

BANACH-VALUED HENSTOCK-KURZWEIL INTEGRABLE
FUNCTIONS ARE MCSHANE INTEGRABLE ON A PORTION

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Abstract. It is shown that a Banach-valued Henstock-Kurzweil integrable function on an m -dimensional compact interval is McShane integrable on a portion of the interval. As a consequence, there exist a non-Perron integrable function $f: [0, 1]^2 \rightarrow \mathbb{R}$ and a continuous function $F: [0, 1]^2 \rightarrow \mathbb{R}$ such that

$$(P) \int_0^x \left\{ (P) \int_0^y f(u, v) \, dv \right\} du = (P) \int_0^y \left\{ (P) \int_0^x f(u, v) \, du \right\} dv = F(x, y)$$

for all $(x, y) \in [0, 1]^2$.

Keywords: Henstock-Kurzweil integral, McShane integral

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1. INTRODUCTION

It is well known that if f is Denjoy-Perron integrable on an interval $[a, b] \subset \mathbb{R}$, then f must be Lebesgue integrable on a portion of $[a, b]$. K. Karták [6] asked whether an analogous result holds for the multiple Perron integral. In a fairly recent paper [1] Buczolicz gave an affirmative answer to this problem using the Henstock-Kurzweil integral. Nevertheless, his proof depends on the measurability of the integrand. Since a Banach-valued Henstock-Kurzweil integrable function need not be strongly measurable, see for instance [4, p. 567], it is natural to ask whether Buczolicz's result holds for Banach-valued Henstock-Kurzweil integrable functions. In this paper we give an affirmative answer to this problem. As an application, we answer another question of K. Karták [6, Problem 9.3] concerning the Perron integral; namely, there exist a non-Perron integrable function $f: [0, 1]^2 \rightarrow \mathbb{R}$ and a continuous function

$F: [0, 1]^2 \rightarrow \mathbb{R}$ such that

$$(P) \int_0^x \left\{ (P) \int_0^y f(u, v) \, dv \right\} du = (P) \int_0^y \left\{ (P) \int_0^x f(u, v) \, du \right\} dv = F(x, y)$$

for all $(x, y) \in [0, 1]^2$.

2. PRELIMINARIES

Unless stated otherwise, the following conventions and notation will be used. The set of all real numbers is denoted by \mathbb{R} , and the ambient space of this paper is \mathbb{R}^m , where m is a fixed positive integer. The norm in \mathbb{R}^m is the maximum norm $\|\cdot\|$, where $\|(x_1, x_2, \dots, x_m)\| = \max_{i=1, \dots, m} |x_i|$. For $x \in \mathbb{R}^m$ and $r > 0$, set $B(x, r) := \{y \in \mathbb{R}^m : \|y - x\| < r\}$. Let $E := \prod_{i=1}^m [a_i, b_i]$ be a fixed non-degenerate interval in \mathbb{R}^m . Let X be a Banach space equipped with a norm $\|\cdot\|$. A function is always X -valued. When no confusion is possible, we do not distinguish between a function defined on a set Z and its restriction to a set $W \subset Z$.

An *interval* in \mathbb{R}^m is the cartesian product of m non-degenerate compact intervals in \mathbb{R} , \mathcal{I} denotes the family of all non-degenerate subintervals of E . For each $I \in \mathcal{I}$, $|I|$ denotes the volume of I .

A *partition* P is a collection $\{(I_i, \xi_i)\}_{i=1}^p$, where I_1, I_2, \dots, I_p are non-overlapping non-degenerate subintervals of E . Given $Z \subseteq E$, a positive function δ on Z is called a *gauge* on Z . We say that a partition $\{(I_i, \xi_i)\}_{i=1}^p$ is

- (i) a partition *in* Z if $\bigcup_{i=1}^p I_i \subseteq Z$,
- (ii) a partition *of* Z if $\bigcup_{i=1}^p I_i = Z$,
- (iii) *anchored* in Z if $\{\xi_1, \xi_2, \dots, \xi_p\} \subset Z$,
- (iv) δ -*fine* if $I_i \subset B(\xi_i, \delta(\xi_i))$ for each $i = 1, 2, \dots, p$,
- (v) *Perron* if $\xi_i \in I_i$ for each $i = 1, 2, \dots, p$,
- (vi) *McShane* if ξ_i need not belong to I_i for all $i = 1, 2, \dots, p$.

According to Cousin's Lemma [8, Lemma 6.2.6], for any given gauge δ on E , δ -fine Perron partitions of E exist. Hence the following definition is meaningful.

Definition 2.1. A function $f: E \rightarrow X$ is said to be Henstock-Kurzweil integrable (McShane integrable, respectively) on E if there exists $A \in X$ with the following property: given $\varepsilon > 0$ there exists a gauge δ on E such that

$$\left\| \sum_{i=1}^p f(\xi_i) |I_i| - A \right\| < \varepsilon$$

for each δ -fine Perron partition (δ -fine McShane partition, respectively) $\{(I_i, \xi_i)\}_{i=1}^p$ of E . We write A as $(\text{HK}) \int_E f$ ($(\text{M}) \int_E f$, respectively).

It is well known that if f is Henstock-Kurzweil integrable on E , then f is Henstock-Kurzweil integrable on each subinterval J of E . Moreover, the interval function $J \mapsto (\text{HK}) \int_J f$ is additive on \mathcal{I} . This interval function is known as the *indefinite Henstock-Kurzweil integral*, or in short the indefinite \mathcal{HK} -integral, of f .

Theorem 2.2 (Saks-Henstock Lemma). *Let $f: E \rightarrow X$ be Henstock-Kurzweil integrable on E and let F be the indefinite \mathcal{HK} -integral of f . Then given $\varepsilon > 0$ there exists a gauge δ on E such that*

$$\left\| \sum_{(I,x) \in P} \{f(x)|I| - F(I)\} \right\| < \varepsilon$$

for each δ -fine Perron partition P in E .

3. BANACH-VALUED HENSTOCK-KURZWEIL INTEGRABLE FUNCTIONS ARE MC SHANE INTEGRABLE ON A PORTION

Theorem 3.1. *Let $f: E \rightarrow X$ be Henstock-Kurzweil integrable on E and let F denote the indefinite Henstock-Kurzweil integral of f . Then the following conditions are equivalent:*

- (i) f is McShane integrable on E ;
- (ii) $\sup \left\| \sum_{i=1}^q F(J_i) \right\|$ is finite, where the supremum is taken over all finite partitions $\{J_1, \dots, J_q\}$ of pairwise non-overlapping subintervals of E .

Proof. Since E is compact, the implication (i) \implies (ii) follows from [11, Lemma 28].

(ii) \implies (i). Assume (ii). If $x \in X^*$, then $x(f)$ is Henstock-Kurzweil integrable on E and the indefinite Henstock-Kurzweil integral of $x(f)$ is of bounded variation on E . The rest of the proof is similar to that of the implication (iii) \implies (i) of [2, Corollary 9]. The proof is complete. \square

In view of [3, Proposition 2B], the next theorem is a mild improvement of [2, Theorem 8].

Theorem 3.2. Let $f: E \rightarrow X$ be Henstock-Kurzweil integrable on E and let F denote the indefinite Henstock-Kurzweil integral of f . Then the following conditions are equivalent:

- (i) f is McShane integrable on E ;
- (ii) F is absolutely continuous on \mathcal{I} , that is, given any $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality $\left\| \sum_{i=1}^p F(I_i) \right\| < \varepsilon$ holds whenever $\{I_1, \dots, I_p\}$ is a finite collection of pairwise non-overlapping subintervals of E with $\sum_{i=1}^p |I_i| < \delta$.

Proof. (i) \implies (ii). This follows from [11, Lemma 28].

(ii) \implies (i). Since E is compact, this follows from Theorem 3.1. \square

It is well known that the real-valued McShane integral is equivalent to the Lebesgue integral. For a proof of this result, see, for example, [10]. Hence the following theorem is a generalization of [1, Theorem]. Recall that a portion of E is a set of the form $E \cap I$, where I is an open interval in \mathbb{R}^m .

Theorem 3.3. If $f: E \rightarrow X$ is Henstock-Kurzweil integrable on E , then f is McShane integrable on a portion of E .

Proof. Since f is assumed to be Henstock-Kurzweil integrable on E , the Saks-Henstock Lemma (Theorem 2.2) holds. Therefore there exists a gauge δ on E such that

$$\left\| \sum_{(I,x) \in P} \{f(x)|I| - F(I)\} \right\| < 1$$

for each δ -fine Perron partition P in E . For each $n \in \mathbb{N}$, we set

$$X_n = \left\{ x \in E: \|f(x)\| < n \text{ and } \delta(x) > \frac{1}{n} \right\}.$$

Clearly $\bigcup_{n \in \mathbb{N}} X_n = E$ and hence by Baire's Category Theorem [5, Theorem 5.2] there exists $N \in \mathbb{N}$ such that X_N is dense on some J belonging to \mathcal{I} . Without loss of generality we may assume that $\text{diam}(J) < 1/N$, where $\text{diam}(J)$ denotes the diameter of J .

Consider any finite collection $\{J_1, \dots, J_q\}$ of pairwise non-overlapping subintervals of J . For each $i \in \{1, \dots, q\}$ we invoke the density of $X_N \cap J$ in J to pick $x_i \in X_N \cap J$. Since $\text{diam}(J) < 1/N$, we see that $\{(J_1, x_1), \dots, (J_q, x_q)\}$ is a $(1/N)$ -fine, and hence δ -fine, Perron partition anchored in $X_N \cap J$. Hence, by our choice of δ ,

$$\left\| \sum_{i=1}^q \{f(x_i)|J_i| - F(J_i)\} \right\| < 1$$

and so

$$\left\| \sum_{i=1}^q F(J_i) \right\| < 1 + \sum_{i=1}^q \|f(x_i)\| |J_i| < 1 + N|J|.$$

As $\{J_1, \dots, J_q\}$ is an arbitrary finite collection of pairwise non-overlapping subintervals of J , an appeal to Theorem 3.1 completes the proof of the theorem. \square

In [7], Kurzweil and Jarník proved that if f is a real-valued Henstock-Kurzweil integrable function on E , then there exists an increasing sequence $\{X_n\}_{n=1}^{\infty}$ of closed sets whose union is E , and for each $n \in \mathbb{N}$, f is Lebesgue integrable on X_n . Hence it is natural to pose the following problem.

Problem 3.4. Let $f: E \rightarrow X$ be Henstock-Kurzweil integrable on E . Can we find an increasing sequence $\{X_n\}_{n=1}^{\infty}$ of closed sets whose union is E , and for each $n \in \mathbb{N}$, f is McShane integrable on X_n ?

4. ON A QUESTION OF K. KARTÁK CONCERNING THE PERRON INTEGRAL

K. Karták posed the following problem for the Perron integral:

Problem 4.1 [6, Problem 9.3]. Is there a function $f: [0, 1]^2 \rightarrow \mathbb{R}$ such that

$$(P) \int_0^x \left\{ (P) \int_0^y f(u, v) dv \right\} du = (P) \int_0^y \left\{ (P) \int_0^x f(u, v) du \right\} dv = F(x, y)$$

for all $(x, y) \in [0, 1]^2$ and that the function F is continuous on $[0, 1]^2$ while f is not Perron integrable on $[0, 1]^2$?

Recall that the real-valued Henstock-Kurzweil integral is equivalent to the Perron integral. Hence we may use the Henstock-Kurzweil integral to answer the above question of K. Karták.

Theorem 4.2. *There exist $f: [0, 1]^2 \rightarrow \mathbb{R}$ and a continuous function $F: [0, 1]^2 \rightarrow \mathbb{R}$ such that*

$$(1) \quad (HK) \int_0^x \left\{ (HK) \int_0^y f(u, v) dv \right\} du = (HK) \int_0^y \left\{ (HK) \int_0^x f(u, v) du \right\} dv = F(x, y)$$

for all $(x, y) \in [0, 1]^2$ but f is not Henstock-Kurzweil integrable on $[0, 1]^2$.

Proof. Let f be given as in [12, Chapter VI]. Then there exist a continuous function $F: [0, 1]^2 \rightarrow \mathbb{R}$ and $f: [0, 1]^2 \rightarrow \mathbb{R}$ such that

$$\frac{\partial^2 F(x, y)}{\partial x \partial y} = \frac{\partial^2 F(x, y)}{\partial y \partial x} = f(x, y)$$

for all $(x, y) \in (0, 1)^2$. Moreover, f is not Lebesgue integrable, and hence not McShane integrable, on any non-degenerate subinterval of $[0, 1]^2$. It is clear that (1) holds for all $(x, y) \in [0, 1]^2$. Using Theorem 3.3 with $E = [0, 1]^2$ and $X = \mathbb{R}$, we conclude that f cannot be Henstock-Kurzweil integrable on $[0, 1]^2$. The proof is complete.

In view of [9, Theorem 4.3] and [9, Theorem 4.1], we see that every real-valued indefinite Henstock-Kurzweil integral generates a σ -finite Henstock variational measure. Thus it is natural to pose the following problem.

Problem 4.3. Let F be given as in Theorem 4.2, and let \tilde{F} be the additive interval function induced by F . Must the Henstock variational measure $V_{\mathcal{HK}}\tilde{F}$ be σ -finite on $[0, 1]^2$?

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