On well/ill posedness of certain problems in fluid mechanics

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Motivation [De Lellis, Székelyhidi]

Incompressible Euler system

$$\partial_t \mathbf{U} + \mathrm{div}_x(\mathbf{U}\otimes\mathbf{U}) + \nabla_x \Pi = 0,\ \ \mathrm{div}_x \mathbf{U} = 0, N = 2,3$$

Equivalent formulation

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{V} = 0,\ \ \mathrm{div}_x \mathbf{U} = 0,\ \ \mathbf{U}\otimes\mathbf{U} - \frac{1}{N}|\mathbf{U}|^2 \mathbb{I} = \mathbb{V}$$

Subsolutions

$$\frac{1}{2}|\mathbf{U}|^2 \leq \frac{N}{2}\lambda_{\max}\left[\mathbf{U}\otimes\mathbf{U} - \mathbb{V}\right] \equiv G(\mathbf{U},\mathbb{V}) \boxed{<} e,\ \ \mathbb{V}\in R^{N\times N}_{0,\mathrm{sym}}$$

Solutions

$$\frac{1}{2}|\mathbf{U}|^2 = e \ \Rightarrow\ \mathbb{V} = \mathbf{U}\otimes\mathbf{U} - \frac{1}{N}|\mathbf{U}|^2\mathbb{I}$$

Oscillatory lemma

Subsolution

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{V} = 0,\ |\mathbf{U}|^2 \leq G(\mathbf{U}, \mathbb{V}) < e$$

Oscillatory perturbation

$$\partial_t \mathbf{u}_\varepsilon + \mathrm{div}_x \mathbb{V}_\varepsilon = 0,\ \mathbf{u}_\varepsilon, \mathbb{V}_\varepsilon \text{ compactly supported}$$

$$G\left(\mathbf{U} + \mathbf{u}_\varepsilon, \mathbb{V} + \mathbb{V}_\varepsilon\right) < e,\ \mathbf{u}_\varepsilon \rightharpoonup 0$$

$$\liminf_{\varepsilon \to 0} \int_B |\mathbf{u}_\varepsilon|^2 \geq \int_B \Lambda\left(e - G(\mathbf{U}, \mathbb{V})\right),\ \Lambda(Z) > 0 \text{ for } Z > 0$$

$$\Rightarrow$$

$$\liminf_{\varepsilon \to 0} \int_B |\mathbf{U} + \mathbf{u}_\varepsilon|^2 \geq \int_B |\mathbf{U}|^2 + \int_B \Lambda\left(e - G(\mathbf{U}, \mathbb{V})\right)$$

Typical results

Good news

The set of subsolutions nonempty $\Rightarrow$ the problem possesses a *global-in-time* solution for *any* initial data

Bad news

The problem possesses *infinitely many* solutions for any initial data

What's wrong? ... more bad news

"Many" solutions violate the energy conservation **but** there is a "large" set of initial data for which the problem admits infinitely many energy conserving (dissipating) solutions

Oscillatory lemma with continuous coefficients

E. Chiodaroli, EF et al.

Hypotheses:

$$U \subset R \times R^N, \ \ N = 2, 3 \text{ bounded open set}$$

$$\tilde{\mathbf{h}} \in C(U; R^N), \ \tilde{\mathbb{H}} \in C(U; R^{N\times N}_{\mathrm{sym},0}), \ \tilde{e}, \ \tilde{r} \in C(U), \ \tilde{r} > 0, \ \tilde{e} \leq \overline{e} \text{ in } U$$

$$\frac{N}{2}\lambda_{\max}\left[\frac{\tilde{\mathbf{h}} \otimes \tilde{\mathbf{h}}}{\tilde{r}} - \tilde{\mathbb{H}}\right] < \tilde{e} \text{ in } U.$$

Conclusion:

$$\mathbf{w}_n \in C^\infty_c(U; R^N),\ \mathbb{G}_n \in C^\infty_c(U; R^{N\times N}_{\mathrm{sym},0}),\ n = 0, 1, \dots$$

$$\partial_t \mathbf{w}_n + \mathrm{div}_x \mathbb{G}_n = 0,\ \mathrm{div}_x \mathbf{w}_n = 0 \text{ in } R \times R^N,$$

$$\frac{N}{2}\lambda_{\max}\left[\frac{(\tilde{\mathbf{h}} + \mathbf{w}_n)\otimes(\tilde{\mathbf{h}} + \mathbf{w}_n)}{\tilde{r}} - (\tilde{\mathbb{H}} + \mathbb{G}_n)\right] < \tilde{e} \text{ in } U,$$

$$\mathbf{w}_n \to 0 \text{ weakly in } L^2(U; R^N)$$

$$\liminf_{n\to\infty} \int_U \frac{|\mathbf{w}_n|^2}{\tilde{r}}\ \mathrm{d}x\mathrm{d}t \geq \Lambda(\overline{e}) \int_U \left(\tilde{e} - \frac{1}{2}\frac{|\tilde{\mathbf{h}}|^2}{\tilde{r}}\right)^2\ \mathrm{d}x\mathrm{d}t$$

Basic ideas of proof

Localization

Localizing the result of DeLellis and Széhelyhidi to “small” cubes by means of scaling arguments

Linearization

Replacing all continuous functions by their means on any of the “small” cubes

Eliminating singular sets

Applying Whitney’s decomposition lemma to the non-singular sets (e.g. out of the vacuum $\{h = 0\}$)

Energy and other coefficients depending on solutions

Applying compactness of the abstract operators in C

Abstract formulation

Variable coefficients "Euler system"

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{H}[\mathbf{v}]) \odot (\mathbf{v} + \mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} + \mathbb{M}[\mathbf{v}] \right) = 0$$

$$\mathrm{div}_x \mathbf{v} = 0,$$

$$\mathbf{v} \odot \mathbf{w} = \mathbf{v} \otimes \mathbf{w} - \frac{1}{2} \mathbf{v} \cdot \mathbf{w} \mathbb{I}$$

Kinetic energy

$$\frac{1}{2} \frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} = E[\mathbf{v}]$$

Data

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Abstract operators

Boundedness

b maps bounded sets in $L^\infty((0,T)\times\Omega;R^N)$ on bounded sets in $C_b(Q,R^M)$

Continuity

$$b[\mathbf{v}_n] \to b[\mathbf{v}] \text{ in } C_b(Q;R^M) \text{ (uniformly for } (t,x)\in Q \text{)}$$

whenever

$$\mathbf{v}_n \to \mathbf{v} \text{ in } C_{\mathrm{weak}}([0,T];L^2(\Omega;R^N))$$

Causality

$\mathbf{v}(t,\cdot)=\mathbf{w}(t,\cdot)$ for $0\le t\le\tau\le T$ implies $b[\mathbf{v}]=b[\mathbf{w}]$ in $[(0,\tau]\times\Omega]$

Results

Result (A)

The set of subsolutions is non-empty $\Rightarrow$ there exists infinitely many weak solutions of the problem with the same initial data

Initial energy jump

$$\frac{1}{2}\frac{|\mathbf{v}_0 + \mathbf{H}[\mathbf{v}_0]|^2}{h[\mathbf{v}_0]} \boxed{<} \liminf_{t\to 0} \frac{1}{2}\frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]}$$

Result (B)

The set of subsolutions is non-empty $\Rightarrow$ there exists a dense set of times such that the values $\mathbf{v}(t)$ give rise to non-empty subsolution set with

$$\frac{1}{2}\frac{|\mathbf{v}_0 + \mathbf{H}[\mathbf{v}_0]|^2}{h[\mathbf{v}_0]} \boxed{=} \liminf_{t\to 0} \frac{1}{2}\frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]}$$

Savage-Hutter model for avalanches

Unknowns

flow height . . . . $h = h(t, x)$
depth-averaged velocity . . . . $\mathbf{u} = \mathbf{u}(t, x)$

$$\partial_t h + \mathrm{div}_x(h\mathbf{u}) = 0$$

$$\partial_t(h\mathbf{u}) + \mathrm{div}_x(h\mathbf{u} \otimes \mathbf{u}) + \nabla_x(ah^2) = h\left(-\gamma\frac{\mathbf{u}}{|\mathbf{u}|} + \mathbf{f}\right)$$

Periodic boundary conditions

$$\Omega = \left([0,1]|_{\{0,1\}}\right)^2$$

Transformation - Step I

Helmholtz decomposition

$$h\mathbf{u} = \mathbf{v} + \mathbf{V} + \nabla_x \Psi$$

where

$$\mathrm{div}_x \mathbf{v} = 0,\ \int_\Omega \Psi \ \mathrm{d}x = 0,\ \int_\Omega \mathbf{v} \ \mathrm{d}x = 0,\ \mathbf{V} \in R^2$$

Fixing h and the potential Ψ

$$\partial_t h + \Delta \Psi = 0$$

$$h(0,\cdot) = h_0,\ -\partial_t h(0,\cdot) = \Delta \Psi_0$$

Problem I

Equation

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{V} + \nabla_x \Psi) \otimes (\mathbf{v} + \mathbf{V} + \nabla_x \Psi)}{h} + \left(a h^2 + \partial_t \Psi \right) \mathbb{I} \right)$$

$$+ \partial_t \mathbf{V}$$

$$= h \left(-\gamma \frac{\mathbf{v} + \mathbf{V} + \nabla_x \Psi}{|\mathbf{v} + \mathbf{V} + \nabla_x \Psi|} + \mathbf{f} \right),$$

Constraints and initial conditions

$$\mathrm{div}_x \mathbf{v} = 0, \ \int_\Omega \mathbf{v}(t, \cdot) \ \mathrm{d}x = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{V}(0) = \mathbf{V}_0$$

Transformation - Step II

Prescribing the kinetic energy

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}+\nabla_x\Psi|^2}{h} = E \equiv \Lambda(t) - ah^2 - \partial_t\Psi$$

Problem II

$$\partial_t\mathbf{v} + \partial_t\mathbf{V}$$

$$+\mathrm{div}_x\left(\frac{(\mathbf{v}+\mathbf{V}+\nabla_x\Psi)\otimes(\mathbf{v}+\mathbf{V}+\nabla_x\Psi)}{h} - \frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}+\nabla_x\Psi|^2}{h}\mathbb{I}\right)$$

$$= -\gamma\left(\frac{h}{2E}\right)^{1/2}(\mathbf{v}+\mathbf{V}+\nabla_x\Psi) + h\mathbf{f}$$

Transformation - Step III

Determining function V

$$\partial_t \mathbf{V} - \left[\frac{1}{|\Omega|} \int_\Omega \gamma \left(\frac{h}{2E} \right)^{1/2} \, \mathrm{d}x \right] \mathbf{V}$$

$$= \frac{1}{|\Omega|} \int_\Omega \left[\gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \nabla_x \Psi) + h\mathbf{f} \right] \, \mathrm{d}x, \;\; \mathbf{V}(0) = \mathbf{V}_0$$

Problem III

Equation

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi) \odot (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi)}{h} \right)$$

$$= -\gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi)$$

$$+ \frac{1}{|\Omega|} \int_\Omega \gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi) \ \mathrm{d}x + h\mathbf{f} - \frac{1}{|\Omega|} \int_\Omega h\mathbf{f} \ \mathrm{d}x$$

$$\mathbf{v} \odot \mathbf{w} = \mathbf{v} \otimes \mathbf{w} - \frac{1}{2} \mathbf{v} \cdot \mathbf{w} \mathbb{I}$$

Transformation - Step IV

Solving elliptic problem

$$\mathrm{div}_x \mathbb{M} \equiv \mathrm{div}_x \left(\nabla_x \mathbf{m} + \nabla_x^t \mathbf{m} - \mathrm{div}_x \mathbf{m} \mathbb{I} \right)$$

$$= -\gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi)$$

$$+ \frac{1}{|\Omega|} \int_\Omega \gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi) \ \mathrm{d}x + h\mathbf{f} - \frac{1}{|\Omega|} \int_\Omega h\mathbf{f} \ \mathrm{d}x,$$

$$\int_\Omega \mathbb{M}(t, \cdot) \ \mathrm{d}x = 0 \text{ for any } t \in [0, T].$$

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Causality

$\mathbf{v}(t,\cdot)=\mathbf{w}(t,\cdot)$ for $0\le t\le\tau\le T$ implies $b[\mathbf{v}]=b[\mathbf{w}]$ in $[(0,\tau]\times\Omega]$

Results Savage-Hutter model

Theorem (with P.Gwiazda and A.Swierczewska-Gwiazda [2015])

(i) Let the initial data

$$h_0 \in C^2(\Omega),\ \mathbf{u}_0 \in C^2(\Omega; R^2), h_0 > 0 \text{ in } \Omega$$

be given, and let $\mathbf{f}$ and a be smooth.
Then the Savage-Hutter system admits infinitely many weak solutions in $(0, T) \times \Omega$.

(ii) Let $T > 0$ and

$$h_0 \in C^2(\Omega),\ \ h_0 > 0$$

be given.
Then there exists

$$\mathbf{u}_0 \in L^\infty(\Omega; R^2)$$

such that the Savage-Hutter system admits infinitely many weak solutions in $(0, T) \times \Omega$ satisfying the energy inequality.

Example II, Euler-Fourier system

(joint work with E.Chiodaroli and O.Kreml [2014])

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = 0$$

Internal energy balance

$$\frac{3}{2}\left[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{u})\right] - \Delta\vartheta = -\varrho\vartheta\mathrm{div}_x\mathbf{u}$$

Example III, Euler-Korteweg-Poisson system

(joint work with D.Donatelli and P.Marcati [2014])

Mass conservation - equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equations - Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho)$$

$$= \boxed{\varrho \nabla_x \left(K(\varrho)\Delta_x \varrho + \frac{1}{2} K'(\varrho)|\nabla_x \varrho|^2 \right)} - \varrho \mathbf{u} + \varrho \nabla_x V$$

Poisson equation

$$\Delta_x V = \varrho - \overline{\varrho}$$

Example IV, Euler-Cahn-Hilliard system

Model by Lowengrub and Truskinovsky

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p_0(\varrho, c) = \mathrm{div}_x \left(\varrho \nabla_x c \otimes \nabla_x c - \frac{\varrho}{2} |\nabla_x c|^2 \mathbb{I} \right)$$

Cahn-Hilliard equation

$$\partial_t(\varrho c) + \mathrm{div}_x(\varrho c \mathbf{u}) = \Delta \left(\mu_0(\varrho, c) - \frac{1}{\varrho} \mathrm{div}_x \left(\varrho \nabla_x c \right) \right)$$

Example V, models of collective behavior

(joint work with J.A. Carrillo, P.Gwiazda, A.Swierczewska–Gwiazda)

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u})$$

$$= -p(\varrho) + \left(1 - H\left(|\mathbf{u}|^2\right)\right) \varrho \mathbf{u}$$

$$-\varrho \nabla_x K * \varrho + \varrho \psi * \left[\varrho \Big(\mathbf{u} - \mathbf{u}(x) \Big) \right]$$