EXISTENCE OF GLOBAL SOLUTIONS TO DIFFERENTIAL INCLUSIONS; A PRIORI BOUNDS

Ovidiu Cârjă, Alina Ilinca Lazu, Iaşi

(Received October 15, 2009)

Abstract. The paper presents an existence result for global solutions to the finite dimensional differential inclusion $y' \in F(y)$, F being defined on a closed set K. A priori bounds for such solutions are provided.

Keywords: differential inclusion, global solution, a priori bound

MSC 2010: 34A60, 34C11

1. Introduction

Let K be a nonempty subset of $\mathbb{R}^p$ and F a multifunction mapping K to subsets of $\mathbb{R}^p$. Consider the initial value problem

$$\begin{cases} y'(t) \in F(y(t)), \\ y(0) = \xi. \end{cases} \tag{1.1}$$

Recall that by a solution of (1.1) on $[0,T]$, with $T > 0$, we mean an absolutely continuous function $y\colon [0,T] \to K$ which satisfies $y(0) = \xi$ and $y'(t) \in F(y(t))$ a.e. for $t \in [0,T]$.

We are interested in establishing conditions for the existence of global solutions to the Cauchy problem (1.1), giving also a priori estimates for such solutions.

The typical condition for the Cauchy problem (1.1) to have a global solution is that the multifunction F has sublinear growth: for some positive constants γ and c and for all $x \in K$ and $v \in F(x)$ we have $\|v\| \leqslant \gamma \|x\| + c$ (see e.g. [5]). A more general condition, that F is "positively sublinear", is presented in [4]. A particular case for the latter is when the "sign condition" holds, that is $\langle x, u\rangle \leqslant 0$ for each $x \in K$ and $u \in F(x)$.

We establish here a result concerning the existence of global solutions to problem (1.1) under weaker hypotheses and we also give precise a priori bounds of the solutions obtained.

The type of the hypotheses we are imposing here, as well as the bound obtained, appear e.g. in [6] for differential equations. Our approach here is different and is based on viability results. Similar approaches are provided in [2], [3] for the infinite dimensional single-valued nonlinear case, but the right hand side is defined on the whole space. Our situation here corresponds to the state constrained case.

We conclude this introduction with some basic definitions and known results that we are going to use in our study. Let $\mathcal{X}$ be a finite dimensional space, $\mathcal{K}$ a nonempty subset of $\mathcal{X}$, $\mathcal{F}\colon \mathcal{K} \rightrightarrows \mathcal{X}$ a given multifunction and let us consider the differential inclusion

$$w'(t) \in \mathcal{F}(w(t)). \tag{1.2}$$

The set $\mathcal{K}$ is viable with respect to $\mathcal{F}$ if for each $\zeta \in \mathcal{K}$ there exists $T > 0$ such that (1.2) has at least one solution $w\colon [0,T] \to \mathcal{K}$ with $w(0) = \zeta$.

A vector $\eta \in \mathcal{X}$ is tangent to the set $\mathcal{K}$ at the point $\zeta \in \mathcal{K}$ if

$$\liminf_{h \downarrow 0} \frac{1}{h} \operatorname{dist}(\zeta + h\eta; \mathcal{K}) = 0.$$

We denote by $\mathcal{T}_{\mathcal{K}}(\zeta)$ the set of all vectors which are tangent to the set $\mathcal{K}$ at the point ζ. Below, we present a well-known characterization of tangent vectors.

Proposition 1.1. *Let $\mathcal{K} \subset \mathcal{X}$ and $\zeta \in \mathcal{K}$. Then $\eta \in \mathcal{T}_{\mathcal{K}}(\zeta)$ if and only if there exist two sequences $(h_n)_n$ in $\mathbb{R}_+$ with $h_n \downarrow 0$ and $(p_n)_n$ in $\mathcal{X}$ with $\lim_n p_n = 0$ such that $\zeta + h_n(\eta + p_n) \in \mathcal{K}$ for each $n \in \mathbb{N}$.*

The following viability result is an important tool in this paper (see [1], [5]; see also [4]).

Theorem 1.1. *Let $\mathcal{K}$ be a nonempty and locally closed subset in $\mathcal{X}$ and $\mathcal{F}$: $\mathcal{K} \rightrightarrows \mathcal{X}$ an upper semicontinuous multifunction with nonempty, compact and convex values. A necessary and sufficient condition for $\mathcal{K}$ to be viable with respect to $\mathcal{F}$ is that*

$$\mathcal{F}(\zeta) \cap \mathcal{T}_{\mathcal{K}}(\zeta) \neq \emptyset \tag{1.3}$$

for each $\zeta \in \mathcal{K}$.

In the end of this section we state the Brézis-Browder ordering principle following the presentation given in [4, Theorem 2.1.1].

Theorem 1.2. *Let $\mathcal{S}$ be a nonempty set, $\preceq \subseteq \mathcal{S} \times \mathcal{S}$ a preorder on $\mathcal{S}$ and let $\mathcal{N}\colon \mathcal{S} \to \mathbb{R} \cup \{+\infty\}$ be a function. Suppose that*

(i) *each increasing sequence in $\mathcal{S}$ is bounded from above;*

(ii) *the function $\mathcal{N}$ is increasing.*

Then for each $\xi_0 \in \mathcal{S}$ there exists an $\mathcal{N}$-maximal element $\overline{\xi} \in \mathcal{S}$ satisfying $\xi_0 \preceq \overline{\xi}$.

Recall that $\overline{\xi} \in \mathcal{S}$ is $\mathcal{N}$-maximal if $\mathcal{N}(\xi) = \mathcal{N}(\overline{\xi})$ for every $\xi \in \mathcal{S}$ with $\overline{\xi} \preceq \xi$.

2. Main result

The main result of this paper is given by the following theorem.

Theorem 2.1. *Let K be a closed subset in $\mathbb{R}^p$ and F an upper semi-continuous multifunction with nonempty compact and convex values which maps bounded subsets in K into bounded subsets in $\mathbb{R}^p$. Suppose that there exists $c > 0$ such that*

$$\inf_{u \in F(x) \cap \mathcal{T}_K(x)} \langle x, u \rangle \leqslant c(1 + \|x\|^2) \tag{2.1}$$

for each $x \in K$. Then for each $\xi \in K \setminus \{0\}$ there exists a solution $y\colon [0, +\infty) \to K$ of (1.1) *with*

$$\|y(t)\| \leqslant \mathrm{e}^{\theta t} \, \|\xi\| \tag{2.2}$$

for all $t \geqslant 0$, where $\theta = c(1 + \|\xi\|^{-2})$.

R e m a r k 2.1. Note that, under the convention that $\inf \emptyset = +\infty$, inequality (2.1) includes the tangency condition: $F(x) \cap \mathcal{T}_K(x) \neq \emptyset$ for each $x \in K$.

R e m a r k 2.2. In the case $0 \in K$ and the initial condition is $\xi = 0$, by virtue of Theorem 1.1 and Remark 2.1, there exists a solution $y(t)$ to the problem (1.1) defined on an interval $[0, T]$, $T > 0$. If $y(t)$ is identically zero on $[0, T]$ then we have a global solution, precisely, $y(t) = 0$ for each $t \geqslant 0$. If there exists $t_0 \in [0, T]$ such that $y(t_0) \neq 0$, then, by Theorem 2.1, we have a global solution of problem (1.1) with the a priori bound

$$\|y(t)\| \leqslant \mathrm{e}^{\overline{\theta}(t - t_0)} \, \|y(t_0)\|$$

for all $t \geqslant t_0$, where $\overline{\theta} = c(1 + \|y(t_0)\|^{-2})$.

First, we prove the following lemma.

Lemma 2.1. *Let K be a locally closed set and F an upper semi-continuous multifunction with nonempty compact and convex values. Suppose that there exists $c > 0$ such that (2.1) holds for each $x \in K$. Then for each $\xi \in K \setminus \{0\}$ and $\alpha > c(1 + \|\xi\|^{-2})$ there exist $T > 0$ and a solution $y\colon [0, T] \to K$ of (1.1) such that*

$$\|y(t)\| < \mathrm{e}^{\alpha t} \|\xi\| \tag{2.3}$$

for all $t \in [0, T]$.

Proof. Let $\alpha > 0$ be such that there exists at least one $\xi \in K \setminus \{0\}$ such that $c(1 + \|\xi\|^{-2}) < \alpha$ and let us consider the set

$$\mathcal{K} = \{(\xi, \lambda);\ \xi \in K \setminus \{0\},\ c(1 + \|\xi\|^{-2}) < \alpha,\ \|\xi\| \leqslant \lambda\}.$$

It is easy to see that $\mathcal{K}$ is a locally closed nonempty set. We define the multifunction $\mathcal{F}\colon \mathcal{K} \rightrightarrows \mathbb{R}^{p+1}$ by

$$\mathcal{F}(y, z) = F(y) \times \{\alpha \|y\|\}$$

for each $(y, z) \in \mathcal{K}$ and we prove that $\mathcal{K}$ is viable with respect to $\mathcal{F}$. To this end we shall apply Theorem 1.1. So, we have to prove that the tangency condition (1.3) holds.

Let $(\xi, \lambda) \in \mathcal{K}$. By (2.1) there exists $u \in F(\xi) \cap \mathcal{T}_K(\xi)$ such that

$$\langle \xi, u \rangle \leqslant c(1 + \|\xi\|^2). \tag{2.4}$$

Moreover, by Proposition 1.1 there exist sequences $(h_n)_n$ and $(p_n)_n$ with $h_n \downarrow 0$, $p_n \to 0$ in $\mathbb{R}^p$ and such that $\xi + h_n(u + p_n) \in K$. By (2.4) we get that

$$\lim_{n\to\infty} \frac{\|\xi + h_n u\| - \|\xi\|}{h_n} \leqslant c\,\frac{1}{\|\xi\|}(1 + \|\xi\|^2).$$

Hence there exist two sequences $r_n \downarrow 0$ and $t_n \downarrow 0$ (a subsequence of h_n) such that

$$\frac{\|\xi + t_n u\| - \|\xi\|}{t_n} \leqslant c\,\|\xi\|\,(1 + \|\xi\|^{-2}) + r_n$$

for any $n = 1, 2, \ldots$. Therefore, we obtain that

$$\begin{aligned} \|\xi + t_n u + t_n p_n\| &\leqslant \|\xi + t_n u\| + t_n \|p_n\| \\ &\leqslant \|\xi\| + t_n \alpha \|\xi\| + t_n (r_n + \|p_n\|) \\ &\leqslant \lambda + t_n \alpha \|\xi\| + t_n (r_n + \|p_n\|) \end{aligned}$$

for any $n = 1, 2, \ldots$. Also, we have that $c(1 + \|\xi + t_n u + t_n p_n\|^{-2}) < \alpha$ and $\xi + t_n u + t_n p_n \neq 0$ for n sufficiently large. So, we conclude that $(u, \alpha \|\xi\|) \in \mathcal{T}_{\mathcal{K}}(\xi, \lambda)$, that is

$\mathcal{F}(\xi,\lambda)\cap\mathcal{T}_{\mathcal{K}}(\xi,\lambda)\neq\emptyset$ and, by Theorem 1.1, we get the viability of $\mathcal{K}$ with respect to $\mathcal{F}$. Then there exist $T>0$ and a solution $y\colon [0,T]\to K$ of (1.1) such that

$$\|y(t)\|\leqslant\|\xi\|+\alpha\int_0^t\|y(s)\|\,\mathrm{d}s$$

for all $t\in[0,T]$ and by the Gronwall inequality we obtain that (2.3) holds for all $t\in[0,T]$. $\square$

We continue with the proof of Theorem 2.1.

Proof. Step 1. We prove that for any $\xi\in K\setminus\{0\}$ and $\alpha>c(1+\|\xi\|^{-2})$ there exists a solution $y\colon [0,+\infty)\to K$ of (1.1) such that (2.3) holds for all $t\geqslant 0$. To this aim, we shall make use of the Brézis-Browder ordering principle, Theorem 1.2.

Let $\xi\in K\setminus\{0\}$ and $\alpha>c(1+\|\xi\|^{-2})$. Let $\mathcal{S}$ be the set of all solutions $y_a(\cdot)$ to the problem (1.1) defined on an interval $[0,a)$ with $a>0$ and satisfying (2.3) for all $t\in[0,a)$. This set is clearly nonempty, as we have already proved in Lemma 2.1. We introduce a preorder on $\mathcal{S}$ as follows. We say that $y_a\preceq y_b$ $(a,b>0)$ if and only if $a\leqslant b$ and $y_a(t)=y_b(t)$ for all $t\in[0,a)$. Let us show that each increasing sequence in $\mathcal{S}$ is bounded from above. Indeed, let $(y_n)_n$ be an increasing sequence in $\mathcal{S}$, $y_n\colon [0,a_n)\to K$, $n=1,2,\ldots$ and define $\tilde{y}\colon [0,\sup_n a_n)\to K$ by $\tilde{y}(t)=y_n(t)$ for any $t\in[0,a_n)$. Then $\tilde{y}(\cdot)$ is well defined, verifies (2.3) for all $t\in[0,\sup_n a_n)$, so $\tilde{y}\in\mathcal{S}$, and $\tilde{y}$ is an upper bound for $(y_n)_n$. Let us introduce an increasing function $\mathcal{N}\colon \mathcal{S}\to\mathbb{R}\cup\{+\infty\}$ defined by $\mathcal{N}(y_a)=a$ for any $y_a\in\mathcal{S}$. Consequently, the set S endowed with the preorder $\preceq$ and the function $\mathcal{N}$ satisfy the hypotheses of the Brézis-Browder ordering principle. Accordingly, there exists an $\mathcal{N}$-maximal element $\overline{y}\in\mathcal{S}$, $\overline{y}\colon [0,\bar{a})\to K$, such that, if $\overline{y}\preceq\tilde{y}$, $\tilde{y}\in\mathcal{S}$, then $\mathcal{N}(\tilde{y})=\mathcal{N}(\overline{y})$.

We will next prove that the solution $\overline{y}\colon [0,\bar{a})\to K$ of the problem (1.1) is noncontinuable. Indeed, let us assume by contradiction that $\overline{y}(\cdot)$ is continuable. Then there exist $\sigma>\bar{a}$ and $\tilde{y}\colon [0,\sigma)\to K$ such that $\tilde{y}(t)=\overline{y}(t)$ for all $t\in[0,\bar{a})$. Then, by continuity, we have that $\|\tilde{y}(\bar{a})\|\leqslant\mathrm{e}^{\alpha\bar{a}}\|\xi\|$. If $\|\tilde{y}(\bar{a})\|<\mathrm{e}^{\alpha\bar{a}}\|\xi\|$, then, by the continuity of $\tilde{y}(\cdot)$, there exists $b\in(\bar{a},\sigma)$ such that $\|\tilde{y}(t)\|<\mathrm{e}^{\alpha t}\|\xi\|$ for any $t\in[\bar{a},b)$, which contradicts the $\mathcal{N}$-maximality of $\overline{y}$. Suppose now that

$$\|\tilde{y}(\bar{a})\|=\mathrm{e}^{\alpha\bar{a}}\|\xi\|. \tag{2.5}$$

Then $\alpha>c(1+\|\tilde{y}(\bar{a})\|^{-2})$ and by Lemma 2.1 and (2.5) there exist $b>\bar{a}$ and a solution $\hat{y}(\cdot)$ of the problem (1.1) with $\|\hat{y}(t)\|\leqslant\mathrm{e}^{\alpha(t-\bar{a})}\|\tilde{y}(\bar{a})\|=\mathrm{e}^{\alpha t}\|\xi\|$ for $t\in[\bar{a},b)$, which also contradicts the $\mathcal{N}$-maximality of $\overline{y}$.

Now, we show that $\overline{y}$ is a global solution, i.e. $\bar{a}=+\infty$. To this aim, let us assume by contradiction that $\bar{a}<+\infty$. As F maps bounded sets in K into bounded sets in $\mathbb{R}^p$ and $\overline{y}$ is bounded on $[0,\bar{a})$, we have that $F(\overline{y}(\cdot))$ is bounded on $[0,\bar{a})$ and therefore the $\lim_{t\uparrow\bar{a}}\overline{y}(t)=y^*$ exists. Since K is closed it follows that $y^*\in K$. By the hypotheses, K is viable with respect to F, therefore $\overline{y}$ can be continued to the right of $\bar{a}$. This contradicts the fact that $\overline{y}$ is noncontinuable.

S t e p 2. We prove that for each $\xi \in K \setminus \{0\}$ and $T > 0$ there exists a solution $y\colon [0,T] \to K$ of (1.1) which satisfies (2.2) for all $t \in [0,T]$.

Let $\xi \in K \setminus \{0\}$ and $\alpha_n > \theta$ with $\alpha_n \to \theta$. By Step 1, for $n = 1, 2, \ldots$ there exists a solution $y_n\colon [0,+\infty) \to K$ of the problem (1.1) such that

$$\|y_n(t)\| \leqslant \mathrm{e}^{\alpha_n t}\,\|\xi\| \tag{2.6}$$

for all $t \geqslant 0$. Let $T > 0$. By (2.6), it follows that $\{y_n;\ n = 1, 2, \ldots\}$ is uniformly bounded on $[0,T]$. As F maps bounded sets in K into bounded sets in $\mathbb{R}^p$ we get that $\{y_n;\ n = 1, 2, \ldots\}$ is equicontinuous on $[0,T]$, and therefore, there exists $y \in C[0,T]$ such that, on a subsequence at least,

$$\lim_n y_n(t) = y(t),$$

uniformly for $t \in [0,T]$. Moreover, y is absolutely continuous on $[0,T]$ and $y(t) \in K$ for all $t \in [0,T]$ as K is a closed set. Passing to the limit for $n \to \infty$ in (2.6) we get that y satisfies the inequality (2.2). Now, by a standard argument, we show that $y'(t) \in F(y(t))$ a.e. for $t \in [0,T]$. So, we have proved that for each $\xi \in K \setminus \{0\}$ and $T > 0$ there exists a solution $y\colon [0,T] \to K$ of (1.1) which verifies (2.2) for all $t \in [0,T]$.

To complete the proof of our theorem, we follow the same arguments as those of the first part of the proof. So, by the Brézis-Browder ordering principle, we obtain a global solution $y\colon [0,+\infty) \to K$ to problem (1.1) which verifies inequality (2.2) for all $t \geqslant 0$. $\square$

A c k n o w l e d g m e n t. This work was supported by CNCSIS grant ID_387/2007.

References

[1] *J. P. Aubin, H. Frankowska*: Set-Valued Analysis. Birkhäuser, Boston, 1990. zbl

[2] *O. Cârjă, A. Lazu*: Lyapunov pairs for continuous perturbations of nonlinear evolutions. Nonlinear Anal., Theory Methods Appl. *71* (2009), 1012–1018. zbl

[3] *O. Cârjă, D. Motreanu*: Characterization of Lyapunov pairs in the nonlinear case and applications. Nonlinear Anal., Theory Methods Appl. *70* (2009), 352–363. zbl

[4] *O. Cârjă, M. Necula, I. I. Vrabie*: Viability, Invariance and Applications. North-Holland Mathematics Studies 207, Elsevier, Amsterdam, 2007. zbl

[5] *F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, P. R. Wolenski*: Nonsmooth Analysis and Control Theory. Graduate Texts in Mathematics 178, Springer, New York, 1998. zbl

[6] *H. O. Fattorini*: Infinite Dimensional Optimization and Control Theory. Encyclopedia of Mathematics and Its Applications 62, Cambridge University Press, Cambridge, 1999. zbl

Authors' addresses: *O. Cârjă*, Department of Mathematics, Al. I. Cuza University, Iaşi 700 506 and Octav Mayer Mathematics Institute, Romanian Academy, Iaşi 700 506, Romania, e-mail: `ocarja@uaic.ro`; *A. I. Lazu*, Department of Mathematics, Gh. Asachi Technical University, 11 Blvd Carol I, Iaşi 700 506, Romania, e-mail: `vieru_alina@yahoo.com`.