DISJOINT AND COMPLETE UNIONS OF INCIDENCE STRUCTURES

František Machala, Olomouc, Marek Pomp, Ostrava

(Received May 2, 1996)

Abstract. Some decompositions of general incidence structures with regard to distinguished components (modular or simple) are considered and several structure theorems for them are deduced.

Keywords: incidence structure (context) and its special cases: complete, open, trivial, regular, simple, modular; onto homomorphisms of incidence structures; union of substructures: disjoint, complete.

MSC 1991: 06B05, 08A35

Definition 1. Let G and M be non-empty sets and $I \subseteq G \times M$. Then the triple $\mathcal{J} = (G, M, I)$ is called an *incidence structure* (a *context*). If $A \subseteq G$, $B \subseteq M$ are non-empty sets, then denote

$$A^{\uparrow} := \{m \in M\,;\, gIm \quad \forall g \in A\},$$

$$B^{\downarrow} := \{g \in G\,;\, gIm \quad \forall m \in B\}.$$

Further notation: $\emptyset^{\uparrow} := M$, $\emptyset^{\downarrow} := G$,

$$\begin{aligned} g^{\uparrow} &:= \{g\}^{\uparrow} \text{ for all } g \in G,\\ m^{\downarrow} &:= \{m\}^{\downarrow} \text{ for all } m \in M,\\ A^{\uparrow\downarrow} &:= (A^{\uparrow})^{\downarrow} \text{ for all } A \subseteq G,\\ B^{\downarrow\uparrow} &:= (B^{\downarrow})^{\uparrow} \text{ for all } B \subseteq M. \end{aligned}$$

(See [3]).

This paper was supported by GAČR Grant 201/95/1631

Definition 2. Let $\mathcal{J} = (G, M, I)$ be an incidence structure. If $G_1 \subseteq G$, $M_1 \subseteq M$ are non-empty subsets and $I_1 = I \cap (G_1 \times M_1)$, then the incidence structure $\mathcal{J}_1 = (G_1, M_1, I_1)$ is called a *substructure* of $\mathcal{J}$.

Definition 3. Let $\mathcal{J} = (G, M, I)$ be an incidence structure. $\mathcal{J}$ is called

1. *complete* if $I = G \times M$,
2. *open* if $g^{\uparrow} \neq M$ for all $g \in G$ and $m^{\downarrow} \neq G$ for all $m \in M$,
3. *trivial* if $|G| = |M| = 1$,
4. *regular* if $g^{\uparrow} \neq \emptyset$ for all $g \in G$ and $m^{\downarrow} \neq \emptyset$ for all $m \in M$,
5. *simple* if $|g^{\uparrow}| = 1$ for all $g \in G$ and $|m^{\downarrow}| = 1$ for all $m \in M$.

Let $\mathcal{J} = (G, M, I)$ be a simple incidence structure. It will be useful to express G and M as indexed families $G = \{g_\nu ; \nu \in T_1\}$, $M = \{m_\mu ; \mu \in T_2\}$ where $g_{\nu_1} = g_{\nu_2}$ iff $\nu_1 = \nu_2$ and $m_{\mu_1} = m_{\mu_2}$ iff $\mu_1 = \mu_2$. By Definition 3, for every $g_i \in G$ there exists exactly one $m_j \in G$ such that $g_i I m_j$, and vice-versa. Hence the map $\alpha\colon T_1 \to T_2$, defined by $\alpha(i) = j$ iff $g_i I m_j$ for all $i \in T_1$, is injective. Assume that there exists an $l \in T_2$, $l \notin \alpha(T_1)$. Then there exists a $g_i \in G$ such that $g_i I m_l$. It follows that $\alpha(i) = l$, a contradiction. Thus $\alpha(T_1) = T_2$ and the map α is a one-to-one map of T_1 onto T_2 so that we can identify both sets of indices. If we denote $p_i := m_{\alpha(i)}$ for all $i \in T_1$, then we have $g_i I p_j \Leftrightarrow g_i I m_{\alpha(j)} \Leftrightarrow \alpha(i) = \alpha(j) \Leftrightarrow i = j$.

Let $\mathcal{J} = (G, M, I)$ be a simple incidence structure. Then T will serve as an index set for elements of G, M such that the relation I is defined by $g_i I m_j$ iff $i = j$. In what follows we will suppose that incidence relations in simple incidence structures are expressed like this.

Definition 4. An incidence structure $\mathcal{J} = (G, M, I)$ is said to be the *union of substructures* $\mathcal{J}_\nu = (G_\nu, M_\nu, I_\nu)$, $\nu \in T$, if $\{G_\nu ; \nu \in T\}$ and $\{M_\nu ; \nu \in T\}$ are decompositions of G and M. In this case we will write $\mathcal{J} = \bigcup\limits_{\nu \in T} \mathcal{J}_\nu$.

R e m a r k 1. If a family $\{P_\nu ; \nu \in T\}$ forms a decomposition of a non-empty set P, then we will write $P = \dot{\bigcup\limits_{\nu \in T}} P_\nu$.

Let $\mathcal{J} = (G, M, I)$ be an incidence structure and $G_\nu \subseteq G$, $M_\nu \subseteq M$ non-empty subsets for all $\nu \in T$. Then denote $\mathcal{J}_{ij} := (G_i, M_j, I_{ij})$ the substructure of $\mathcal{J}$, where $I_{ij} = I \cap (G_i \times M_j)$ for $i, j \in T$. Moreover, put $\mathcal{J}_{ii} = \mathcal{J}_i$ and $I_{ii} = I_i$ for all $i \in T$.

Theorem 1. *If $\mathcal{J} = \bigcup\limits_{\nu \in T} \mathcal{J}_\nu$ as in Definition 4, then $I = \dot{\bigcup\limits_{i,j \in T}} I_{ij}$.*

P r o o f . Consider the substructures $\mathcal{J}_{ij}$ of $\mathcal{J}$, $i, j \in T$. Then $\bigcup\limits_{i,j \in T} I_{ij} \subseteq I$. Let $(g, m) \in I$. Since $G = \dot{\bigcup\limits_{\nu \in T}} G_\nu$ and $M = \dot{\bigcup\limits_{\nu \in T}} M_\nu$, there exist $i, j \in T$ such that

$g \in G_i$ and $m \in M_j$. Then $(g,m) \in I_{ij}$, $I = \bigcup_{i,j \in T} I_{ij}$. If $(g,m) \in I_{i_1 j_1} \cap I_{i_2 j_2}$, then $(g,m) \in (G_{i_1} \times M_{j_1}) \cap (G_{i_2} \times M_{j_2})$ and $g \in G_{i_1} \cap G_{i_2}$, $m \in M_{j_1} \cap M_{j_2}$, a contradiction. Thus $I = \dot{\bigcup}_{i,j \in T} I_{ij}$. □

Definition 5. Let an incidence structure $\mathcal{J} = (G,M,I)$ be the union of substructures $\mathcal{J}_\nu$, $\nu \in T$. This union is called *disjoint* if $I_{ij} = \emptyset$ for distinct $i,j \in T$, and will be denoted by $\mathcal{J} = \dot{\bigcup}_{\nu \in T} \mathcal{J}_\nu$. The union is called *complete* if $I_{ij} = G_i \times M_j$ for distinct $i,j \in T$, and will be denoted by $\mathcal{J} = \overline{\bigcup}_{\nu \in T} \mathcal{J}_\nu$.

Remark 2. 1. Let $\mathcal{J} = \bigcup_{\nu \in T} \mathcal{J}_\nu$. Then $\mathcal{J} = \dot{\bigcup}_{\nu \in T} \mathcal{J}_\nu$ iff $I = \dot{\bigcup}_{\nu \in T} I_\nu$ and $\mathcal{J} = \overline{\bigcup}_{\nu \in T} \mathcal{J}_\nu$ iff $I = (\bigcup_{\nu \in T} I_\nu) \cup (\bigcup_{i,j \in T} (G_i \times M_j))$ where $i \neq j$.

2. If $|T| = 1$, then $\mathcal{J} = \dot{\bigcup} \mathcal{J} = \overline{\bigcup} \mathcal{J}$. Let $\mathcal{J} = (G,M,I)$ be a simple incidence structure, where $G = \{g_\nu\,;\, \nu \in T\}$, $M = \{m_\nu\,;\, \nu \in T\}$ and $g_i I m_j$ iff $i = j$. If $\mathcal{J}_\nu = (\{g_\nu\}, \{m_\nu\}, I_\nu)$, $\nu \in T$, are substructures of $\mathcal{J}$ then $\mathcal{J}$ is the disjoint union of substructures $\mathcal{J}_\nu$, $\nu \in T$.

3. If $\mathcal{J}$ is a disjoint union of substructures $\mathcal{J}_\nu$, $\nu \in T$ then $\mathcal{J}$ is regular iff $\mathcal{J}_\nu$ are regular for all $\nu \in T$. If $\mathcal{J}$ is a complete union of substructures $\mathcal{J}_\nu$, $\nu \in T$, then $\mathcal{J}$ is open iff $\mathcal{J}_\nu$ are open for all $\nu \in T$.

Remark 3. If an incidence structure $\mathcal{J}$ is a union of substructures $\mathcal{J}_\nu$, $\nu \in T$ then write operators $\uparrow$, $\downarrow$ as right superscripts ($X^\uparrow$) for the incidence relation I in $\mathcal{J}$ and as left superscripts (${}^\uparrow X$) for incidence relations I_ν in substructures $\mathcal{J}_\nu$. Furthermore, write $G^\nu = G - G_\nu$ and $M^\nu = M - M_\nu$ for all $\nu \in T$.

Theorem 2. *Let $\mathcal{J} = (G,M,I)$ be the disjoint union of substructures $\mathcal{J}_\nu$, $\nu \in T$. If $A \subseteq G_i$, $A \neq \emptyset$ and $B \subseteq M_i$, $B \neq \emptyset$ for some $i \in T$ then $A^\uparrow = {}^\uparrow A$, $A^{\uparrow\downarrow} = {}^{\downarrow\uparrow} A$ and $B^\downarrow = {}^\downarrow B$, $B^{\downarrow\uparrow} = {}^{\uparrow\downarrow} B$, respectively. If $a \in G_i$, $b \in G_j$ and $m \in M_i$, $n \in M_j$ for $i,j \in T$, $i \neq j$, then $\{a,b\}^\uparrow = \emptyset$ and $\{m,n\}^\downarrow = \emptyset$, respectively.*

Proof. Let $A \subseteq G_i$, $A \neq \emptyset$. Then $m \in A^\uparrow$ iff aIm for all $a \in A$. Since $I = \dot{\bigcup}_{\nu \in T} I_\nu$, we obtain $aI_i m$ for all $a \in A$, $A^\uparrow = {}^\uparrow A$ and $A^\uparrow \subseteq M_i$. Similarly we obtain $B^\downarrow = {}^\downarrow B$, $B^\downarrow \subseteq G_i$. This yields $A^{\uparrow\downarrow} = {}^{\downarrow\uparrow} A$ and $B^{\downarrow\uparrow} = {}^{\uparrow\downarrow} B$.

Let $a \in G_i$, $b \in G_j$, $i \neq j$. If $m \in \{a,b\}^\uparrow$ then aIm and bIm, hence $m \in M_i \cap M_j$, which is a contradiction to $M_i \cap M_j = \emptyset$. Similarly we proceed when elements $m \in M_i$, $n \in M_j$ are under consideration. □

Theorem 3. *Let an incidence structure $\mathcal{J}$ be the complete union of substructures $\mathcal{J}_\nu$, $\nu \in T$.*

1. *If $A \subseteq G_i$ and $B \subseteq M_i$, $i \in T$, then $A^{\uparrow} = M^i \cup {}^{\uparrow}A$ and $B^{\downarrow} = G^i \cup {}^{\downarrow}B$. If the incidence structure $\mathcal{J}$ is open then $A^{\uparrow\downarrow} = {}^{\downarrow\uparrow}A$ and $B^{\downarrow\uparrow} = {}^{\uparrow\downarrow}B$.*
2. *Let $a \in G_i$ and $b \in G_j$ for distinct $i, j \in T$. Then $\{a,b\}^{\uparrow} = (M^i \cap M^j) \cup {}^{\uparrow}a \cup {}^{\uparrow}b$. If the incidence structure $\mathcal{J}$ is open then $\{a,b\}^{\uparrow\downarrow} = {}^{\downarrow\uparrow}a \cup {}^{\downarrow\uparrow}b$. Let $m \in M_i$, $n \in M_j$, $i \neq j$, $i,j \in T$. Then $\{m,n\}^{\uparrow} = (G^i \cap G^j) \cup {}^{\downarrow}m \cup {}^{\downarrow}n$. If $\mathcal{J}$ is open then $\{m,n\}^{\downarrow\uparrow} = {}^{\uparrow\downarrow}m \cup {}^{\uparrow\downarrow}n$.*

Proof. Let $g \in G$. Since $G = \dot{\bigcup\limits_{\nu \in T}} G_\nu$, there exists $l \in T$ such that $g \in G_l$. By Definition 1, $g^{\uparrow} = \{m \in M;\, gIm\}$ and from $I = (\bigcup\limits_{\nu \in T} I_\nu) \cup (\bigcup\limits_{i,j \in T} (G_i \times M_j))$ where $i \neq j$, we obtain $g^{\uparrow} = M^l \cup {}^{\uparrow}g$. Similarly, for $m \in M$ there exists $k \in T$ such that $m \in M_k$ and $m^{\downarrow} = G^k \cup {}^{\downarrow}m$.

1. Let $A \subseteq G_i$ and $A = \emptyset$. Then $A^{\uparrow} = M = M^i \cup M_i = \mathrm{M}^i \cup {}^{\uparrow}\emptyset = M^i \cup {}^{\uparrow}A$. If $A \neq \emptyset$ then $A^{\uparrow} = \bigcap\limits_{a \in A} a^{\uparrow} = \bigcap\limits_{a \in A} (M^i \cup {}^{\uparrow}a) = M^i \cup (\bigcap\limits_{a \in A} {}^{\uparrow}a) = M^i \cup {}^{\uparrow}A$.

Let $\mathcal{J}$ be an open incidence structure. Then $(M^i)^{\downarrow} = G_i$ for all $i \in T$. We obtain $A^{\uparrow\downarrow} = (A^{\uparrow})^{\downarrow} = (M^i \cup {}^{\uparrow}A)^{\downarrow} = (M^i)^{\downarrow} \cap ({}^{\uparrow}A)^{\downarrow}$. As ${}^{\uparrow}A \subseteq M_i$, we have $({}^{\uparrow}A)^{\downarrow} = G^i \cup {}^{\downarrow\uparrow}A$ and $A^{\uparrow\downarrow} = G_i \cap (G^i \cup {}^{\downarrow\uparrow}A) = (G_i \cap G^i) \cup (G_i \cap {}^{\downarrow\uparrow}A) = {}^{\downarrow\uparrow}A$.

If $B \subseteq M_i$ then the proof is similar.

2. Let $a \in G_i$, $b \in G_j$, $i \neq j$. Then $\{a,b\}^{\uparrow} = a^{\uparrow} \cap b^{\uparrow} = (M^i \cup {}^{\uparrow}a) \cap (M^j \cup {}^{\uparrow}b) = (M^i \cap M^j) \cup (M^j \cap {}^{\uparrow}a) \cup (M^i \cap {}^{\uparrow}b) \cup ({}^{\uparrow}a \cap {}^{\uparrow}b)$. Since $M^j \cap {}^{\uparrow}a = {}^{\uparrow}a$, $M^i \cap {}^{\uparrow}b = {}^{\uparrow}b$, ${}^{\uparrow}a \cap {}^{\uparrow}b = \emptyset$ we have $\{a,b\}^{\uparrow} = (M^i \cap M^j) \cup {}^{\uparrow}a \cup {}^{\uparrow}b$.

Let $\mathcal{J}$ be an open incidence structure. For every $i,j \in T$ we obtain $(M^i \cap M^j)^{\downarrow} = (\bigcup\limits_{l \neq i,j} M_l)^{\downarrow} = G_i \cup G_j$. Hence, $\{a,b\}^{\uparrow\downarrow} = (\{a,b\}^{\uparrow})^{\downarrow} = ((M^i \cap M^j) \cup {}^{\uparrow}a \cup {}^{\uparrow}b)^{\downarrow} = (M^i \cap M^j)^{\downarrow} \cap ({}^{\uparrow}a)^{\downarrow} \cap ({}^{\uparrow}b)^{\downarrow} = (G_i \cup G_j) \cap (G^i \cup {}^{\downarrow\uparrow}a) \cap (G^j \cup {}^{\downarrow\uparrow}b) = [(G_i \cup G_j) \cap (G^i \cap G^j)] \cup [(G_i \cup G_j) \cap {}^{\downarrow\uparrow}a] \cup [(G_i \cup G_j) \cap {}^{\downarrow\uparrow}b]$. Now, $(G_i \cup G_j) \cap (G^i \cap G^j) = (G_i \cup G_j) \cap (\bigcup\limits_{l \neq i,j} G_l) = \emptyset$. By virtue of ${}^{\downarrow\uparrow}a \subseteq \mathrm{G}_i$, ${}^{\downarrow\uparrow}b \subseteq G_j$, it follows that $(G_i \cup G_j) \cap {}^{\downarrow\uparrow}a = {}^{\downarrow\uparrow}a$, $(G_i \cup G_j) \cap {}^{\downarrow\uparrow}b = {}^{\downarrow\uparrow}b$. Thus $\{a,b\}^{\uparrow\downarrow} = {}^{\downarrow\uparrow}a \cup {}^{\downarrow\uparrow}b$.

For $m \in M_i$ and $n \in M_j$ the proof is similar. □

Definition 6. Let $\mathcal{J} = (G, M, I)$, $\mathcal{J}_1 = (G_1, M_1, I_1)$ be incidence structures. A map $\varphi\colon G \cup M \to G_1 \cup M_1$ is called a *homomorphism* of $\mathcal{J}$ onto $\mathcal{J}_1$ if

1. $\varphi(G) := \{\varphi(g);\, g \in G\} = G_1, \varphi(M) := \{\varphi(m);\, m \in M\} = M_1$,
2. $aIm \Longrightarrow \varphi(a) I_1 \varphi(m)$,
3. for $a' I_1 m'$ there are elements $a \in G$, $m \in M$ such that aIm, $\varphi(a) = a'$ and $\varphi(m) = m'$.

Remark 4. 1. Let $\mathcal{J} = (G, M, I)$ be an incidence structure and let $\overline{G}, \overline{M}$ be decompositions of G, M. Put $\mathcal{R} = (\overline{G}, \overline{M})$ and consider the incidence structure

$\mathcal{J}_{\mathcal{R}} = (\overline{G}, \overline{M}, I_{\mathcal{R}})$ where $\bar{g} I_{\mathcal{R}} \overline{m}$ iff there is an $h \in \bar{g}$ with $n \in \overline{m}$, hIm for every $\bar{g} \in \overline{G}$, $\overline{m} \in \overline{M}$. The map $\varphi_{\mathcal{R}}$ defined by

$$\varphi_{\mathcal{R}} : \begin{cases} g \mapsto \bar{g} & \forall g \in G, \\ m \mapsto \overline{m} & \forall m \in M, \end{cases}$$

is a homomorphism of $\mathcal{J}$ onto $\mathcal{J}_{\mathcal{R}}$. (See [1], Theorem 1.)

2. Let φ be an incidence structure homomorphism of $\mathcal{J} = (G, M, I)$ onto $\mathcal{J}_1 = (G_1, M_1, I_1)$. If we put $\bar{g} = \{h \in G;\ \varphi(h) = \varphi(g)\}$, $\overline{m} = \{n \in M;\ \varphi(n) = \varphi(m)\}$ then $G_\varphi = \{\bar{g};\ g \in G\}$ is a decomposition of the set G and $M_\varphi = \{\overline{m};\ m \in M\}$ is a decomposition of the set M. If we denote $\mathcal{R}_\varphi = (G_\varphi, M_\varphi)$ then the map ξ defined by

$$\xi : \begin{cases} \bar{g} \mapsto \varphi(g) & \forall \bar{g} \in G_\varphi, \\ \overline{m} \mapsto \varphi(m) & \forall \overline{m} \in M_\varphi, \end{cases}$$

is an isomorphism (i.e., both sided homomorphism) between $\mathcal{J}_{\mathcal{R}_\varphi}$ and $\mathcal{J}_1$. (See [1], Theorem 1.)

Theorem 4. *Let $\mathcal{J} = (G, M, I)$ be an incidence structure. Then the following conditions are equivalent.*

1. *$\mathcal{J}$ is the disjoint union of substructures $\mathcal{J}_\nu = (G_\nu, M_\nu, I_\nu)$, $\nu \in T$, where $|T| \geqslant 2$ and $I_\nu \neq \emptyset$ for all $\nu \in T$.*
2. *There exists a homomorphism of $\mathcal{J}$ onto a simple non-trivial incidence structure.*

Proof. 1. $\Longrightarrow$ 2. Let the assumption 1 hold. Then the sets $\overline{G} = \{G_\nu;\ \nu \in T\}$, $\overline{M} = \{M_\nu;\ \nu \in T\}$ are decompositions of the sets G, M. Put $\mathcal{R} = (\overline{G}, \overline{M})$ and consider the incidence structure $\mathcal{J}_{\mathcal{R}} = (\overline{G}, \overline{M}, I_{\mathcal{R}})$ from Remark 4. We will prove that $\mathcal{J}_{\mathcal{R}}$ is a simple incidence structure. Let $G_i \in \overline{G}$. Then there exist $g \in G_i$ and $m \in M_i$ such that gI_im, because $I_i \neq \emptyset$. By Theorem 1, we have gIm and by Remark 4, we obtain $G_i I_{\mathcal{R}} M_i$ and $|G_i^{\uparrow}| \geqslant 1$. Similarly we get $|M_j^{\downarrow}| \geqslant 1$ for every $M_j \in \overline{M}$. Now suppose that $G_i I_{\mathcal{R}} M_j$ for $i, j \in T$. Then there exist $g \in G_i$ and $m \in M_j$ such that gIm, and according to Definition 5 and Remark 2 there exists an $l \in T$ such that $g \in G_l$, $m \in M_l$ and gI_lm. But $g \in G_i \cap G_l$ and $m \in M_j \cap M_l$, which means that $i = j = l$ so that $|G_i^{\uparrow}| = 1$. Similarly we obtain $|M_j^{\downarrow}| = 1$ for all $M_j \in \overline{M}$. Thus $\mathcal{J}_{\mathcal{R}}$ is simple. Because of $|T| \geqslant 2$, we have $|\overline{G}| \geqslant 2$, $|\overline{M}| \geqslant 2$ and $\mathcal{J}_{\mathcal{R}}$ is not trivial.

According to Remark 4 the map $\varphi_{\mathcal{R}} : \mathcal{J} \to \mathcal{J}_{\mathcal{R}}$ is a homomorphism of $\mathcal{J}$ onto $\mathcal{J}_{\mathcal{R}}$.

2. $\Longrightarrow$ 1. Let $\varphi\colon \mathcal{J} \to \mathcal{J}'$ be a homomorphism of $\mathcal{J}$ onto a simple incidence structure $\mathcal{J}' = (G', M', I')$. Suppose that $G' = \{g'_\nu;\ \nu \in T\}$, $M' = \{m'_\nu;\ \nu \in T\}$ and $g'_i I' m'_j$ iff $i = j$. Since $\mathcal{J}'$ is non-trivial, it follows that $|T| \geqslant 2$.

By Remark 4, we obtain the structure $\mathcal{J}_{\mathcal{R}_\varphi} = (G_\varphi, M_\varphi, I_{\mathcal{R}_\varphi})$, where $\mathrm{G}_\varphi = \{\bar{g};\ g \in G\}$, $M_\varphi = \{\overline{m};\ m \in M\}$ and $\bar{g} I_{\mathcal{R}_\varphi} \overline{m}$ iff there are $h \in \bar{g}$, $n \in \overline{m}$ such that hIn. Furthermore, put $G_i := \bar{g}$ iff $\varphi(g) = g_i'$ and $M_i := \overline{m}$ iff $\varphi(m) = m_i'$ and consider substructures $\mathcal{J}_i = (G_i, M_i, I_i)$, where $I_i = I \cap (G_i \times M_i)$ for all $i \in T$. Then $\varphi(G_i) = g_i'$, $\varphi(M_i) = m_i'$ and $g_i' I' m_i'$. By Condition 3 from Definition 6 there exist $g \in G_i$, and $m \in M_i$ such that gIm. Then $gI_i m$ and hence $I_i \neq \emptyset$ for all $i \in T$.

We will prove that $\mathcal{J} = \dot{\bigcup\limits_{\nu \in T}} \mathcal{J}_\nu$. Since G_φ, M_φ are decompositions of G, M, the sets $\{G_\nu;\ \nu \in T\}$ and $\{M_\nu;\ \nu \in T\}$ are decompositions of G, M, too. Now the set $\{I_\nu;\ \nu \in T\}$ is a decomposition of the set I. We have gIm so that $\varphi(g) I' \varphi(m)$. If $\varphi(g) = g_i'$ then $\varphi(m) = m_i'$ and $(g, m) \in G_i \times M_i$. This yields $(g, m) \in I_i$ and $I_i \subseteq I$ for all $i \in T$. From $G_i \cap G_j = \emptyset$ and $M_i \cap M_j = \emptyset$ for $i \neq j$, we get $I = \dot{\bigcup\limits_{\nu \in T}} I_\nu$. $\square$

R e m a r k 5. There exists a homomorphism of an arbitrary incidence structure with non-empty incidence relation onto a trivial simple incidence structure.

Theorem 5. *Every regular incidence structure is a homomorphic image of a certain simple incidence structure.*

P r o o f. Let $\mathcal{J} = (G, M, I)$ be a regular incidence structure. Set $G = \{g_\nu;\ \nu \in P_1\}$, $M = \{m_\mu;\ \mu \in P_2\}$ and define the set $U \subseteq P_1 \times P_2$ by $(i,j) \in U$ iff $g_i I m_j$. Let $U = \{u_\xi;\ \xi \in T\}$. We consider the map $\alpha\colon U \to P_1$, given by $\alpha(i,j) = i$ for all $(i,j) \in U$. If $i \in P_1$, then $|g_i^{\uparrow}| \neq \emptyset$ because $\mathcal{J}$ is regular. Hence there exists $m_j \in M$ such that $g_i I m_j$. It follows that $(i,j) \in U$, $\alpha(i,j) = i$ and so α is a map onto P_1. For every $i \in P_1$, put $\alpha^{-1}(i) = U_i = \{u_\eta;\ \eta \in T_i\}$ where $T_i \subseteq T$. Similarly, define a map $\beta\colon U \to P_2$ such that $\beta(i,j) = j$. This map is onto. Denote $\beta^{-1}(j) = U^j = \{u_\kappa;\ \kappa \in T^j\}$ where $T^j \subseteq T$.

Now consider the simple incidence structure $\mathcal{J}_1 = (G_1, M_1, I_1)$ where $G_1 = \{b_\xi;\ \xi \in T\}$, $M_1 = \{p_\xi;\ \xi \in T\}$ and $b_i I_1 p_j$ iff $i = j$. Put $\bar{b}_i = \{b_\xi;\ \xi \in T_i\}$ for $i \in P_1$ and $\bar{p}_j = \{p_\xi;\ \xi \in T^j\}$ for $j \in P_2$.

The family $\{\bar{b}_i;\ i \in P_1\}$ forms a decomposition of G_1. If $b_l \in G_1$ then $l \in T$, and there exists a $u_l \in U$. We express it as $u_l = (p, q)$ so that $\alpha(u_l) = p$, $u_l \in U_p$ and consequently, $l \in T_p$, $b_l \in \bar{b}_p$, $G_1 = \bigcup\limits_{i \in T_1} \bar{b}_i$. If $b_l \in \bar{b}_{i_1} \cap \bar{b}_{i_2}$ then $l \in T_{i_1} \cap T_{i_2}$ and $u_l \in U_{i_1} \cap U_{i_2}$, which yields $i_1 = i_2$. Obviously, $\bar{b}_i \neq \emptyset$ for all $i \in P_1$. Similarly one can prove that the family $\{\overline{m}_j;\ j \in P_2\}$ forms a decomposition of M_1.

It is clear that

$$u_l = (i,j),\ l \in T \Leftrightarrow u_l \in U_i \cap U^j \Leftrightarrow l \in T_i \cap T^j \Leftrightarrow b_l \in \bar{b}_i,\ p_l \in \bar{p}_j.$$

Finally consider the map $\varphi\colon G_1 \cup M_1 \to G \cup M$ given by $\varphi(b_i) = g_j$ iff $b_i \in \bar{b}_j$ for all $b_i \in G_1$ and $\varphi(p_i) = m_j$ iff $p_i \in \bar{p}_j$ for all $p_i \in M_1$. We claim that φ

is a homomorphism of $\mathcal{J}_1$ onto $\mathcal{J}$: In deed, first it is obvious that $\varphi(G_1) = G$, $\varphi(M_1) = M$. If $b_l I_1 p_k$ then $l = k$. If $\varphi(b_l) = g_i$ then $b_l \in \bar{b}_i$ and similarly for $\varphi(p_l) = m_j$, $p_l \in \bar{p}_j$. This implies $u_l = (i,j) \in U$ and we obtain $g_i I m_j$, $\varphi(b_l) I \varphi(p_l)$.

If $g_i I m_j$ then there exists an $l \in T$ with $u_l = (i,j)$ and it follows that $b_l \in \bar{b}_i$, $p_l \in \bar{p}_j$. This yields $\varphi(b_l) = g_i$, $\varphi(p_l) = m_j$ and $b_l I_1 p_l$. $\square$

Modular incidence structures have been defined in [2]:

Definition 7. An incidence structure $\mathcal{J} = (G, M, I)$ is said to be *modular* if it satisfies the following conditions:

(M1) $\{a,b\}^{\uparrow} \neq \emptyset \quad \forall a,b \in G,$

(M2) $\{m,n\}^{\downarrow} \neq \emptyset \quad \forall m,n \in M,$

(M3) $a,b \in G,\ x \in \{a,b\}^{\uparrow\downarrow},\ x \neq a \Longrightarrow \{a,x\}^{\uparrow} \subseteq \{a,b\}^{\uparrow},$

(M4) $m,n \in M,\ y \in \{m,n\}^{\downarrow\uparrow},\ y \neq m \Longrightarrow \{m,y\}^{\downarrow} \subseteq \{m,n\}^{\downarrow}.$

Theorem 6. *Let an incidence structure $\mathcal{J} = (G, M, I)$ be the complete union of incidence structures $\mathcal{J}_\nu = (G_\nu, M_\nu, I_\nu)$ where $\nu \in T$ and $|T| > 1$. Then the following two conditions are equivalent:*

1. *$\mathcal{J}$ is open modular.*
2. *$|G| \geqslant 3$ and each of $\mathcal{J}_\nu$ is either open modular, or simple non-trivial, or a trivial incidence structure with empty incidence relation.*

P r o o f. 1. $\Longrightarrow$ 2. As $\mathcal{J}$ is open, all substructures $\mathcal{J}_\nu$ are open by Remark 2. Since $|T| > 1$, we have $|G| \geqslant 2$ and $|M| \geqslant 2$. Suppose that $|G| = 2$, $G = \{a,b\}$. It follows that $\mathcal{J}_1 = (\{a\}, M_1, I_1)$, $\mathcal{J}_2 = (\{b\}, M_2, I_2)$ where $M = M_1 \dot\cup M_2$. Moreover, $\mathcal{J}_{12} = (\{a\}, M_2, I_{12})$, $\mathcal{J}_{21} = (\{b\}, M_1, I_{21})$ where $I_{12} = \{a\} \times M_2$, $I_{21} = \{b\} \times M_1$. Since $\mathcal{J}_1, \mathcal{J}_2$ are open, $I_1 = I_2 = \emptyset$ and $|m^{\downarrow}| = 1$ for all $m \in M$. But $\mathcal{J}$ is modular so that, according to Theorem 3 of [2], $\mathcal{J}$ is not open, which is a contradiction. Hence $|G| \geqslant 3$ and similarly, $|M| \geqslant 3$.

Let $\mathcal{J}_i = (G_i, M_i, I_i)$, $i \in T$, be substructures of $\mathcal{J}$.

(1) Let $|G_i| = 1$. Then $G_i = \{a\}$ for some $a \in G$. Furthermore, suppose that $I_i \neq \emptyset$. Then there exists an $m \in M_i$ such that $a I_i m$ and it follows that $\{a\} = {}^{\downarrow}m$. According to Theorem 3, $m^{\downarrow} = G^i \cup {}^{\downarrow}m = G^i \cup G_i = G$. We have obtained a contradiction to Condition 1. Therefore $I_i = \emptyset$.

Let m, n be distinct elements of M_i. Then ${}^{\downarrow}m = \emptyset = {}^{\downarrow}n$ and $m^{\downarrow} = n^{\downarrow} = G^i$, in contradiction to Theorem 4 of [2]. Thus $m = n$ and $|M_i| = 1$. Hence $\mathcal{J}_i$ is trivial and its incidence relation is empty. The case $|M_i| = 1$ can be considered analogously.

(2) Let $|G_i| > 1$. Then $|M_i| > 1$, too. Suppose that ${}^{\uparrow}a = \emptyset$ for some $a \in G_i$. By Theorem 3 we have $a^{\uparrow} = M^i \cup {}^{\uparrow}a = M^i$. Since $|G_i| > 1$, there exists a $b \in G_i$, $b \neq a$ and from $b^{\uparrow} = M^i \cup {}^{\uparrow}b$ we get $a^{\uparrow} \subseteq b^{\uparrow}$. But this is a contradiction to Theorem 4 of [2], so that $|{}^{\uparrow}a| \geqslant 1$. Similarly we prove $|{}^{\downarrow}m| \geqslant 1$.

(a) Suppose that $|{}^{\uparrow}a| = 1$ for some $a \in G_i$. Then there exists an $m \in M_i$ such that aI_im and ${}^{\uparrow}a = \{m\}$. Further suppose that there exists a $b \in G_i$, $b \neq a$ such that bI_im. Then $m \in {}^{\uparrow}b$ and ${}^{\uparrow}a \subseteq {}^{\uparrow}b$. Since $a^{\uparrow} = M^i \cup {}^{\uparrow}a$ and $b^{\uparrow} = M^i \cup {}^{\uparrow}b$, we have $a^{\uparrow} \subseteq b^{\uparrow}$, which is again a contradiction to Theorem 4 of [2]. This implies $|{}^{\downarrow}m| = 1$ and ${}^{\downarrow}m = \{a\}$.

Let n be an arbitrary element of M_i, $n \neq m$. Then $n \notin {}^{\uparrow}a$. Suppose there exist distinct $b, c \in G_i$, such that bI_in, cI_in. Clearly ${}^{\uparrow}\{a,b\} = \emptyset$ and by Theorem 3, $\{a,b\}^{\uparrow} = M^i$. Now ${}^{\downarrow\uparrow}\{a,b\} = G_i$ and $c \in {}^{\downarrow\uparrow}\{a,b\}$. By Theorem 3 it follows that ${}^{\downarrow\uparrow}\{a,b\} = \{a,b\}^{\uparrow\downarrow}$. Hence $c \in \{a,b\}^{\uparrow\downarrow}$ and from $n \notin M^i$, one gets $n \notin \{a,b\}^{\uparrow}$. Moreover, $n \in \{b,c\}^{\uparrow}$, hence $\{b,c\}^{\uparrow} \nsubseteq \{b,a\}^{\uparrow}$, which is a contradiction to (M3). From $|{}^{\downarrow}n| \geqslant 1$ we obtain $|{}^{\downarrow}n| = 1$.

Let b be an arbitrary element of G_i, $b \neq a$. Suppose there exist distinct $n, p \in M_i$ such that bI_in, bI_ip. Then ${}^{\downarrow}\{m,n\} = \emptyset$ and ${}^{\uparrow\downarrow}\{m,n\} = M_i$, and therefore $p \in {}^{\uparrow\downarrow}\{m,n\} = \{m,n\}^{\downarrow\uparrow}$. Moreover, $b \in \{n,p\}^{\downarrow}$ and $b \notin \{m,n\}^{\downarrow}$ so that $\{n,p\}^{\downarrow} \nsubseteq \{m,n\}^{\downarrow}$, in contradiction to (M4). Hence $|{}^{\uparrow}b| = 1$ and $\mathcal{J}_i$ is simple.

Similarly we prove that $|{}^{\downarrow}m| = 1$ implies that $\mathcal{J}_i$ is simple.

(b) Let us suppose that there exists $a \in G_i$ such that $|{}^{\uparrow}a| > 1$. Then by part (a) $|{}^{\uparrow}x| > 1$ for all $x \in G_i$ and $|{}^{\downarrow}m| > 1$ for all $m \in M_i$. We prove that every incidence structure $\mathcal{J}_i$ satisfies conditions (M1)–(M4).

To (M1): Let $a, b \in G_i$ such that ${}^{\uparrow}\{a,b\} = \emptyset$. Then ${}^{\downarrow\uparrow}\{a,b\} = \{a,b\}^{\uparrow\downarrow} = G_i$ and for arbitrary $x \in G_i$ we obtain $x \in \{a,b\}^{\uparrow\downarrow}$. As $\mathcal{J}$ is modular, (M3) implies $\{x,a\}^{\uparrow} \subseteq \{a,b\}^{\uparrow}$ whenever $x \neq a$, in other words $M^i \cup {}^{\uparrow}\{x,a\} \subseteq M^i \cup {}^{\uparrow}\{a,b\}$. As ${}^{\uparrow}\{a,b\} = \emptyset$, we obtain ${}^{\uparrow}\{x,a\} = \emptyset$. By $|{}^{\uparrow}a| > 1$, there exists an $m \in M_i$ such that aI_im. As $|{}^{\downarrow}m| > 1$, there exists a $c \in G_i$, $c \neq a$ such that cI_im. Hence $m \in {}^{\uparrow}\{c,a\}$, which is a contradiction. Then ${}^{\uparrow}\{a,b\} \neq \emptyset$.

Condition (M2) can be proved similarly as (M1).

To (M3): Let $a, b \in G_i$ and $c \in {}^{\downarrow\uparrow}\{a,b\}$, $c \neq a$. Then $c \in \{a,b\}^{\uparrow\downarrow}$. By (M3), $\{c,a\}^{\uparrow} \subseteq \{a,b\}^{\uparrow}$ i.e. $M^i \cup {}^{\uparrow}\{c,a\} \subseteq M^i \cup {}^{\uparrow}\{a,b\}$. If $x \in {}^{\uparrow}\{c,a\}$ then $x \in M^i \cup {}^{\uparrow}\{a,b\}$ and, regarding $x \notin M^i$, we obtain $x \in {}^{\uparrow}\{a,b\}$. It follows that ${}^{\uparrow}\{c,a\} \subseteq {}^{\uparrow}\{a,b\}$.

Condition (M4) can be proved similarly as (M3).

2. $\Longrightarrow$ 1. Each of $\mathcal{J}_\nu$, $\nu \in T$ is an open and consequently $\mathcal{J}$ is open. We show that $\mathcal{J}$ satisfies conditions (M1)–(M4).

To (M1): Let a, b be elements of G such that $a, b \in G_i$ for some $i \in T$. By virtue of $|T| > 1$, it follows that $M^i \neq \emptyset$ and $\{a, b\}^{\uparrow} = M^i \cup {}^{\uparrow}\{a, b\} \neq \emptyset$.

Let $a \in G_i$, $b \in G_j$ where $i \neq j$ and let $|T| = 2$. Then $\mathcal{J} = \mathcal{J}_1 \dot{\cup} \mathcal{J}_2$. According to the hypothesis $|G| \geqslant 3$ both structures $\mathcal{J}_1$ and $\mathcal{J}_2$ are non-trivial. Hence, for instance, $\mathcal{J}_1$ is simple non-trivial or modular and so regular. If $a \in G_1$ and $b \in G_2$ then ${}^{\uparrow}a \neq \emptyset$ and, by Theorem 3, $\{a, b\}^{\uparrow} = (M^i \cap M^j) \cup {}^{\uparrow}a \cup {}^{\uparrow}b = {}^{\uparrow}a \cup {}^{\uparrow}b \neq \emptyset$. If $|T| > 2$ then $M^i \cap M^j \neq \emptyset$ and again $\{a, b\}^{\uparrow} \neq \emptyset$.

The condition (M2) can be proved similarly as the condition (M1).

To (M3): Let a, b be elements of G and $c \in \{a, b\}^{\uparrow\downarrow}$, $c \neq a$. We have to prove that $\{a, c\}^{\uparrow} \subseteq \{a, b\}^{\uparrow}$.

(a) Let $a, b \in G_i$ for a certain $i \in T$. Then $\{a, b\}^{\uparrow\downarrow} = {}^{\downarrow\uparrow}\{a, b\}$. If $\mathcal{J}_i$ is trivial with $I_i = \emptyset$ then $G_i = \{a\}$, $c = a = b$ and ${}^{\uparrow}\{a, c\} = {}^{\uparrow}\{a, b\} = \emptyset$. Further, $\{a, c\}^{\uparrow} = M^i = \{a, b\}^{\uparrow}$. If $\mathcal{J}_i$ is simple then, because of $a \neq c$, it follows that ${}^{\uparrow}\{a, c\} = \emptyset$ and ${}^{\uparrow}\{a, c\} \subseteq {}^{\uparrow}\{a, b\}$. If $\mathcal{J}_i$ is modular then we obtain the same conclusion as a consequence of (M3). Hence $\{a, c\}^{\uparrow} = M^i \cup {}^{\uparrow}\{a, c\} \subseteq M^i \cup {}^{\uparrow}\{a, b\} = \{a, b\}^{\uparrow}$.

(b) Let $a \in G_i$, $b \in G_j$, $i \neq j$.

If $x, y \in G_l$ for an arbitrary $l \in T$ then ${}^{\uparrow}y \subseteq {}^{\uparrow}x$ iff $y = x$. If $\mathcal{J}_l$ is simple then ${}^{\uparrow}\{x, y\} = {}^{\uparrow}x \cap {}^{\uparrow}y = \emptyset$ for $x \neq y$ and (M3) is valid. If $\mathcal{J}_l$ is modular, then $\mathcal{J}_l$ is open and we obtain (M3) by Theorem 4 of [1].

By the hypothesis $c \in \{a, b\}^{\uparrow\downarrow}$. That means, by Theorem 3, $c \in {}^{\downarrow\uparrow}a \cup {}^{\downarrow\uparrow}b$. Since ${}^{\downarrow\uparrow}a \cap {}^{\downarrow\uparrow}b = \emptyset$, c belongs to exactly one of the sets ${}^{\downarrow\uparrow}a$ and ${}^{\downarrow\uparrow}b$. Let $c \in {}^{\downarrow\uparrow}a$. Hence ${}^{\uparrow}a \subseteq {}^{\uparrow}c$ and $a = c$. This yields $\{a, c\}^{\uparrow} = (M^i \cap M^j) \cup {}^{\uparrow}a \cup {}^{\uparrow}c = (M^i \cap M^j) \cup {}^{\uparrow}a \subseteq (M^i \cap M^j) \cup {}^{\uparrow}a \cup {}^{\uparrow}b = \{a, b\}^{\uparrow}$.

Condition (M4) can be proved similarly as (M3). □

Remark 6. Let $\mathcal{J} = (G, M, I)$ be a simple incidence structure with $|G| \geqslant 3$. We put $G = \{g_\nu\,;\, \nu \in T\}$, $M = \{m_\nu\,;\, \nu \in T\}$, $g_i I m_j$ iff $i = j$. If $\mathcal{J}'$ is a complementary incidence structure on $\mathcal{J}$ (i.e. $\mathcal{J}' = (G, M, (G \times M) - I)$), then $\mathcal{J}$ is open modular.

Remark 7. According to Theorem 6, we can extend every open modular incidence structure with help of other open modular or non-trivial simple incidence structures or of trivial ones the incidence relations of which is empty, to a new incidence structure which is open modular, too.

References

[1] *Machala, F.*: Über Homomorphismen der Kontexte. Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. *114* (1994), 95–104.
[2] *Machala, F.*: Modulare Inzidenzstrukturen. Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. *34* (1995), 137–145.
[3] *Wille, R.*: Restructuring lattice theory: an approach based on hierarchies of concepts. Ordered sets (I. Rival, ed.). Reidel, Dordrecht-Boston, 1982, pp. 445–470.
[4] *Ganter, B.; Wille, R.*: Formale Begriffsanalyse. Mathematische Grundlangen. Springer, 1996, p. 296.

Authors' addresses: *František Machala,* Department of Algebra and Geometry, Faculty of Science, Palacký University, Tomkova 40, Hejčín, 779 00 Olomouc, Czech Republic, e-mail: `machala@risc.upol.cz`; *Marek Pomp* Department of Mathematics, Faculty of Science, University of Ostrava, Bráfova 6, 701 03 Ostrava, Czech Republic, e-mail: `pomp@osu.cz`.