MULTIPLIERS OF TEMPERATE DISTRIBUTIONS

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Abstract. Spaces $\mathcal{O}_q$, $q \in \mathbb{N}$, of multipliers of temperate distributions introduced in an earlier paper of the first author are expressed as inductive limits of Hilbert spaces.

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We denote by $L_{\rm loc}$ the space of all locally Lebesgue integrable functions $f\colon \mathbb{R}^n \to \mathbb{R}$ and by $\mathcal{D}$ the space of all C^∞-functions, defined on $\mathbb{R}^n$, with a compact support. For $\alpha \in \mathbb{N}^n$, $x \in \mathbb{R}^n$, we write $|\alpha| = \sum\limits_{i=1}^n \alpha_i$, $x^\alpha = \prod\limits_{i=1}^n x_i^{\alpha_i}$, and $D^\alpha = \partial|\alpha|/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\dots\partial x_n^{\alpha_n}$. It is convenient to use a weight function $w(x) = \bigl(1 + \sum\limits_{i=1}^n x_i^2\bigr)^{1/2}$ and a constant $r = 1 + [\frac12 n]$, where $[t]$ is the greatest integer less or equal to t, $t \in \mathbb{R}$.

A function $f \in L_{\rm loc}$ has a generalized derivative $g \in L_{\rm loc}$ of order $\alpha \in \mathbb{N}^n$ if for all $\varphi \in \mathcal{D}$ we have $\int_{\mathbb{R}^n} f D^\alpha \varphi \,\mathrm{d}x = (-1)^{|\alpha|} \int_{\mathbb{R}^n} g\varphi \,\mathrm{d}x$. We denote by $\mathcal{S}_k$, $k \in \mathbb{N}$, the space of all functions $f\colon \mathbb{R}^n \to \mathbb{R}$ which have generalized derivatives of all orders less or equal to k and satisfy $\|f\|_k = \sum\limits_{|\alpha+\beta|\leqslant k} (\int_{\mathbb{R}^n} |x^\alpha D^\beta f(x)|^2 \,\mathrm{d}x)^{1/2} < +\infty$. Each space $\mathcal{S}_k$ with the norm $f \longmapsto \|f\|_k$ is Hilbert and the Schwartz space $\mathcal{S}$ of rapidly decreasing functions is the projective limit $\operatorname{proj} \mathcal{S}_k$, see [6]. We denote by $\mathcal{S}_{-k}$, $k \in \mathbb{N}$, the strong dual of $\mathcal{S}_k$. Then the space $\mathcal{S}'$ of temperate distributions, defined by Schwartz, is the inductive limit $\operatorname{ind} \mathcal{S}_{-k}$, see [5].

Let $\mathcal{L}_\beta(\mathcal{S}_p, \mathcal{S}_q)$ be the space of all continuous linear operators from $\mathcal{S}_p$ into $\mathcal{S}_q$ equipped with the bounded topology. For any $p, q \in \mathbb{N}$, $p \geqslant q$, we denote by $\mathcal{O}_{p,q}$ the set of all functions $u\colon \mathbb{R}^n \to \mathbb{R}$ for which the mapping $f \mapsto uf\colon \mathcal{S}_p \to \mathcal{S}_q$ is continuous. Then $\mathcal{O}_{p,q}$ is a closed subspace of the Banach space $\mathcal{L}_\beta(\mathcal{S}_p,\mathcal{S}_q)$ and as such it is also Banach. We denote its norm by $\|\cdot\|_{p,q}$. Evidently $\mathcal{O}_{p,q}\subset\mathcal{O}_{p+1,q}$, $p\geqslant q$, and for every $u\in\mathcal{O}_{p,q}$, we have $\|u\|_{p+1,q}\leqslant\|u\|_{p,q}$. Hence the identity map $\mathrm{id}\colon\mathcal{O}_{p,q}\to\mathcal{O}_{p+1,q}$ is continuous and the inductive limit $\mathrm{ind}\{\mathcal{O}_{p,q};\ p\to\infty\}$ makes sense. We denote it by $\mathcal{O}_q$. It was proved in [6] that $\mathcal{O}_q$ is the set of all functions $u\colon\mathbb{R}^n\to\mathbb{R}$ for which $f\mapsto uf$ is a continuous mapping from $\mathcal{S}_{-q}$ into $\mathcal{S}'$.

Finally, we use two classical Banach spaces of functions, measurable on $\mathbb{R}^n$, namely L^1 and L^∞. The norm in L^∞ is denoted by $\|u\|_\infty=\operatorname{ess\,sup}\{|u(x)|;\ x\in\mathbb{R}^n\}$.

Lemma 1. *$\mathcal{S}_r\subset L^\infty$ and the identity map $\mathrm{id}\colon\mathcal{S}_r\to L^\infty$ is continuous.*

P r o o f. The Fourier transformation $u\mapsto\hat u=\int_{\mathbb{R}^n}u(x)\exp(-2\pi\mathrm{i}x,\xi)\,\mathrm{d}x$ is a topological isomorphism on $\mathcal{S}_r$. Hence the Fourier transformation $\hat u$ of a function $u\in\mathcal{S}_r$ is also in $\mathcal{S}_r$ and $\int_{\mathbb{R}^n}|\hat u|\,\mathrm{d}\xi=\int_{\mathbb{R}^n}|w^{-r+r}\hat u|\,\mathrm{d}\xi\leqslant\|w^{-r}\|_0\cdot\|w^r\hat u\|_0\leqslant\|w^{-r}\|_0\cdot\|\hat u\|_r$.

Then the function u, as an inverse Fourier transformation of $\hat u\in L^1$, is uniformly continuous on $\mathbb{R}^n$, hence measurable, and bounded by the constant $\|w^{-r}\|_0\cdot\|\hat u\|_r$.

Finally, $\mathrm{id}\colon\mathcal{S}_r\to L^\infty$ is the composition of three continuous maps $u\mapsto\hat u\mapsto\hat u\mapsto u\colon\mathcal{S}_r\to\mathcal{S}_r\to L^1\to L^\infty$. □

Lemma 2. *For any $k\in\mathbb{N}$, there exists a constant $C_k>0$ such that $\|w^{k-|\alpha|}D^\alpha u\|_\infty\leqslant C_k\cdot\|u\|_{k+r}$ for any $\alpha\in\mathbb{N}^n$, $|\alpha|\leqslant k$, and any $u\in\mathcal{S}_{k+r}$.*

P r o o f. Take $k\in\mathbb{N}$, $\alpha\in\mathbb{N}^n$, $|\alpha|\leqslant k$, and $u\in\mathcal{S}_{k+r}$. Then $u_\alpha=w^{k-|\alpha|}D^\alpha u\in\mathcal{S}_r$ and, by Lemma 1, there exists a constant $C_\alpha>0$, which does not depend on the choice of u, such that $\|u_\alpha\|_\infty\leqslant C_\alpha\|u_\alpha\|_r\leqslant C_\alpha\|u\|_{k+r}$. Lemma 2 holds for $C_k=\max\{C_\alpha;\ \alpha\in\mathbb{N}^n,|\alpha|\leqslant k\}$. □

Definition. For any $p,q\in\mathbb{N}$, let $H_{p,q}$ be the space $\{u\colon\mathbb{R}^n\to\mathbb{R};\ \forall\alpha\in\mathbb{N}^n$, $|\alpha|\leqslant q$, $\exists$ generalized derivative $D^\alpha u$ and $\|w^{-p}D^\alpha u\|_0<\infty\}$ with the scalar product $\langle u,v\rangle=\sum\limits_{|\alpha|\leqslant q}\int_{\mathbb{R}^n}w^{-2p}D^\alpha uD^\alpha v\,\mathrm{d}x$. We denote the correponding norm by $|\!|\!|\cdot|\!|\!|_{p,q}$.

For any $q\in\mathbb{N}$, we have $H_{1,q}\subset H_{2,q}\subset\ldots$ with the inclusions continuous. Hence the inductive limit $\mathrm{ind}\{H_{p,q};\ p\to\infty\}$ makes sense. We denote it by H_q.

Lemma 3. *Let a function $u\colon\mathbb{R}^n\to\mathbb{R}$ have the generalized derivative $\partial u/\partial x_1$ and $v\in\mathcal{S}$. Then the generalized derivative $\partial/\partial x_1(uv)$ also exists and equals to $(\partial u/\partial x_1)v+u(\partial v/\partial x_1)$.*

P r o o f. Take $\varphi \in \mathcal{D}$. Then $\varphi v \in \mathcal{D}$ and

$$\begin{aligned}\int_{\mathbb{R}^n}\Big(\frac{\partial u}{\partial x_1}v+u\frac{\partial v}{\partial x_1}\Big)\varphi\,\mathrm{d}x &= \int_{\mathbb{R}^n}\frac{\partial u}{\partial x_1}(v\varphi)\,\mathrm{d}x+\int_{\mathbb{R}^n}u\frac{\partial v}{\partial x_1}\varphi\,\mathrm{d}x\\ &= -\int_{\mathbb{R}^n}u\frac{\partial}{\partial x_1}(v\varphi)\,\mathrm{d}x+\int_{\mathbb{R}^n}u\frac{\partial v}{\partial x_1}\varphi\,\mathrm{d}x\\ &= -\int_{\mathbb{R}^n}u\Big(\frac{\partial}{\partial x_1}(v\varphi)-\frac{\partial v}{\partial x_1}\varphi\Big)\,\mathrm{d}x=-\int_{\mathbb{R}^n}(uv)\frac{\partial\varphi}{\partial x_1}\,\mathrm{d}x.\end{aligned}$$

□

Lemma 4. *Let $u \in \mathcal{O}_{p,q}, p, q \in \mathbb{N}, p \geqslant q$. Then $\frac{\partial u}{\partial x_1} \in \mathcal{O}_{p+1,q-1}$ and $\|u\|_{p+1,q-1} \leqslant \|u\|_{pq}$.*

P r o o f. Take $v \in \mathcal{S}$. Then $\|\frac{\partial u}{\partial x_1}v\|_{q-1} = \|\frac{\partial}{\partial x_1}(uv) - u\frac{\partial v}{\partial x_1}\|_{q-1} \leqslant \|uv\|_q + \|u\frac{\partial v}{\partial x_1}\|_q \leqslant \|u\|_{p,q}(\|v\|_p + \|\frac{\partial v}{\partial x_1}\|_p) \leqslant \|u\|_{p,q}\cdot 2\|v\|_{p+1}$. Since the space $\mathcal{S}$ is dense in $\mathcal{S}_{p+1}$, the proof is complete. □

Proposition 1. *$H_{p,q} \subset \mathcal{O}_{p+q+r,q}$ for any $p, q \in \mathbb{N}$. The identity map* id: $H_{p,q} \to \mathcal{O}_{p+q+r,q}$ *is continuous.*

P r o o f. Take $u \in H_{p,q}$ and put, for brevity, $s = p+q+r$. Since the space $\mathcal{S}$ is dense in $\mathcal{S}_s$, it is sufficient to show that there is a constant $C > 0$, which does not depend on u, such that $\sup\{\|uv\|_q;\ v \in \mathcal{S}, \|v\|_s \leqslant 1\} \leqslant C\cdot |\!|\!| u |\!|\!|_{p,q}$. □

By Lemma 3, for any $v \in \mathcal{S}$ and any $\alpha \in \mathbb{N}^n$, $|\alpha| \leqslant q$, the generalized derivative $D^\alpha(uv)$ exists and can be computed by Leibniz's rule. There are constants $A, B > 0$, independent on u and on $\alpha \in \mathbb{N}^n$, $|\alpha| \leqslant q$, such that $\|w^{q-|\alpha|}D^\alpha(uv)\|_0 \leqslant A\sum\limits_{\beta+\gamma=\alpha}\|w^{q-|\alpha|}D^\beta u D^\gamma v\|_0$ and $\sum\limits_{|\alpha|\leqslant q}\sum\limits_{\beta\leqslant\alpha}\|w^{-p}D^\beta u\|_0 \leqslant B\cdot |\!|\!| u |\!|\!|_{p,q}$.

Now for the constant C_k from Lemma 2, where $k = p+q$, and for $v \in \mathcal{S}, \|u\|_s \leqslant 1$, we have

$$\begin{aligned}\|uv\|_q &\leqslant \sum_{|\alpha|\leqslant q}\|w^{q-|\alpha|}D^\alpha(uv)\|_0 \leqslant A\sum_{|\alpha|\leqslant q}\sum_{\beta+\gamma=\alpha}\|w^{q-|\alpha|}D^\beta u D^\gamma v\|_0\\ &\leqslant A\sum_{|\alpha|\leqslant q}\sum_{\beta+\gamma=\alpha}\|w^{-p}D^\beta u\|_0\cdot\|w^{p+q-|\alpha|}D^\gamma v\|_\infty\\ &\leqslant A\sum_{|\alpha|\leqslant q}\sum_{\beta+\gamma=\alpha}\|w^{-p}D^\beta u\|_0\cdot\|w^{p+q-|\gamma|}D^\gamma v\|_\infty\\ &\leqslant AB\cdot |\!|\!| u |\!|\!|_{p,q}\cdot\sum_{|\gamma|\leqslant q}\|w^{p+q-|\gamma|}D^\gamma v\|_\infty \leqslant AB\cdot |\!|\!| u |\!|\!|_{p,q}\cdot C_{p+q}\Big(\sum_{|\gamma|\leqslant q}1\Big)\|v\|_s.\end{aligned}$$

This implies $\|u\|_{p+q+r,q} \leqslant ABC_{p+q}\Big(\sum\limits_{|\gamma|\leqslant q}1\Big)\cdot |\!|\!| u |\!|\!|_{p,q}$.

Lemma 5. *Let a function* $\varphi \in \mathcal{D}(\mathbb{R})$ *be even with* $\operatorname{supp}\varphi \subset [-2,2]$, $0 \leqslant \varphi(t) \leqslant 1$ *for* $t \in \mathbb{R}$, *and* $\varphi(t) = 1$ *for* $t \in [-1,1]$. *For* $\lambda \geqslant 0$ *put*

$$\varphi_\lambda(t) = \begin{cases} 1 & \text{if } |t| \leqslant \lambda + 1, \\ \varphi(t - \lambda \operatorname{sgn} t) & \text{if } |t| \geqslant \lambda + 1 \end{cases}$$

and $\psi_\lambda(x) = \prod\limits_{i=1}^{n} \varphi_\lambda(x_i)$ *for* $x \in \mathbb{R}^n$. *Then* $\sup\{\|w^{-k-r}\psi_\lambda\|_k;\ \lambda \geqslant 0\} < \infty$ *for any* $k \in \mathbb{N}$.

P r o o f. It holds

$$\begin{aligned}\|w^{-k-r}\psi_\lambda\|_k &= \sum_{|\alpha+\beta| \leqslant k} \|x^\alpha D^\beta(w^{-k-r}\psi_\lambda)\|_0 \\ &\leqslant \sum_{|\alpha+\beta| \leqslant k} \Big\| x^\alpha \sum_{\gamma+\delta=\beta} \Big[\begin{matrix}\beta \\ \gamma, \delta\end{matrix}\Big] D^\gamma w^{-k-r} \cdot D^\delta \psi_\lambda \Big\|_0.\end{aligned}$$

Each function $D^\delta \psi_\lambda$ is bounded by a constant independent on λ, hence it is sufficient to show that $\int_{\mathbb{R}^n} |x^\alpha D^\gamma w^{-k-r}|^2 \, \mathrm{d}x < +\infty$ for any $\alpha, \gamma \in \mathbb{N}^n$, $|\alpha+\gamma| \leqslant k$. Since $|D^\gamma w^{-k-r}(x)| \leqslant (k+r)^{|\gamma|} w^{-k-r}(x)$ for any $x \in \mathbb{R}^n$, we have $\int_{\mathbb{R}^n} |x^\alpha D^\gamma w^{-k-r}|^2 \, \mathrm{d}x \leqslant (k+r)^{2k} \int_{\mathbb{R}^n} |x^\alpha w^{-k-r}|^2 \, \mathrm{d}x \leqslant (k+r)^{2k} \int_{\mathbb{R}^n} w^{-2r} < +\infty$. $\square$

Proposition 2. $\mathcal{O}_{p,q} \subset H_{p+q+r,q}$ *for any* $p, q \in \mathbb{N}$, $p \geqslant q$. *The identity map* $\mathrm{id}\colon \mathcal{O}_{p,q} \to H_{p+q+r,q}$ *is continuous.*

P r o o f. Put, for brevity, $s = p+q+r$, $B(\lambda) = \{x \in \mathbb{R}^n;\ \|x\| \leqslant \lambda\}$ for $\lambda > 0$, and take $u \in \mathcal{O}_{p,q}$. Let the functions $\psi_\lambda, \lambda \geqslant 0$ be the same as in Lemma 5. $\square$

By Lemma 4, we have $D^\alpha u \in \mathcal{O}_{p+|\alpha|,q-|\alpha|}$ and $\|D^\alpha u\|_{p+|\alpha|,q-|\alpha|} \leqslant \|u\|_{p,q}$ for any $\alpha \in \mathbb{N}^n$, $|\alpha| \leqslant q$. Put $C = \sup\{\|w^{-s}\psi_\lambda\|_{p+|\alpha|};\ \lambda \geqslant 0\}$. By Lemma 5, C is a finite constant. Take $\lambda \geqslant 0$. Then $(\int_{B(\lambda)} |w^{-s} D^\alpha u|^2 \, \mathrm{d}x)^{1/2} \leqslant \|\psi_\lambda w^{-s} D^\alpha u\|_0 \leqslant \|D^\alpha u\|_{p+|\alpha|,q-|\alpha|} \cdot \|\psi_\lambda w^{-s}\|_{p+|\alpha|} \leqslant \|u\|_{p,q} \cdot \|\psi_\lambda w^{-s}\|_{p+|\alpha|} \leqslant C \cdot \|u\|_{p,q}$.

Since this inequality holds for all $\lambda \geqslant 0$, we have $\|w^{-s} D^\alpha u\|_0 \leqslant C \cdot \|u\|_{p,q}$ for any $\alpha \in \mathbb{N}^n$, $|\alpha| \leqslant q$, which implies $|\!|\!|u|\!|\!|_{p,q} \leqslant C \cdot \sum\limits_{|\alpha| \leqslant q} \|u\|_{p,q} \leqslant qn^q C \cdot \|u\|_{p,q}$.

Theorem. *For any* $q \in \mathbb{N}$, *the spaces* $\mathcal{O}_q$ *and* H_q *are the same. Their inductive topologies are the same, too.*

P r o o f. Take $q \in \mathbb{N}$. By Propositions 1 and 2, for any $p, q \in \mathbb{N}$, $p \geqslant q$, we have $H_{p,q} \subset \mathcal{O}_{p+q+r,q} \subset \mathcal{O}_q$ and $\mathcal{O}_{p,q} \subset H_{p+q+r,q} \subset H_q$ with all four inclusions continuous. Hence the identity map $\mathrm{id}\colon H_q \to \mathcal{O}_q$ is a topological isomorphism. $\square$

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