Two-sided bounds on eigenvalues of elliptic operators

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Laplace eigenvalue problem

Classical formulation

$$\begin{aligned} -\Delta u_i &= \lambda_i u_i && \text{in } \Omega \\ u_i &= 0 && \text{on } \partial\Omega \end{aligned}$$

Countable sequence of eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$$

Weak formulation

$$\lambda_i \in \mathbb{R},\ u_i \in H_0^1(\Omega): \quad (\nabla u_i, \nabla v) = \lambda_i (u_i, v) \quad \forall v \in H_0^1(\Omega)$$

Finite element method

$$V_h = \{v_h \in H_0^1(\Omega) : v_h|_K \in P_1(K),\ \forall K \in \mathcal{T}_h\}$$

$$\Lambda_{h,i} \in \mathbb{R},\ u_{h,i} \in V_h : \quad (\nabla u_{h,i}, \nabla v_h) = \Lambda_{h,i}(u_{h,i}, v_h) \quad \forall v_h \in V_h$$

Upper bound: $\lambda_i \le \Lambda_{h,i}, \quad i = 1, 2, \ldots, \dim V_h$

Laplace eigenvalue problem

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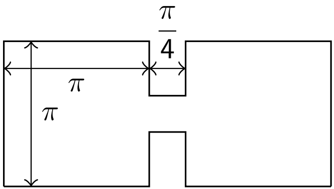
Lower bound: $\quad ? \le \lambda_i \le \Lambda_{h,i}, \quad i = 1, 2, \ldots, \dim V_h$

Example – dumbbell

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$



[Trefethen, Betcke 2006]

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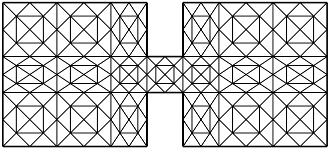
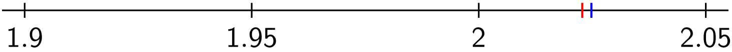
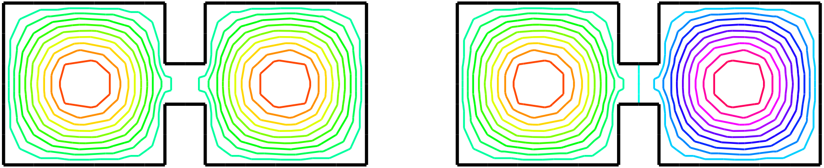




$\lambda_1 \approx 2.02280$ $\lambda_2 \approx 2.02481$



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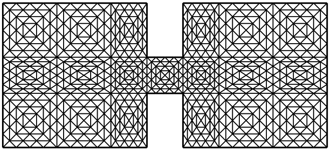

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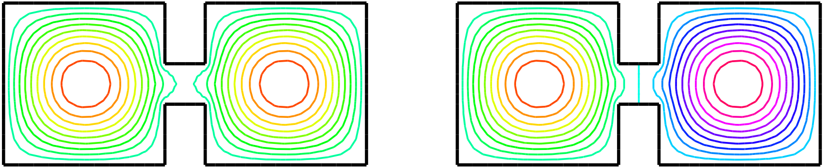

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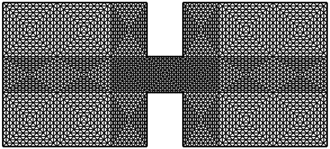

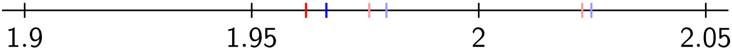

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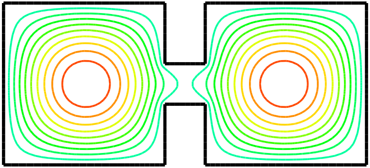

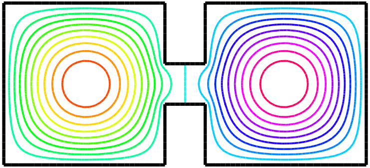

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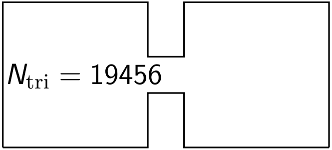

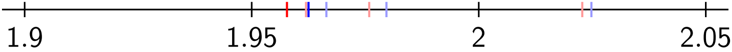

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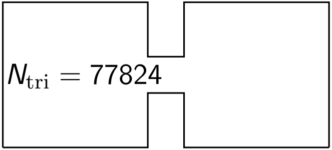

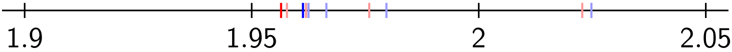

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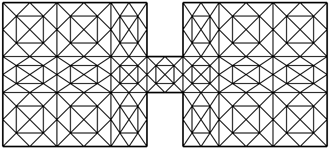
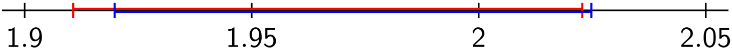
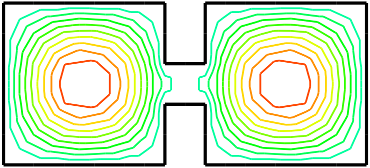
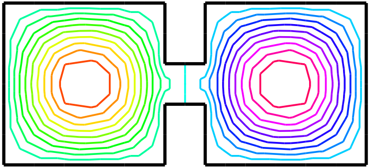




$$1.91067 \le \lambda_1 \le 2.02280 \qquad 1.91981 \le \lambda_2 \le 2.02481$$





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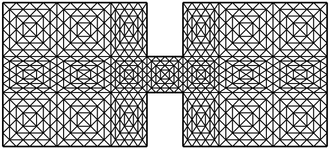

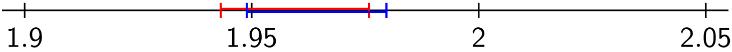

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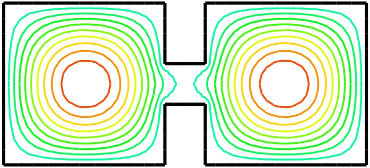

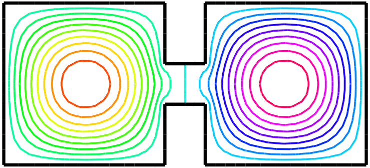

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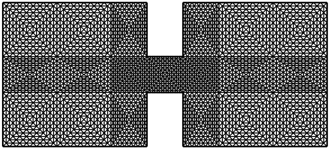

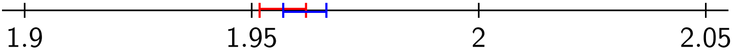

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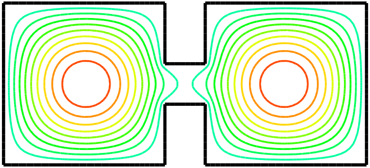

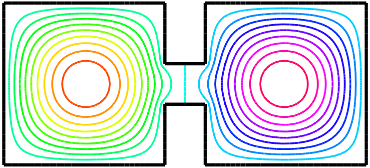

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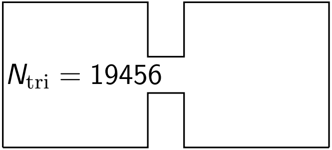

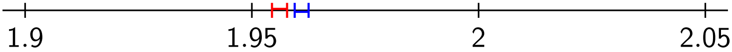

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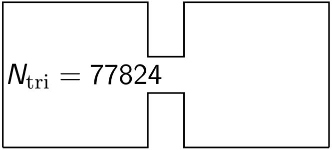

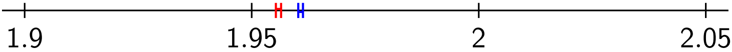

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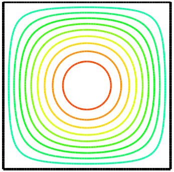
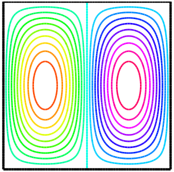
$1.95532 \leq \lambda_1 \leq 1.95646 \quad 1.96025 \leq \lambda_2 \leq 1.96129$

Example: Square

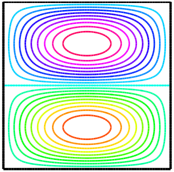
$\lambda_1 = 2,\ u_1(x, y) = \sin(x)\sin(y)$

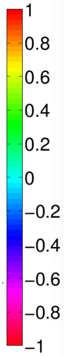


$\lambda_2 = 5,\ u_2(x, y) = \sin(2x)\sin(y)$



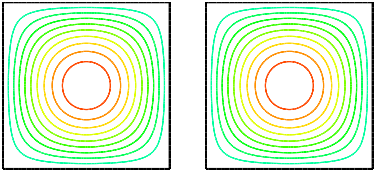
$\lambda_3 = 5,\ u_3(x, y) = \sin(x)\sin(2y)$



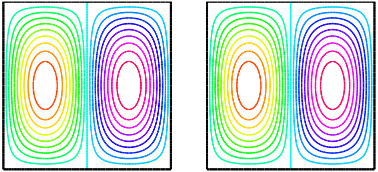


Example: Two squares

$\lambda_1 = 2$



$\lambda_2 = 5$



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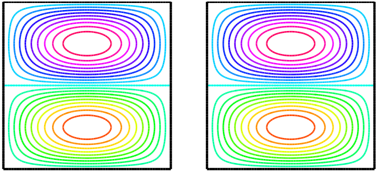

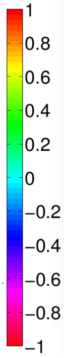

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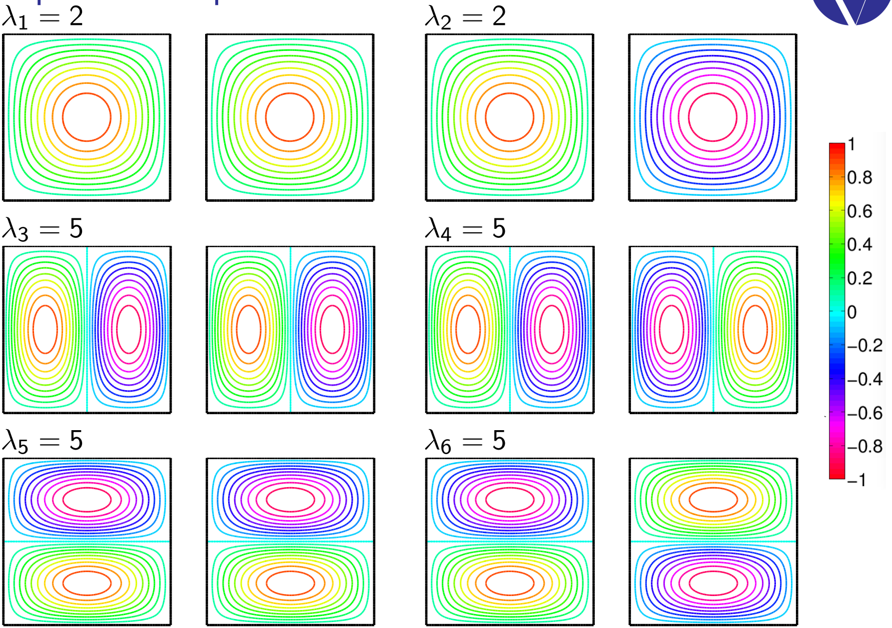
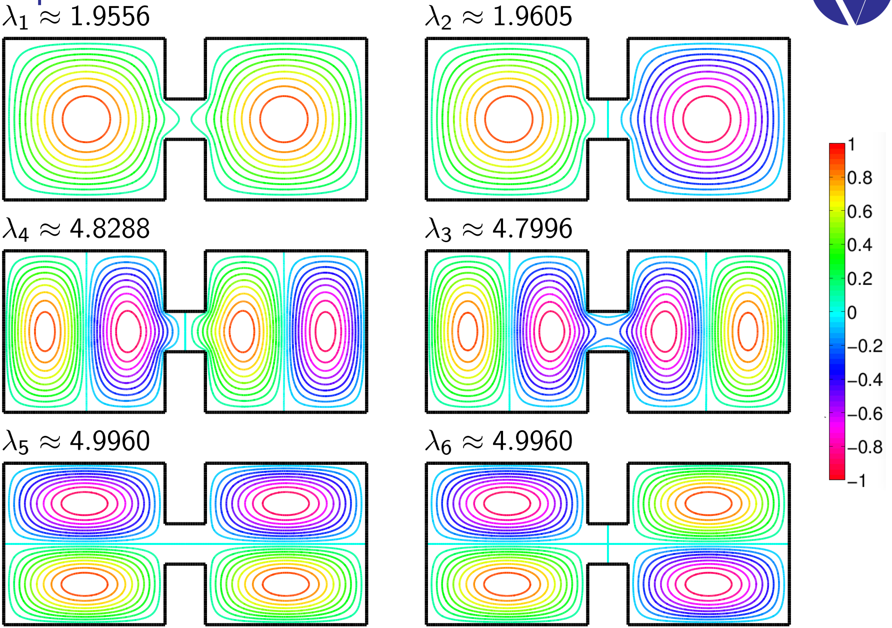


Example: Dumbbell



Lower bounds on eigenvalues

Old problem:
Temple 1928, Weinstein 1937, Kato 1949, Lehmann 1949, 1950,
...

Weinstein method

V ... Hilbert space
$A : D(A) \to V$ linear, symmetric operator

Eigenvalue problem:
Find $u_i \in D(A) \setminus \{0\}$ and $\lambda_i \in \mathbb{R}$: $\quad Au_i = \lambda_i u_i$

Assume: $\{u_i\}$ form ON basis in V

Theorem: Let $u_* \in D(A) \setminus \{0\}$ and $\lambda_* \in \mathbb{R}$ be arbitrary and $\delta = \|Au_* - \lambda_* u_*\| / \|u_*\|$.
Then exists λ_i: $\quad \lambda_* - \delta \leq \lambda \leq \lambda_* + \delta$.

Proof: $\|Au_* - \lambda_* u_*\|^2 = \sum_{j=1}^{\infty} \langle Au_* - \lambda_* u_*, u_j \rangle^2 =$

$$\sum_{j=1}^{\infty} |\lambda_j - \lambda_*|^2 \langle u_*, u_j \rangle^2 \geq \min_j |\lambda_j - \lambda_*|^2 \|u_*\|^2$$

Thus, there exists λ_i : $|\lambda_i - \lambda_*| \leq \delta$. $\square$

Weak form

Eigenvalue problem: Find $u_i \in V \setminus \{0\}$ and $\lambda_i \in \mathbb{R}$:

$$a(u_i, v) = \lambda_i b(u_i, v) \quad \forall v \in V.$$

Properties:

- $0 < \lambda_1 \leq \lambda_2 \leq \cdots$
- $b(u_i, u_j) = \delta_{ij}$
- $\|v\|_b^2 = \sum_{j=1}^{\infty} |b(v, u_j)|^2$
- $\|v\|_a^2 = \sum_{j=1}^{\infty} \lambda_j |b(v, u_j)|^2$

Weak form

Theorem: Let $u_* \in V \setminus \{0\}$ and $\lambda_* \in \mathbb{R}$ be arbitrary and $w \in V$ be given by

$$a(w, v) = a(u_*, v) - \lambda_* b(u_*, v) \quad \forall v \in V.$$

Then

$$\min_j \frac{|\lambda_j - \lambda_*|^2}{\lambda_j} \leq \frac{\|w\|_a^2}{\|u_*\|_b^2}.$$

Proof:

$$\|w\|_a^2 = \sum_{j=1}^{\infty} \lambda_j |b(w, u_j)|^2 = \sum_{j=1}^{\infty} \frac{|a(w, u_j)|^2}{\lambda_j}$$

$$= \sum_{j=1}^{\infty} \frac{|a(u_*, u_j) - \lambda_* b(u_*, u_j)|^2}{\lambda_j} = \sum_{j=1}^{\infty} \frac{|\lambda_j - \lambda_*|^2}{\lambda_j} |b(u_*, u_j)|^2$$

Thus,

$$\|w\|_a^2 \geq \min_j \frac{|\lambda_j - \lambda_*|^2}{\lambda_j} \sum_{j=1}^{\infty} |b(u_*, u_j)|^2$$

Weak form

Corollary: If

$$\sqrt{\lambda_{i-1}\lambda_i} \leq \lambda_* \leq \sqrt{\lambda_i\lambda_{i+1}}$$

and

$$\|w\|_a \leq \eta$$

then

$$\ell_i \leq \lambda_i,$$

where $\ell_i = \dfrac{1}{4\|u_*\|_b^2}\left(-\eta + \sqrt{\eta^2 + 4\lambda_*\|u_*\|_b^2}\right)^2.$

Complementary upper bound on the residual

Theorem: Let $\mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)$ be such that $-\operatorname{div} \mathbf{q} = \lambda_* u_*$ then

$$\|w\|_a \leq \eta = \|\nabla u_* - \mathbf{q}\|.$$

Proof:

$$\begin{aligned} a(w, v) = (\nabla u_*, \nabla v) - \lambda_*(u_*, v) - (\operatorname{div} \mathbf{q}, v) - (\mathbf{q}, \nabla v) \\ = (\nabla u_* - \mathbf{q}, \nabla v) - (\lambda_* u_* + \operatorname{div} \mathbf{q}, v) \\ \leq \|\nabla u_* - \mathbf{q}\| \|v\| \end{aligned}$$

$\square$

[Braess 2007]

Flux reconstruction

- FEM eigenpairs: $\Lambda_{h,i} \in \mathbb{R}$, $u_{h,i} \in V_h$, $i = 1, 2, \dots, m$
- Flux reconstruction: $\mathbf{q}_{h,i} = \sum_{\mathbf{z} \in \mathcal{N}_h} \mathbf{q}_{\mathbf{z},i}$ [Braess, Schöberl 2006]
- Local mixed FEM: $\mathbf{q}_{\mathbf{z},i} \in \mathbf{W}_{\mathbf{z}}$, $d_{\mathbf{z},i} \in P_1^*(\mathcal{T}_{\mathbf{z}})$

$$\begin{aligned} (\mathbf{q}_{\mathbf{z},i}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} - (d_{\mathbf{z},i}, \operatorname{div} \mathbf{w}_h)_{\omega_{\mathbf{z}}} &= (\psi_{\mathbf{z}} \nabla u_{h,i}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} && \forall \mathbf{w}_h \in \mathbf{W}_{\mathbf{z}} \\ -(\operatorname{div} \mathbf{q}_{\mathbf{z},i}, \varphi_h)_{\omega_{\mathbf{z}}} &= (r_{\mathbf{z},i}, \varphi_h)_{\omega_{\mathbf{z}}} && \forall \varphi_h \in P_1^*(\mathcal{T}_{\mathbf{z}}) \end{aligned}$$

where

- $\omega_{\mathbf{z}}$ is the patch of elements around vertex $\mathbf{z} \in \mathcal{N}_h$
- $\mathcal{T}_{\mathbf{z}}$ is the set of elements in $\omega_{\mathbf{z}}$
- $\mathbf{W}_{\mathbf{z}} = \{\mathbf{w}_h \in \mathbf{H}(\operatorname{div}, \omega_{\mathbf{z}}) : \mathbf{w}_h|_K \in \mathbf{RT}_1(K)\ \forall K \in \mathcal{T}_{\mathbf{z}}$ and $\mathbf{w}_h \cdot \mathbf{n}_{\omega_{\mathbf{z}}} = 0$ on $\Gamma_{\omega_{\mathbf{z}}}^{\mathrm{ext}}\}$
- $P_1^*(\mathcal{T}_{\mathbf{z}}) = \begin{cases} \{v_h \in P_1(\mathcal{T}_{\mathbf{z}}) : \int_{\omega_{\mathbf{z}}} v_h \,\mathrm{d}x = 0\} & \text{for } \mathbf{z} \in \mathcal{N}_h \setminus \partial\Omega \\ P_1(\mathcal{T}_{\mathbf{z}}) & \text{for } \mathbf{z} \in \mathcal{N}_h \cap \partial\Omega \end{cases}$
- $r_{\mathbf{z},i} = \Lambda_{h,i} \psi_{\mathbf{z}} u_{h,i} - \nabla \psi_{\mathbf{z}} \cdot \nabla u_{h,i}$

Flux reconstruction

- FEM eigenpairs: $\Lambda_{h,i} \in \mathbb{R}$, $u_{h,i} \in V_h$, $i = 1, 2, \dots, m$
- Flux reconstruction: $\mathbf{q}_{h,i} = \sum_{\mathbf{z} \in \mathcal{N}_h} \mathbf{q}_{\mathbf{z},i}$ [Braess, Schöberl 2006]
- Local mixed FEM: $\mathbf{q}_{\mathbf{z},i} \in \mathbf{W}_{\mathbf{z}}$, $d_{\mathbf{z},i} \in P_1^*(\mathcal{T}_{\mathbf{z}})$

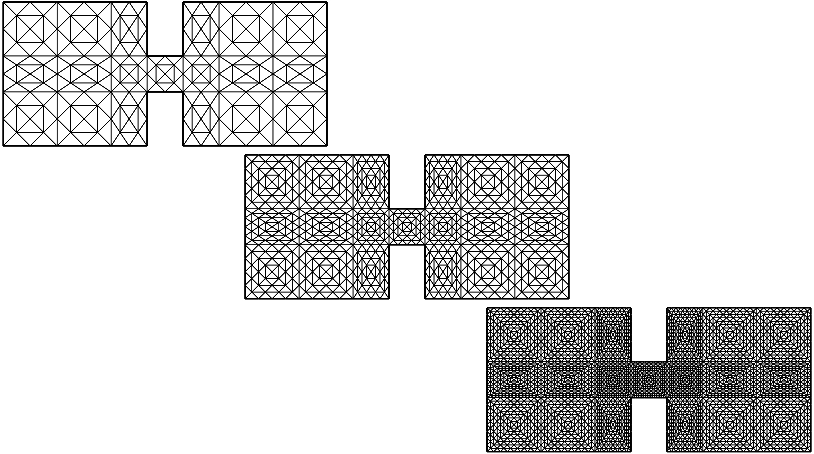
$$
\begin{aligned}
(\mathbf{q}_{\mathbf{z},i}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} - (d_{\mathbf{z},i}, \operatorname{div} \mathbf{w}_h)_{\omega_{\mathbf{z}}} &= (\psi_{\mathbf{z}} \nabla u_{h,i}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} \quad && \forall \mathbf{w}_h \in \mathbf{W}_{\mathbf{z}} \\
-(\operatorname{div} \mathbf{q}_{\mathbf{z},i}, \varphi_h)_{\omega_{\mathbf{z}}} &= (r_{\mathbf{z},i}, \varphi_h)_{\omega_{\mathbf{z}}} && \forall \varphi_h \in P_1^*(\mathcal{T}_{\mathbf{z}})
\end{aligned}
$$

- Error estimator: $\eta_i = \| \nabla u_{h,i} - \mathbf{q}_{h,i} \|_{L^2(\Omega)}$
- Lower bound: $\ell_i = \left(-\eta_i + \sqrt{\eta_i^2 + 4\Lambda_{h,i}} \right)^2 / 4$
- Provided $\Lambda_{h,i} \le \sqrt{\lambda_i \lambda_{i+1}}$

Example: Dumbbell – speed of convergence

$$
\begin{aligned}
-\Delta u_i &= \lambda_i u_i && \text{in } \Omega = \text{dumbbell} \\
u_i &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

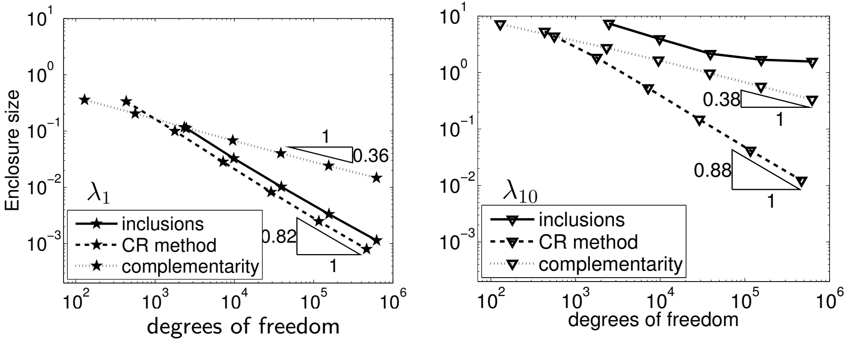
Uniformly refined meshes:



Example: Dumbbell – speed of convergence

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega = \text{dumbbell}$$
$$u_i = 0 \quad \text{on } \partial\Omega$$

Eigenvalue enclosure sizes:



[Behnke, Mertins, Plum, Wieners 2000], [Carstensen, Gedicke 2014]

Quadratically convergent bound

Theorem: Let $u_* \in V \setminus \{0\}$ be arbitrary and let $\lambda_* = \|u_*\|_a^2 / \|u_*\|_b^2$. Let there be $\nu \in \mathbb{R}$ such that

$$\lambda_{i-1} < \lambda_* < \nu \leq \lambda_{i+1}$$

for a fixed index i. Let $\|w\|_a \leq \eta$. Then

$$L_i \leq \lambda_i,$$

where

$$L_i = \lambda_* \left(1 + \frac{\nu}{\lambda_*(\nu - \lambda_*)} \frac{\eta^2}{\|u_*\|_b^2}\right)^{-1}.$$

Conclusions

- Good for general symmetric elliptic operators.
- Mixed boundary conditions.
- Standard conforming finite element technology.
- Natural for adaptive refinement.

Thank you for your attention

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