ON THE WARD THEOREM FOR $\mathcal{P}$-ADIC-PATH BASES ASSOCIATED WITH A BOUNDED SEQUENCE

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Abstract. In this paper we prove that each differentiation basis associated with a $\mathcal{P}$-adic path system defined by a bounded sequence satisfies the Ward Theorem.

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1. Introduction

In this paper we prove that each $\mathcal{P}$-adic-path system associated with a bounded sequence $\mathcal{P}$ defines a differentiation basis for which the Ward Theorem holds true. As an application of this result, we find a full descriptive characterization for the $\mathcal{P}$-adic-path integral defined by a bounded sequence $\mathcal{P}$.

2. Preliminaries

We introduce some notation. If $E \subset \mathbb{R}$, then $|E|$ and $|E|_e$ denote respectively the Lebesgue measure and the outer Lebesgue measure of E. The terms "almost everywhere" (briefly a.e.) and "measurable" are always used in the sense of the Lebesgue measure. Let

$$\mathcal{P} = \{p_j\}_{j=0}^{\infty} \tag{1}$$

be a fixed sequence of integers with $p_j > 1$ for $j = 0, 1, \ldots$.

We set $m_0 = 1$, $m_k = \prod_{j=0}^{k-1} p_j$ for $k \geqslant 1$. We call the closed intervals

$$\left[\frac{r}{m_k}, \frac{r+1}{m_k}\right] = \triangle_r^{(k)}, \quad r = 0, 1, \dots, m_k - 1$$

for fixed $k = 0, 1, \dots$ the $\mathcal{P}$-adic intervals (or simply $\mathcal{P}$-intervals) of rank k. In what follows we denote by the symbol $\triangle^{(k)}$ any $\mathcal{P}$-adic interval of rank k.

Let $Q_{\mathcal{P}}$ be the set of all *$\mathcal{P}$-adic rational points of* $[0,1]$, i.e. the points of the form $\frac{r}{m_k}$ with $0 \leqslant r \leqslant m_k$ and $k = 0, 1, \dots$. The complement in $[0,1]$ is the set of all *$\mathcal{P}$-adic irrational points in* $[0,1]$. For each $\mathcal{P}$-adic irrational point $x \in [0,1]$ there exists only one $\mathcal{P}$-adic interval $\triangle^{(k)}(x) = [a_k(x), b_k(x)]$ of rank k containing x, so that $x \in \triangle^{(k)}(x)$. We call the sequence

$$\{[a_k(x), b_k(x)]\}_{k=0}^{\infty}$$

of nested intervals, the *basic sequence convergent to x*. With each $\mathcal{P}$-adic rational point x we can associate two decreasing sequences of $\mathcal{P}$-intervals for which x is the common end-point, starting with some k. So, for such a point we have two basic sequences convergent to x: the left one and the right one. Now we can define the $\mathcal{P}$-adic paths. If x is a $\mathcal{P}$-adic irrational point we set $\mathcal{P}_x^- = \{a_k(x)\}$ and $\mathcal{P}_x^+ = \{b_k(x)\}$. The set $\mathcal{P}_x = \mathcal{P}_x^+ \cup \mathcal{P}_x^- \cup \{x\}$ is the *$\mathcal{P}$-adic path leading to x*. If x is a $\mathcal{P}$-adic rational point we denote by $\mathcal{P}_x^-$ and $\mathcal{P}_x^+$ respectively the sequences of left and right end-points of the intervals of the left and right basic sequence. The *$\mathcal{P}$-adic-path system* is the collection $\mathcal{P} = \{\mathcal{P}_x \colon x \in [0,1]\}$.

We call *$\mathcal{P}$-adic-path intervals of rank k attached to a point* $x \in [0,1]$ the intervals $[x, b_k(x)]$ or $[a_k(x), x]$, where $b_k(x) \in \mathcal{P}_x^+$ and $a_k(x) \in \mathcal{P}_x^-$, if $x \in (0,1)$; the interval $[x, b_k(x)]$, if $x = 0$, and the interval $[a_k(x), x]$, if $x = 1$. We denote by the symbol $I^{(k)}(x)$ any $\mathcal{P}$-adic path interval of rank k attached to the point x.

Let $F \colon [0,1] \longrightarrow \mathbb{R}$ be a pointwise function. We also can view F as an additive interval-function if we write $F(I) = F(b) - F(a)$ for each subinterval $I = [a,b]$ of $[0,1]$.

Given a function $F \colon [0,1] \longrightarrow \mathbb{R}$ and a point $x \in [0,1]$, we say that F is *$\mathcal{P}$-adic-path continuous at x* if

$$\lim_{\substack{y \to x \\ y \in \mathcal{P}_x}} F(y) = F(x).$$

We say that F is *$\mathcal{P}$-adic-path differentiable at x* if

$$\lim_{\substack{y \to x \\ y \in \mathcal{P}_x}} \frac{F(y) - F(x)}{y - x} = \lim_{k \to \infty} \frac{F(I^{(k)}(x))}{|I^{(k)}(x)|}$$

exists and is finite. Then we write $F'_{\mathcal{P}}(x) = f(x)$. We also define the *lower* and the *upper $\mathcal{P}$-adic-path derivative* respectively as follows:

$$\begin{aligned}\underline{F}'_{\mathcal{P}}(x) &= \liminf_{\varrho\to 0}\left\{\frac{F(y)-F(x)}{y-x}\colon\ 0<|y-x|<\varrho,\ y\in\mathcal{P}_x\right\}\\ &= \liminf_{k\to\infty}\left\{\frac{F(I^{(k)}(x))}{|I^{(k)}(x)|}\right\},\\ \overline{F'}_{\mathcal{P}}(x) &= \limsup_{\varrho\to 0}\left\{\frac{F(y)-F(x)}{y-x}\colon\ 0<|y-x|<\varrho,\ y\in\mathcal{P}_x\right\}\\ &= \limsup_{k\to\infty}\left\{\frac{F(I^{(k)}(x))}{|I^{(k)}(x)|}\right\}.\end{aligned}$$

3. The Ward Theorem for a $\mathcal{P}$-adic-path system

We recall that a differentiation basis $\mathcal{B}$ satisfies the Ward Theorem whenever *each function is $\mathcal{B}$-differentiable almost everywhere on the set of all points at which at least one of its extreme $\mathcal{B}$-derivatives is finite.*

In this section we will show that the Ward Theorem holds true for each differentiation basis associated with a $\mathcal{P}$-adic-path system defined by a bounded sequence.

We need the following lemma:

Lemma 3.1. *Let $G\colon [0,1] \longrightarrow \mathbb{R}$ be a function and E a subset of $[0,1]$ with $|E|_e > 0$. If the sequence (1) is bounded by $p = \sup\{p_j\}$, and for some positive number $a > 0$ the inequality*

$$0 < \underline{G}'_{\mathcal{P}}(x) < a \tag{2}$$

holds at every point x of E, then for each $\varepsilon > 0$ there exists a $\mathcal{P}$-adic interval $\triangle^{(k)}$ for which we have

$$|\triangle^{(k)}| < \varepsilon,\ |E\cap\triangle^{(k)}|_e > (1-\varepsilon)|\triangle^{(k)}| \text{ and } G(\triangle^{(k)}) < a\cdot p|\triangle^{(k)}|. \tag{3}$$

Proof. By the definition of derivative and by condition (2), for each $x \in E$ there exists $\sigma(x) > 0$ such that $G(I) > 0$ for each $\mathcal{P}$-adic-path interval I attached to x with $|I| < \sigma(x)$.

Let $E_n = \{x \in E\colon \sigma(x) > \frac{1}{n}\}$. It is clear that $E = \bigcup\limits_{n=1}^{\infty} E_n$ and that there exists $\overline{n} \in \mathbb{N}$ such that $|E_{\overline{n}}|_e > 0$. For a fixed $\varepsilon > 0$ we take $\overline{\sigma} \leqslant \min\{\frac{1}{\overline{n}}, \varepsilon, \frac{1}{p}\}$. So we have

$$G(I) > 0 \tag{4}$$

whenever I is a $\mathcal{P}$-adic-path interval attached to $x \in E_{\overline{n}}$ with $|I| < \overline{\sigma}$.

Let $x_0 \in E_{\overline{n}}$ be a point of density for the set $E_{\overline{n}}$. We can assume that x_0 is a $\mathcal{P}$-adic irrational point.

By virtue of $\underline{G}'_P(x) < a$ and by the density we can determine a $\mathcal{P}$-adic-path interval J attached to x_0 such that

$$|J| < \overline{\sigma},\ |J \cap E_{\overline{n}}|_e > (1-\overline{\sigma}^2)|J| \text{ and } G(J) < a|J|. \tag{5}$$

It follows in particular that

$$|E_{\overline{n}} \cap I|_e > (1-\overline{\sigma})|I| \tag{6}$$

for any interval $I \subset J$ such that

$$|I| > \overline{\sigma}|J|. \tag{7}$$

In fact, the inclusion

$$E_{\overline{n}} \cap J \subset (E_{\overline{n}} \cap I) \cup (J \setminus I)$$

and (5), (7) imply that

$$\begin{aligned} |E_{\overline{n}} \cap I|_e &\geqslant |E_{\overline{n}} \cap J|_e - |J| + |I| > (1-\overline{\sigma}^2)|J| - |J| + |I| \\ &= |I| - \overline{\sigma}^2|J| > |I| - \overline{\sigma}|I| = |I|(1-\overline{\sigma}). \end{aligned}$$

Let $\triangle_j^{(k)}$, $j = 1, 2, \ldots, m$, be $\mathcal{P}$-adic intervals of minimal rank k contained in J and put

$$K = \overline{J \setminus \left(\bigcup_{j=1}^{m} \triangle_j^{(k)} \right)}.$$

We note that K is a $\mathcal{P}$-adic -path interval attached to x_0.

For any $\mathcal{P}$-adic interval $\triangle_j^{(k)} \subset J$ we get

$$|\triangle_j^{(k)}| = \frac{1}{m_k} = \frac{1}{p_k}|\triangle^{(k-1)}| \geqslant \frac{1}{p_k}|J| \geqslant \frac{1}{p}|J| \geqslant \overline{\sigma}|J|, \tag{8}$$

where $\triangle^{(k-1)}$ is the $\mathcal{P}$-adic interval of rank $k-1$ with $\triangle^{(k-1)} \supset J$.

By (6) applied to $\triangle_j^{(k)}$ instead of I, we have

$$|E \cap \triangle_j^{(k)}|_e \geqslant |E_{\overline{n}} \cap \triangle_j^{(k)}|_e > (1-\overline{\sigma})|\triangle_j^{(k)}| > (1-\varepsilon)|\triangle_j^{(k)}|. \tag{9}$$

As $\triangle_j^{(k)} \subset J$ we note that $|\triangle_j^{(k)}| \leqslant |J| \leqslant \overline{\sigma} < \varepsilon$.

By (9) we deduce in particular that

$$E_{\overline{n}} \cap \triangle_j^{(k)} \neq \emptyset \quad \text{for each } j = 1, \dots, m. \tag{10}$$

Now we can write J as

$$J = K \cup \left(\bigcup_{j=1}^{m} \triangle_j^{(k)} \right) \tag{11}$$

Because of (10) we can represent $\triangle_j^{(k)} = I_j^{(k)^-} \cup I_j^{(k)^+}$, where $I_j^{(k)^-}$ and $I_j^{(k)^+}$ are the two $\mathcal{P}$-adic-path intervals of rank k attached to some $x_j \in \triangle_j^{(k)} \cap E_{\overline{n}}$.

By (4) applied to $I_j^{(k)^-}$ and $I_j^{(k)^+}$ and by the additivity of the interval-function G it follows that

$$G(\triangle_j^{(k)}) = G(I_j^{(k)^-}) + G(I_j^{(k)^+}) > 0. \tag{12}$$

Because $x_0 \in E_{\overline{n}} \cap K$, $K \subset J$, $|K| < |J| < \overline{\sigma}$, we can directly apply inequality (4) to K; so we get

$$G(K) > 0. \tag{13}$$

Using (11) as a representation of J we have

$$G(J) = G(K) + \sum_{j=1}^{m} G(\triangle_j^{(k)}), \tag{14}$$

each of the terms of the sum on the right hand side being positive.

Then for any $j = 1, \dots, m$ we get by (13), (14), (5), (8)

$$G(\triangle_j^{(k)}) < G(J) < a|J| < a \cdot p|\triangle_j^{(k)}|. \tag{15}$$

So we can take any $\triangle_j^{(k)}$ as $\triangle^{(k)}$ in the claim of the lemma and this completes the proof. □

Lemma 3.2. *Let $G\colon [0,1] \longrightarrow \mathbb{R}$ be a function, $E \subset [0,1]$, $\triangle^{(h)}$ a $\mathcal{P}$-adic interval of rank h, $\varepsilon > 0$ and b arbitrary fixed numbers. Suppose that*

(i) $|E \cap \triangle^{(h)}|_e > (1-\varepsilon)|\triangle^{(h)}|$,

(ii) *$G(I) > 0$ for each $\mathcal{P}$-adic interval I such that $I \subset \triangle^{(h)}$ and $I \cap E \neq \emptyset$,*

(iii) *$\overline{G'_{\mathcal{P}}}(x) > b$ for each $x \in E$.*

Then

$$G(\triangle^{(h)}) > \frac{b}{p}(1-p\varepsilon)|\triangle^{(h)}|$$

where $p = \sup\{p_i\}$ of the sequence $\mathcal{P}$.

P r o o f. Since the proof is very similar to that of Lemma 11.8 in [7] where it was formulated for the ordinary interval basis, we omit it. We only observe that in our case, instead of the result 11.9 of [7] it needs to use the following lemma.

Lemma 3.3. *Let G, E and $\triangle^{(h)}$ be as in Lemma 3.2. Given any $\eta > 0$, we can associate with any point $x \in E$ a $\mathcal{P}$-adic interval $\triangle^{(k)}(x) \subset \triangle^{(h)}$ of rank k such that*

$$x \in \triangle^{(k)}(x),\ G(\triangle^{(k)}(x)) > \frac{b}{p}|\triangle^{(k)}(x)|,\ |\triangle^{(k)}(x)| < \eta. \tag{16}$$

P r o o f. Let x be a fixed point of E. By (iii) it follows that there exists at least one $\mathcal{P}$-adic-path interval J attached to x such that

$$G(J) > b|J| \quad \text{and} \quad |J| < \frac{\eta}{p}. \tag{17}$$

Let $k+1$ be the minimal rank greater than h (i.e. $k+1 > h$), such that $\triangle^{(k+1)} \subset J$. Then we take $\triangle^{(k)}(x) = \triangle^{(k)} \supset \triangle^{(k+1)}$. By the construction it follows that $x \in \triangle^{(k)}$. Since $k \geqslant h$ we have $\triangle^{(k)} \subset \triangle^{(h)}$. Also $|\triangle^{(k)}| = p_k|\triangle^{(k+1)}| \leqslant p|J| < p\frac{\eta}{p} = \eta$. We define $J' = \overline{\triangle^{(k)} - J}$ and get

$$x \in J',\ J' \subset \triangle^{(h)} \text{ and } x \in J' \cap E.$$

Applying (ii) we have

$$G(J') > 0. \tag{18}$$

Then by the additivity of G and by (18) we get

$$G(\triangle^{(k)}) = G(J') + G(J) \geqslant G(J) > b|J| \geqslant b|\triangle^{(k+1)}| = \frac{b|\triangle^{(k)}|}{p_k} \geqslant \frac{b}{p}|\triangle^{(k)}|,$$

and this completes the proof.

Theorem 3.1. *Let the sequence $\mathcal{P}$ be bounded. Any function $F\colon [0,1] \to \mathbb{R}$ is $\mathcal{P}$-adic-path derivable at almost all points x at which $\overline{F'}_{\mathcal{P}}(x) < +\infty$ or $\underline{F}'_{\mathcal{P}}(x) > -\infty$.*

Proof. First we define

$$D = \{x \in [0,1] \colon \underline{F}'_{\mathcal{P}}(x) > -\infty\}$$

and a subset A of D,

$$A = \{x \in D \colon \overline{F'}_{\mathcal{P}}(x) > \underline{F}'_{\mathcal{P}}(x)\}.$$

If we suppose that $|A|_e > 0$, then we can determine a number $a > 0$ and a set $B \subset A$, $|B|_e > 0$ such that $\underline{F}'_{\mathcal{P}}(x) \neq \infty$ and

$$\overline{F'}_{\mathcal{P}}(x) - \underline{F}'_{\mathcal{P}}(x) > a \text{ at each point } x \in B. \tag{19}$$

Given a positive number ε, we set

$$B_q = \{x \in B \colon q\varepsilon < \underline{F}'_{\mathcal{P}}(x) \leqslant (q+1)\varepsilon\}. \tag{20}$$

Let q_0 be an integer for which $|B_{q_0}|_e > 0$. We can determine a number $\sigma > 0$ and a set $E \subset B_{q_0}$, $|E|_e > 0$ such that $F(I) > q_0\varepsilon|I|$ for each $\mathcal{P}$-adic-path interval I attached to $x \in E$ such that $|I| < \sigma$ (so $E \cap I \neq \emptyset$).

Now we define an additive interval-function G by

$$G(I) = F(I) - q_0\varepsilon|I|$$

for any interval $I \subset [0,1]$. Thus the function G fulfils

$$0 < \underline{G}'_{\mathcal{P}}(x) < 2\varepsilon \tag{21}$$

and by (19) and (20)

$$\overline{G'}_{\mathcal{P}}(x) = \overline{F'}_{\mathcal{P}}(x) - q_0\varepsilon > \underline{F}'_{\mathcal{P}}(x) + a - q_0\varepsilon > a \tag{22}$$

at any point $x \in E$. Moreover,

$$G(I) > 0 \tag{23}$$

for any $\mathcal{P}$-adic-path interval I attached to $x \in E$ such that $|I| < \sigma$.

We note that, splitting eventually the $\mathcal{P}$-adic interval, we can state that (23) is still true for a $\mathcal{P}$-adic interval I' such that $|I'| < \sigma$ and $I' \cap E \neq \emptyset$.

By Lemma 3.1 there exists a $\mathcal{P}$-adic interval $\triangle^{(h)}$ of rank h such that

$$|\triangle^{(h)}| < \varepsilon < \sigma,\ |E \cap \triangle^{(h)}|_e > (1-\varepsilon)|\triangle^{(h)}| \tag{24}$$

and

$$G(\triangle^{(h)}) < 2\varepsilon p|\triangle^{(h)}|. \tag{25}$$

From (24), (22) and (23) and using Lemma 3.2 we get

$$G(\triangle^{(h)}) > \frac{a}{p}(1-p\varepsilon)|\triangle^{(h)}|.$$

Thus $\frac{a}{p}(1-p\varepsilon) < 2\varepsilon p$ for each $\varepsilon > 0$ and this is impossible. Hence $|A|_e = 0$, i.e. $\overline{F'}_{\mathcal{P}}(x) = \underline{F'}_{\mathcal{P}}(x)$ for almost all x for which $\underline{F'}_{\mathcal{P}}(x) > -\infty$.

We have only to prove that the set

$$C = \{x \in [0,1]\colon\ F'_{\mathcal{P}}(x) = +\infty\}$$

is of measure zero, i.e. $|C|_e = 0$.

If we suppose $|C|_e > 0$, then as in the proof of Lemma 3.1 there exists a number $\eta > 0$ such that $F(I) > 0$ whenever I is a $\mathcal{P}$-adic-path interval attached to a point x of C with $|I| < \eta$. Splitting, if necessary, the interval we can write the previous statement for a $\mathcal{P}$-adic interval. If we denote by R any $\mathcal{P}$-adic interval of rank h such that $|R \cap C| > (1 - \frac{1}{2p})|R|$ and $|R| < \eta$, and use a density argument, from Lemma 3.2 we get that $F(R) > \frac{b}{2p}|R|$ for every positive real number b, and this is a contradiction.

4. Application to the $\mathcal{P}$-adic-path integral

In this section, as an application of the Ward Theorem, we find a full descriptive characterization of the $\mathcal{P}$-adic-path integral, in the case the sequence $\mathcal{P}$ is bounded. We recall some definitions.

Given a positive function $\delta\colon [0,1] \to \mathbb{R}$ we call the collection C_δ of interval-point pairs (I,x) with $x \in I \subset [0,1]$ and $I = [y,z]$ where $y,z \in \mathcal{P}_x$, $y \leqslant x \leqslant z$ and $0 < z - y < \delta(x)$ the *$\mathcal{P}$-adic-path-full cover of* $[0,1]$ *associated to* δ. If all (I,x) in the collection C_δ have the point $x \in E \subset [0,1]$ then we will write $C_\delta(E)$.

A *partition of* $[0,1]$ is a family of interval-point pairs $\{(I_j,x_j)\}_{j=1}^n$ for which $x_j \in I_j \subset [0,1]$ and $\mathring{I}_j \cap \mathring{I}_i = \emptyset$ for $i \neq j$ and

$$\bigcup_{j=1}^{n} I_j = [0,1].$$

Proposition 4.1. *Let $\mathcal{P} = \{\mathcal{P}_x \colon x \in [0,1]\}$ be the system of $\mathcal{P}$-adic paths. If C_δ is a $\mathcal{P}$-adic-path-full cover of the interval $[0,1]$, then C_δ must contain a partition of every subinterval of $[0,1]$.*

(A version of this proposition is in Lemma 1.2.1 and Corollary 1.2.2 of [6].)

Definition 4.1. A function $f\colon [0,1] \to \mathbb{R}$ is said to be *$\mathcal{P}$-adic-path integrable* (briefly *$H_{\mathcal{P}}$-integrable*) on $[0,1]$ to A, if for every $\varepsilon > 0$ there is a $\mathcal{P}$-adic-path-full cover C_δ of $[0,1]$ such that for any partition $D = \{([u,v],x)\}$ from C_δ we have

$$\Bigl|\sum f(x)(v-u) - A\Bigr| < \varepsilon.$$

We denote the number A by the symbol $(H_{\mathcal{P}}) \int_0^1 f = A$.

The $\mathcal{P}$-adic-path integral has the following properties (see [4]):

(p_1) *If f is $\mathcal{P}$-adic-path integrable on $[0,1]$, then it is also $\mathcal{P}$-adic-path integrable on each subinterval of $[0,1]$.*

Therefore the indefinite $\mathcal{P}$-adic-path integral $F(x) = (H_{\mathcal{P}}) \int_0^x f$ is defined for any $x \in [0,1]$.

(p_2) *The $\mathcal{P}$-adic-path indefinite integral F of f is $\mathcal{P}$-adic-path continuous at each $x \in [0,1]$, and it is $\mathcal{P}$-adic path differentiable a.e. with $F'_{\mathcal{P}}(x) = f(x)$ a.e. on $[0,1]$.*

In order to study the primitives of the $\mathcal{P}$-adic path integral it is useful to introduce the following notion of variational measure (see [1], [2] and [8]).

Given a function $F\colon [0,1] \to \mathbb{R}$, a set $E \subset R$ and a $\mathcal{P}$-adic-path-full cover $C_\delta(E)$ on $[0,1]$, we define the *δ-variation of F on E* by

$$\mathrm{Var}\,(C_\delta(E), F) = \sup \sum_{(I,x)\in\pi} |F(I)|,$$

where the "sup" is taken over all π partitions of $[0,1]$ from $C_\delta(E)$.

Then we define *$\mathcal{P}$-adic-path variational measure* by

$$V_F^{\mathcal{P}}(E) = \inf \mathrm{Var}\,(C_\delta(E), F),$$

where the "inf" is taken over all $\mathcal{P}$-adic-path-full covers $C_\delta(E)$.

We observe that $V_F^{\mathcal{P}}$ is a metric outer measure on $[0,1]$ (see [8]). So its restriction to Borel sets is a measure.

We recall that a measure μ is said to be *absolutely continuous* with respect to the Lebesgue measure if $|N| = 0$ implies $\mu(N) = 0$.

In [4] Theorem 4, using the equivalent definition of "strong Lusin condition" in the place of "variational measure absolutely continuous" the following property is proved:

(p$_3$) *A function $F\colon [0,1] \to \mathbb{R}$ is the indefinite $\mathcal{P}$-adic-path integral of a function f if and only if F generates a variational measure absolutely continuous with respect to the Lebesgue measure and F is $\mathcal{P}$-adic-path differentiable a.e. with $F'_{\mathcal{P}}(x) = f$ a.e. on $[0,1]$.*

The above property is also called a *partial descriptive characterization.* This means that we need the hypothesis of $\mathcal{P}$-adic-path differentiability of F a.e.

A descriptive characterization of the $H_{\mathcal{P}}$-integral is called a *full descriptive characterization* if no differentiability assumption is supposed a priori.

We need the following results.

Theorem 4.1. *Let F be a function $\mathcal{P}$-adic-path continuous on $[0,1]$ and let $E \subset [0,1]$ be a closed set. If the variational measure $V_F^{\mathcal{P}}$ is σ-finite on all negligible Borel subsets of E then it is σ-finite on E.*

The proof follows as in [3] Theorem 4.3, with minor changes. Hence we omit it.

Corollary 4.2. *Let F be a function on $[0,1]$ and let $E \subset [0,1]$ be a closed set. If the variational measure $V_F^{\mathcal{P}}$ is absolutely continuous on E, then it is σ-finite.*

Proof. Since the measure $V_F^{\mathcal{P}}$ is absolutely continuous, hence F is $\mathcal{P}$-adic path continuous and we can apply Theorem 4.1.

Proposition 4.2. *Let F be a function on $[0,1]$ and let $E \subset [0,1]$ be a Borel subset of $[0,1]$. If the variational measure $V_F^{\mathcal{P}}$ is σ-finite on E, then the extreme $\mathcal{P}$-adic-path derivative is finite almost everywhere on E.*

The proof follows as that of [1] p. 6 or [8] p. 850, where it is written for the ordinary extreme derivative.

Proposition 4.3. *Let the sequence $\mathcal{P}$ be bounded, let $F\colon [0,1] \to \mathbb{R}$ be a function. If the variational measure $V_F^{\mathcal{P}}$ is absolutely continuous on $[0,1]$, then F is $\mathcal{P}$-adic-path differentiable a.e. on $[0,1]$.*

Proof. Let $D_\infty = \{x \in [0,1]\colon \overline{F'}_{\mathcal{P}}(x) = +\infty\} \cup \{x \in [0,1]\colon \underline{F'}_{\mathcal{P}}(x) = -\infty\}$. First we will prove that $|D_\infty| = 0$. The absolute continuity of $V_F^{\mathcal{P}}$ implies the $\mathcal{P}$-adic-path continuity of F on $[0,1]$. Then the function F is measurable and also the lower and upper $\mathcal{P}$-adic-path derivatives are measurable. Therefore the set D_∞ is measurable. Let K be any closed subset of D_∞. By Corollary 4.2 we have that the variational measure $V_F^{\mathcal{P}}$ is σ-finite on $[0,1]$. Then $V_F^{\mathcal{P}}$ is σ-finite on K and by Proposition 4.2, $|K| = 0$. Since this is true for every closed subset of D_∞, also $|D_\infty| = 0$. Now, by recalling that the sequence $\mathcal{P}$ is bounded, as an application of Theorem 3.1 we get the result.

Theorem 4.3. *Let the sequence $\mathcal{P}$ be bounded. A function $F\colon [0,1] \to \mathbb{R}$ is the indefinite $\mathcal{P}$-adic-path integral of a function f if and only if the variational measure $V_F^{\mathcal{P}}$ is absolutely continuous on $[0,1]$.*

Proof. It follows at once from the property (p_3) and from Proposition 4.3.

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