplicity with the only limit point at infinity. By (13), (14) we obtain the inequality

$$\lambda_j(\alpha)\geqslant -M_\varepsilon+(1-|\alpha|\|T\|)\Big(\frac{\|u_{j,\alpha}\|_{M_\varepsilon}}{\|u_{j,\alpha}\|_{L_2(\Omega)}}\Big)^2\geqslant -M_\varepsilon$$

where $u_{j,\alpha}$ is the corresponding eigenfunction. Therefore $\lambda_j(\alpha)\to\infty$, $j\to\infty$.

The eigenvalue λ_1 is simple. So the self-adjoint operator $(I+\alpha T)^{-1}B$ satisfies the conditions of the asymptotic perturbation theory ([7], Chapter 8, Paragraph 2, Theorem 2.6). It means that the eigenfunction $u_{1,\alpha}$ depends continuously on α in the space $H^1(\Omega)$. By ([11], Chapter 3, Paragraph 5, Theorem 4) the trace of $u_{1,\alpha}$ on Γ depends continuously on α in the space $L_2(\Gamma)$. Now it follows from (10) that

$$\lambda_1'(\alpha)=\lim_{\alpha_1\to\alpha}\frac{\lambda_1(\alpha_1)-\lambda_1(\alpha)}{\alpha_1-\alpha}=\frac{\int_\Gamma u_{1,\alpha}^2\,\mathrm{d}s}{\int_\Omega u_{1,\alpha}^2\,\mathrm{d}x}.$$

By ([11], Chapter 4, Paragraph 2, Theorem 4) $u_{1,\alpha}\in H^2(\Omega)$ and satisfies equation (1) almost everywhere and the boundary condition in the sense of trace (the so-called strong solution). In the case $\int_\Gamma u_{1,\alpha}^2\,\mathrm{d}s=0$ we have by (2)

$$u_{1,\alpha}=\frac{\partial u_{1,\alpha}}{\partial\nu}=0\quad\text{on }\Gamma.$$

Applying the uniqueness theorem for the Cauchy problem for second-order elliptic equations ([8], Chapter 1, Paragraph 3), we get $u_{1,\alpha}=0$ in Ω. So, $\int_\Gamma u_{1,\alpha}^2\,\mathrm{d}s>0$ and we proved the inequality $\lambda_1'(\alpha)>0$.

Taking into account (7), for $\alpha_2>\alpha_1$ we have $\lambda_1(\alpha_2)>\lambda_1(\alpha_1)$ and $\lambda_1(\alpha)<\lambda_1^D$ for all α.

To prove the concavity of $\lambda_1(\alpha)$ consider the inequality

$$\begin{aligned}\lambda_1(\beta\alpha_1+(1-\beta)\alpha_2)&=\inf_{v\in H^1(\Omega)}\frac{\int_\Omega|\nabla v|^2\,\mathrm{d}x+(\beta\alpha_1+(1-\beta)\alpha_2)\int_\Gamma v^2\,\mathrm{d}s}{\int_\Omega v^2\,\mathrm{d}x}\\&\geqslant\beta\inf_{v\in H^1(\Omega)}\frac{\int_\Omega|\nabla v|^2\,\mathrm{d}x+\alpha_1\int_\Gamma v^2\,\mathrm{d}s}{\int_\Omega v^2\,\mathrm{d}x}\\&\quad+(1-\beta)\inf_{v\in H^1(\Omega)}\frac{\int_\Omega|\nabla v|^2\,\mathrm{d}x+\alpha_2\int_\Gamma v^2\,\mathrm{d}s}{\int_\Omega v^2\,\mathrm{d}x}\\&=\beta\lambda_1(\alpha_1)+(1-\beta)\lambda_1(\alpha_2),\quad 0<\beta<1.\end{aligned}$$

This completes the proof of Theorem 1. □

4. Operator approach

The proof of Theorem 2 is based on an estimate with respect to the parameter α of the norm of a certain operator acting in the $L_2(\Omega)$ space. This operator is a difference between operators associated with the Robin and Dirichlet problems. Now, using compactness and positivity of these operators we can apply estimates to eigenvalues by the norm of a difference operator (Theorem 3 below).

Let us consider the boundary value problem

$$-\Delta u + u = h \quad \text{in } \Omega, \tag{17}$$

$$\frac{\partial u}{\partial \nu} + \alpha u = 0 \quad \text{on } \Gamma,\ \alpha > 0. \tag{18}$$

For $h(x) \in L_2(\Omega)$ a weak solution $u(x) \in H^1(\Omega)$ of the problem (17), (18) satisfying the integral identity

$$\int_\Omega ((\nabla u, \nabla v) + uv)\,\mathrm{d}x + \alpha \int_\Gamma uv\,\mathrm{d}s = \int_\Omega hv\,\mathrm{d}x \tag{19}$$

for all $v \in H^1(\Omega)$. By definition, introduce a scalar product in the space $H^1(\Omega)$

$$(u, v)_{H^1(\Omega),\alpha} = \int_\Omega ((\nabla u, \nabla v) + uv)\,\mathrm{d}x + \alpha \int_\Gamma uv\,\mathrm{d}s \tag{20}$$

and the corresponding norm

$$\|u\|^2_{H^1(\Omega),\alpha} = (u, u)_{H^1(\Omega),\alpha}.$$

Using (19), (20), we obtain the relation

$$(u, v)_{H^1(\Omega),\alpha} = (h, v)_{L_2(\Omega)}. \tag{21}$$

Hence, consider a linear functional $l_h(v) = (h, v)_{L_2(\Omega)}$ in the $H^1(\Omega)$ space. The functional $l_h(v)$ is bounded: $|l_h(v)| \leqslant \|h\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)}$. Now, by the Riesz lemma there exists a unique function $u \in H^1(\Omega)$ satisfying the integral identity (19). Applying (21) with $v = u$, we obtain $\|u\|^2_{H^1(\Omega),\alpha} \leqslant \|h\|_{L_2(\Omega)}\|u\|_{H^1(\Omega),\alpha}$. Therefore,

$$\|u\|_{L_2(\Omega)} \leqslant \|u\|_{H^1(\Omega),\alpha} \leqslant \|h\|_{L_2(\Omega)}, \tag{22}$$

and we can define a bounded linear operator $A_\alpha\colon\ L_2(\Omega) \to L_2(\Omega)$ such that $u = A_\alpha h$ and $\|A_\alpha\| \leqslant 1$. Moreover, the space $H^1(\Omega)$ in a bounded domain Ω with C^2-class boundary embeds compactly into the space $L_2(\Omega)$ ([6], Theorem 1.1.1). It means that the operator A_α is compact. Note that

$$
(23) \qquad (h, A_\alpha g)_{L_2(\Omega)} = \int_\Omega h A_\alpha g \,\mathrm{d}x = \int_\Omega hv \,\mathrm{d}x
$$
$$
= \int_\Omega ((\nabla u, \nabla v) + uv)\,\mathrm{d}x + \alpha \int_\Gamma uv \,\mathrm{d}s
$$
$$
= \int_\Omega ug \,\mathrm{d}x = (A_\alpha h, g)_{L_2(\Omega)}, \qquad f, g \in L_2(\Omega),
$$

with $u = A_\alpha h$, $v = A_\alpha g$, $u, v \in H^1(\Omega)$. The relation (23) means that A_α is a self-adjoint operator. Now, by the relation (23) we have

$$
(h, A_\alpha h)_{L_2(\Omega)} = \int_\Omega uh \,\mathrm{d}x = \int_\Omega (|\nabla u|^2 + u^2)\,\mathrm{d}x + \alpha \int_\Gamma u^2 \,\mathrm{d}s = \|u\|^2_{H^1(\Omega),\alpha} > 0, \ h \neq 0.
$$

Hence, the operator A_α is positive. Now, A_α is a self-adjoint positive compact operator in the Hilbert space $H = L_2(\Omega)$. By the well-known theorem ([6], Theorem 1.2.1), A_α has a sequence of eigenvalues $\{\mu_k(\alpha)\}$, $k = 1, 2, \ldots$ with finite multiplicities such that $0 < \mu_k(\alpha) \leqslant 1$, $\mu_k(\alpha) \searrow 0$, $k \to \infty$. Let us denote by $u_{k,\alpha} \in L_2(\Omega)$ the corresponding eigenfunction satisfying $A_\alpha u_{k,\alpha} = \mu_k(\alpha) u_{k,\alpha}$. Thus, $\mu_k(\alpha)(u_{k,\alpha}, v)_{H^1(\Omega),\alpha} = (u_{k,\alpha}, v)_{L_2(\Omega)}$ and

$$
\mu_k(\alpha)\Big(\int_\Omega ((\nabla u_{k,\alpha}, \nabla v) + u_{k,\alpha} v)\,\mathrm{d}x + \alpha \int_\Gamma u_{k,\alpha} v \,\mathrm{d}s\Big) = \int_\Omega u_{k,\alpha} v \,\mathrm{d}x.
$$

It is readily seen that $\mu_k(\alpha) = (\lambda_k(\alpha) + 1)^{-1}$. Let us note that for $\alpha > 0$ we have $\mu_k(\alpha) \leqslant (\lambda_1(\alpha) + 1)^{-1} < 1$, so $\|A_\alpha\| < 1$.

Furthermore, consider the Dirichlet problem

$$
(24) \qquad -\Delta u + u = h \quad \text{in } \Omega,
$$
$$
(25) \qquad u = 0 \quad \text{on } \Gamma.
$$

For $h \in L_2(\Omega)$ a weak solution $u(x) \in \mathring{H}^1(\Omega)$ of the problem (24), (25) satisfies the integral identity

$$
(26) \qquad \int_\Omega ((\nabla u, \nabla v) + uv)\,\mathrm{d}x = \int_\Omega hv \,\mathrm{d}x
$$

for all $v \in \mathring{H}^1(\Omega)$. By definition, introduce a scalar product in the space $\mathring{H}^1(\Omega)$

$$
(27) \qquad (u, v)_{\mathring{H}^1(\Omega)} = \int_\Omega ((\nabla u, \nabla v) + uv)\,\mathrm{d}x
$$

and the corresponding norm

$$\|u\|^2_{\mathring{H}^1(\Omega)} = (u,u)_{\mathring{H}^1(\Omega)}.$$

Using (26), (27), we obtain the relation

$$(u,v)_{\mathring{H}^1(\Omega)} = (h,v)_{L_2(\Omega)}. \tag{28}$$

Hence, consider a linear functional $l_h(v) = (h,v)_{L_2(\Omega)}$ in the $\mathring{H}^1(\Omega)$ space. The functional $l_h(v)$ is bounded: $|l_h(v)| \leqslant \|h\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)}$. Now, by the Riesz lemma there exists a unique function $u \in \mathring{H}^1(\Omega)$ satisfying the integral identity (26). Using (26) with $v = u$, we obtain $\|u\|^2_{\mathring{H}^1(\Omega)} \leqslant \|h\|_{L_2(\Omega)}\|u\|_{\mathring{H}^1(\Omega)}$. Therefore,

$$\|u\|_{L_2(\Omega)} \leqslant \|u\|_{\mathring{H}^1(\Omega)} \leqslant \|h\|_{L_2(\Omega)}, \tag{29}$$

and we can define the bounded linear operator $A^D\colon L_2(\Omega) \to L_2(\Omega)$ such that $u = A^D h$ and $\|A\| \leqslant 1$. Moreover, the space $\mathring{H}^1(\Omega)$ in the bounded domain Ω embeds compactly into the space $L_2(\Omega)$ ([6], Theorem 1.1.1) so the operator A^D is compact. Note that

$$\begin{aligned}(h, A^D g)_{L_2(\Omega)} &= \int_\Omega hA^D g\,\mathrm{d}x = \int_\Omega hv\,\mathrm{d}x = \int_\Omega ((\nabla u, \nabla v) + uv)\,\mathrm{d}x \\ &= \int_\Omega ug\,\mathrm{d}x = (A^D h, g)_{L_2(\Omega)}, \quad f, g \in L_2(\Omega),\end{aligned} \tag{30}$$

with $u = A^D h$, $v = A^D g$, $u, v \in \mathring{H}^1(\Omega)$. The relation (30) means that A^D is a self-adjoint operator. Now, by (30) we have

$$(h, A^D h)_{L_2(\Omega)} = \int_\Omega uh\,\mathrm{d}x = \int_\Omega (|\nabla u|^2 + u^2)\,\mathrm{d}x = \|u\|^2_{\mathring{H}^1(\Omega)} > 0, \quad h \neq 0.$$

Hence, the operator A^D is positive. Now, A^D is a self-adjoint positive compact operator in the Hilbert space $H = L_2(\Omega)$. By the well-known theorem ([6], Theorem 1.2.1) there exists a sequence of eigenvalues $\{\mu_k^D\}$, $k = 1, 2, \ldots$, with finite multiplicities such that $0 < \mu_k^D \leqslant 1$, $\mu_k^D \searrow 0$, $k \to \infty$ of the operator A^D. Denote by $u_k^D \in L_2(\Omega)$ the corresponding eigenfunctions satisfying $A^D u_k^D = \mu_k^D u_k^D$. Thus, $\mu_k^D (u_k^D, v)_{\mathring{H}^1(\Omega)} = (u_k^D, v)_{L_2(\Omega)}$ and

$$\mu_k^D \int_\Omega ((\nabla u_k^D, \nabla v) + u_k^D v)\,\mathrm{d}x = \int_\Omega u_k^D v\,\mathrm{d}x.$$

Hence, $\mu_k^D = (\lambda_k^D + 1)^{-1}$. Let us note that $\mu_k^D \leqslant (\lambda_1^D + 1)^{-1} < 1$ so $\|A^D\| < 1$.

Now we obtain an estimate of the norm $\|A_\alpha - A^D\|_{L_2(\Omega)\to L_2(\Omega)}$ for large positive values of α.

Let us remark that in domains with C^2-class boundary surface the functions $u = A_\alpha h$ and $v = A^D h$ are strong solutions and belong to $H^2(\Omega)$ ([11], Chapter 4, Paragraph 2, Theorem 4). Moreover, the estimate

$$\|v\|_{H^2(\Omega)} \leqslant C_2\|h\|_{L_2(\Omega)} \tag{31}$$

holds. Now we use the estimate (15) with $\varepsilon = 1$:

$$\|v\|_{L_2(\Gamma)} \leqslant C_3\|v\|_{H^1(\Omega)}. \tag{32}$$

Combining (31) and (32) we have the inequality

$$\|\nabla v\|_{L_2(\Gamma)} \leqslant C_4\|v\|_{H^2(\Omega)}. \tag{33}$$

Since $\left|\frac{\partial v}{\partial \nu}\right|_\Gamma \leqslant |\nabla v|$ on Γ, from (33) we obtain the estimate

$$\|\frac{\partial v}{\partial \nu}\|_{L_2(\Gamma)} \leqslant C_5\|h\|_{L_2(\Omega)}. \tag{34}$$

Let $w = (A^D - A_\alpha)h$. By (17), (18), (24), (25) the function w is a solution of the boundary value problem

$$-\Delta w + w = 0 \quad \text{in } \Omega, \tag{35}$$

$$\frac{\partial w}{\partial \nu} + \alpha w = \frac{\partial v}{\partial \nu} \quad \text{on } \Gamma. \tag{36}$$

Multiplying the equation (35) by w and integrating on Ω with respect to the boundary condition (36), for $\alpha > 0$ we get the relation

$$\int_\Omega (|\nabla w|^2 + w^2)\,\mathrm{d}x + \frac{1}{\alpha}\int_\Gamma \Big(\frac{\partial w}{\partial \nu}\Big)^2 \mathrm{d}s = \frac{1}{\alpha}\int_\Gamma \frac{\partial w}{\partial \nu}\frac{\partial v}{\partial \nu}\,\mathrm{d}s. \tag{37}$$

From (37) we obtain the inequality

$$\|w\|^2_{L_2(\Omega)} + \frac{1}{\alpha}\Big\|\frac{\partial w}{\partial \nu}\Big\|^2_{L_2(\Gamma)} \leqslant \frac{1}{\alpha}\Big\|\frac{\partial w}{\partial \nu}\Big\|_{L_2(\Gamma)}\Big\|\frac{\partial v}{\partial \nu}\Big\|_{L_2(\Gamma)}$$

and, consequently,

$$\|w\|^2_{L_2(\Omega)} + \frac{1}{\alpha}\Big\|\frac{\partial w}{\partial \nu}\Big\|^2_{L_2(\Gamma)} \leqslant \frac{1}{2\alpha}\Big\|\frac{\partial w}{\partial \nu}\Big\|^2_{L_2(\Gamma)} + \frac{1}{2\alpha}\Big\|\frac{\partial v}{\partial \nu}\Big\|^2_{L_2(\Gamma)}.$$

Therefore, we have the estimate

$$\|w\|_{L_2(\Omega)} \leqslant \frac{1}{\sqrt{2\alpha}} \Big\| \frac{\partial v}{\partial \nu} \Big\|_{L_2(\Gamma)}, \quad \alpha > 0. \tag{38}$$

Combining (38) with (34), we get

$$\|w\|_{L_2(\Omega)} \leqslant C_6 \alpha^{-1/2} \|h\|_{L_2(\Omega)}, \quad \alpha > 0,$$

with the constant C_6 independent of α. Thus, for all $h \in L_2(\Omega)$ we have the estimate

$$\|(A^D - A_\alpha)h\|_{L_2(\Omega)} \leqslant C_6 \alpha^{-1/2} \|h\|_{L_2(\Omega)}$$

and

$$\|A^D - A_\alpha\| \leqslant C_6 \alpha^{-1/2}, \quad \alpha > 0. \tag{39}$$

To prove the inequalities (9) we need the following statement (see [6], Theorem 2.3.1).

Theorem 3. *Let T_1 and T_2 be two self-adjoint, compact and positive operators on a separable Hilbert space H. Let $\mu_k(T_1)$ and $\mu_k(T_2)$ be their k-th respective eigenvalues. Then*

$$|\mu_k(T_1) - \mu_k(T_2)| \leqslant \|T_1 - T_2\| = \sup_{h \in H} \frac{\|(T_1 - T_2)h\|}{\|h\|}. \tag{40}$$

Now we apply this theorem to the operators $T_1 = A_\alpha$, $T_2 = A^D$. Then by the relations

$$\mu_k(\alpha) = \frac{1}{\lambda_k(\alpha) + 1}, \quad \mu_k^D = \frac{1}{\lambda_k^D + 1},$$

and inequalities (39), (40) we get the estimate

$$\Big| \frac{1}{\lambda_k(\alpha) + 1} - \frac{1}{\lambda_k^D + 1} \Big| \leqslant C_6 \alpha^{-1/2}. \tag{41}$$

Therefore,

$$|\lambda_k^D - \lambda_k(\alpha)| \leqslant C_6 \alpha^{-1/2} (\lambda_k^D + 1)(\lambda_k(\alpha) + 1) \tag{42}$$

and taking into account the inequalities $\lambda_k(\alpha) \leqslant \lambda_k^D$, we obtain the estimate

$$0 \leqslant \lambda_k^D - \lambda_k(\alpha) \leqslant C_6 \alpha^{-1/2} (\lambda_k^D + 1)^2 \leqslant C_1 \alpha^{-1/2} (\lambda_k^D)^2. \tag{43}$$

Proof of Theorem 2 is completed. □

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