Numerical lower bounds on eigenvalues of elliptic operators

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What do we do in numerics?

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We solve problems approximately.

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What should we do?

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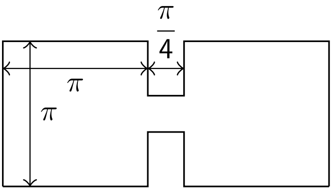
We solve problems approximately.

What should we do?

Solve problems within a given accuracy.

Example – dumbbell

$$
\begin{aligned}
-\Delta u_i &= \lambda_i u_i \quad \text{in } \Omega \\
u_i &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$



[Trefethen, Betcke 2006]

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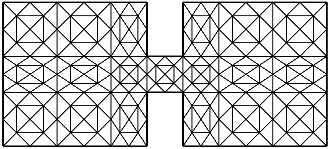
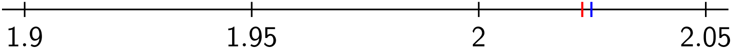
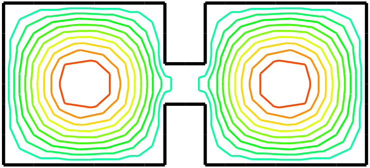
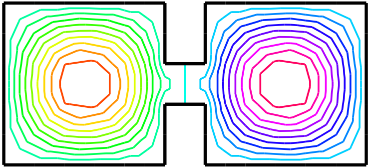




$\lambda_1 \approx 2.02280 \qquad \lambda_2 \approx 2.02481$





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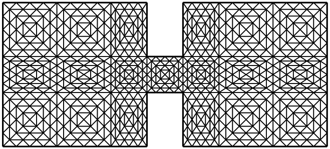

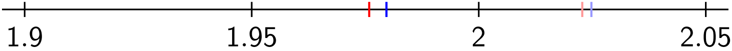

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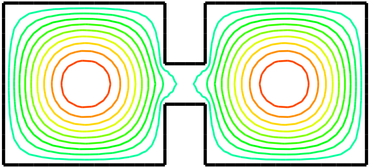

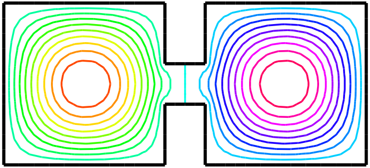

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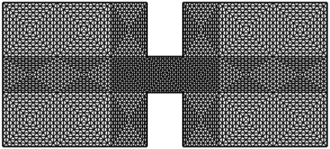

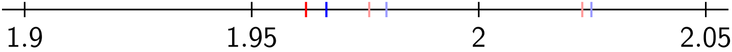

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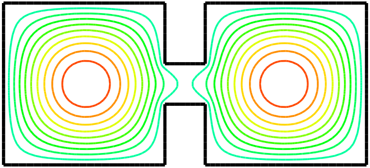

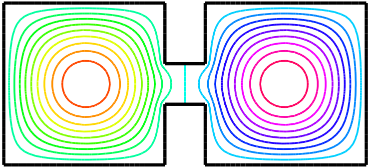

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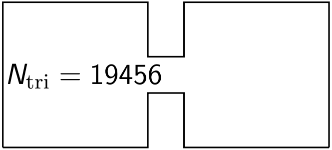

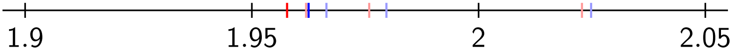

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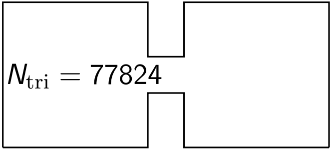

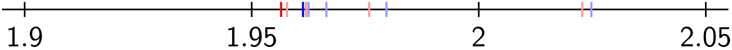

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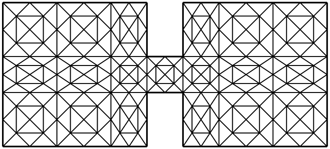

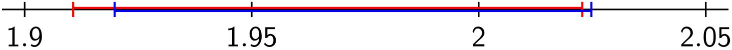

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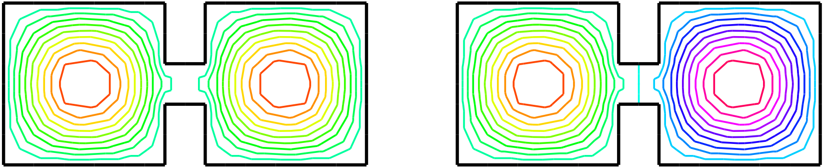

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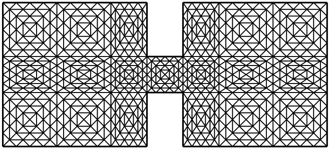

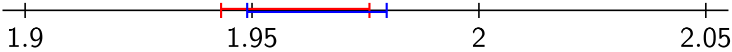

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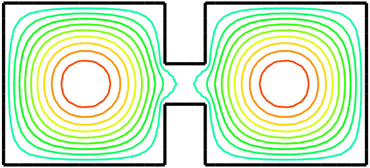

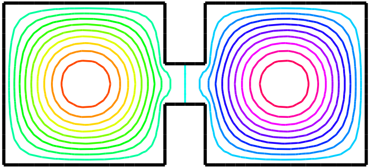

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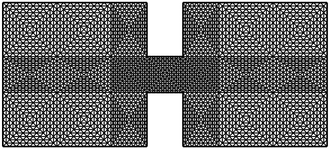

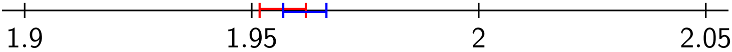

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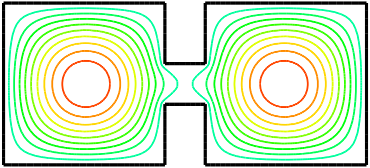

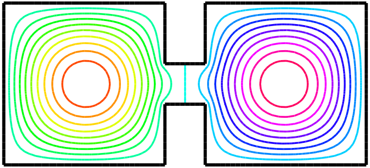

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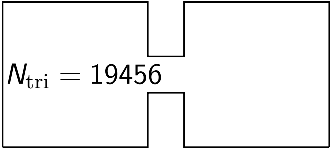

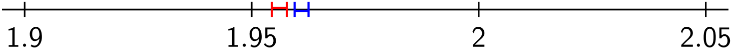

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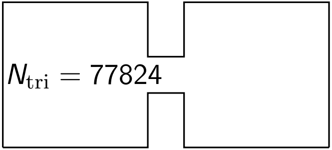

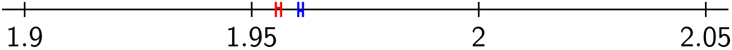

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$1.95532 \leq \lambda_1 \leq 1.95646$	$1.96025 \leq \lambda_2 \leq 1.96129$

Upper bounds on eigenvalues

Laplace eigenvalue problem

$$
\begin{aligned}
-\Delta u_i &= \lambda_i u_i && \text{in } \Omega \\
u_i &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

Weak formulation

$\lambda_i \in \mathbb{R},\ u_i \in H^1_0(\Omega): \quad (\nabla u_i, \nabla v) = \lambda_i (u_i, v) \quad \forall v \in H^1_0(\Omega)$

Finite element method

$V_h = \{v_h \in H^1_0(\Omega) : v_h|_K \in P_1(K),\ \forall K \in \mathcal{T}_h\}$

$\Lambda_{h,i} \in \mathbb{R},\ u_{h,i} \in V_h: \quad (\nabla u_{h,i}, \nabla v_h) = \Lambda_{h,i}(u_{h,i}, v_h) \quad \forall v_h \in V_h$

Lower bound?

$$
? \le \lambda_i \le \Lambda_{h,i}, \quad i = 1, 2, \dots, m
$$

Lower bounds on eigenvalues

Old problem:

Temple 1928, Kato 1949, Lehmann 1949, 1950, Harrell 1978, ...

Methods based on FEM:

1. Eigenvalue inclusions [Behnke, Mertins, Plum, Wieners 2000]
   based on [Behnke, Goerish 1994] and [Plum 1997]
2. Crouzeix–Raviart elements [Carstensen, Gedicke 2013]
3. Complementarity based [Šebestová, Vejchodský 2016]

Method 1. Eigenvalue inclusions

Input: Rough lower bounds: $\underline{\lambda}_2 \leq \lambda_2, \dots, \underline{\lambda}_{m+1} \leq \lambda_{m+1}$,

Algorithm:

- FEM eigenpairs: $\Lambda_{h,i} \in \mathbb{R}$, $u_{h,i} \in V_h$, $i = 1, 2, \dots, m$
- Mixed FEM problem: $\boldsymbol{\sigma}_{h,i} \in \mathbf{W}_h$, $q_{h,i} \in Q_h$, $i = 1, 2, \dots, m$
  $\mathbf{W}_h = \{\boldsymbol{\sigma}_h \in \mathbf{H}(\operatorname{div}, \Omega) : \boldsymbol{\sigma}_h|_K \in \mathbf{RT}_k(K) \quad \forall K \in \mathcal{T}_h\}$
  $Q_h = \{q_h \in L^2(\Omega) : q_h|_K \in P_k(K) \quad \forall K \in \mathcal{T}_h\}$

$$
\begin{aligned}
(\boldsymbol{\sigma}_{h,i}, \mathbf{w}_h) + (q_{h,i}, \operatorname{div} \mathbf{w}_h) &= 0 && \forall \mathbf{w}_h \in \mathbf{W}_h, \\
(\operatorname{div} \boldsymbol{\sigma}_{h,i}, \varphi_h) &= (-u_{h,i}, \varphi_h) && \forall \varphi_h \in Q_h,
\end{aligned}
$$

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- FEM eigenpairs: $\Lambda_{h,i} \in \mathbb{R}$, $u_{h,i} \in V_h$, $i = 1, 2, \ldots, m$
- Mixed FEM problem: $\boldsymbol{\sigma}_{h,i} \in \mathbf{W}_h$, $q_{h,i} \in Q_h$, $i = 1, 2, \ldots, m$
- For $n = 1, 2, \ldots, m$ do

$$\gamma = \|u_{h,n} + \operatorname{div} \boldsymbol{\sigma}_{h,n}\|_{L^2(\Omega)}, \qquad \rho = \underline{\lambda}_{n+1} + \gamma$$

$$\mathbf{M}_{ij} = (\nabla u_{h,i}, \nabla u_{h,j}) + (\gamma - \rho)(u_{h,i}, u_{h,j})$$

$$\mathbf{N}_{ij} = (\nabla u_{h,i}, \nabla u_{h,j}) + (\gamma - 2\rho)(u_{h,i}, u_{h,j}) + \rho^2(\boldsymbol{\sigma}_{h,i}, \boldsymbol{\sigma}_{h,j}) + (\rho^2/\gamma)(u_{h,i} + \operatorname{div} \boldsymbol{\sigma}_{h,i}, u_{h,j} + \operatorname{div} \boldsymbol{\sigma}_{h,j})$$

$$\mu_1 \le \cdots \le \mu_n : \qquad \mathbf{M}\mathbf{y}_i = \mu_i \mathbf{N}\mathbf{y}_i, \qquad i = 1, 2, \ldots, n$$

If $\mathbf{N}$ is s.p.d. and if $\mu_i < 0$ then

$$\ell_{j,n}^{\mathrm{incl}} = \rho - \gamma - \rho / (1 - \mu_{n+1-j}) \le \lambda_j, \quad j = 1, 2, \ldots, n.$$

end for

$$\ell_j^{\mathrm{incl}} = \max\{\ell_{j,n}^{\mathrm{incl}},\ n = j, j+1, \ldots, m\} \le \lambda_j, \quad j = 1, 2, \ldots, m$$

Method 2. Crouzeix–Raviart elements

Crouzeix–Raviart finite elements

$V_h^{\mathrm{CR}} = \{v_h \in P_1(\mathcal{T}_h) : v_h \text{ continuous in midpoints of all } \gamma \in \mathcal{E}_h\}$
Find $0 \neq u_{h,i}^{\mathrm{CR}} \in V_h^{\mathrm{CR}}$, $\lambda_{h,i}^{\mathrm{CR}} \in \mathbb{R}$:

$$(\nabla u_{h,i}^{\mathrm{CR}}, \nabla v_h) = \lambda_{h,i}^{\mathrm{CR}} (u_{h,i}^{\mathrm{CR}}, v_h) \quad \forall v_h \in V_h^{\mathrm{CR}}.$$

Lower bound (no round-off errors)

$$\ell_i^{\mathrm{CR}} = \frac{\lambda_{h,i}^{\mathrm{CR}}}{1 + \kappa^2 \lambda_{h,i}^{\mathrm{CR}} h_{\max}^2} \leq \lambda_i \quad \forall i = 1, 2, \ldots$$

where

- $\kappa^2 = 1/8 + j_{1,1}^{-2} \leq 0.1932$
- $h_{\max} = \max_{K \in \mathcal{T}_h} \operatorname{diam} K$

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$$(\nabla u_{h,i}^{\mathrm{CR}}, \nabla v_h) = \lambda_{h,i}^{\mathrm{CR}} (u_{h,i}^{\mathrm{CR}}, v_h) \quad \forall v_h \in V_h^{\mathrm{CR}}.$$

Lower bound (inexact solver: $\mathbf{A}\tilde{\mathbf{u}}_i^{\mathrm{CR}} \approx \tilde{\lambda}_{h,i}^{\mathrm{CR}} \mathbf{B}\tilde{\mathbf{u}}_i^{\mathrm{CR}}$)

$$\tilde{\ell}_i^{\mathrm{CR}} = \frac{\tilde{\lambda}_{h,i}^{\mathrm{CR}} - \|\mathbf{r}\|_{\mathbf{B}^{-1}}}{1 + \kappa^2 \left(\tilde{\lambda}_{h,i}^{\mathrm{CR}} - \|\mathbf{r}\|_{\mathbf{B}^{-1}} \right) h_{\max}^2} \leq \lambda_i \quad \forall i = 1, 2, \ldots$$

where

- $\kappa^2 = 1/8 + j_{1,1}^{-2} \leq 0.1932$
- $h_{\max} = \max_{K \in \mathcal{T}_h} \operatorname{diam} K$
- $\mathbf{r} = \mathbf{A}\tilde{\mathbf{u}}_i^{\mathrm{CR}} - \tilde{\lambda}_{h,i}^{\mathrm{CR}} \mathbf{B}\tilde{\mathbf{u}}_i^{\mathrm{CR}}$

Provided

- $\|\mathbf{r}\|_{\mathbf{B}^{-1}} < \tilde{\lambda}_{h,i}^{\mathrm{CR}}$
- $\tilde{\lambda}_{h,i}^{\mathrm{CR}}$ is closer to $\lambda_{h,i}^{\mathrm{CR}}$ than to any other discrete eigenvalue $\lambda_{h,j}^{\mathrm{CR}}$, $j \neq i$

Method 2. Crouzeix–Raviart elements

Upper bound

- $\mathcal{T}_h^*$ is the red refinement of $\mathcal{T}_h$
- $u_{h,i}^* = \mathcal{I}_{\mathrm{CM}} \tilde{u}_{h,i}^{\mathrm{CR}}$ for $i = 1, 2, \dots, m$
- $\mathbf{S}, \mathbf{Q} \in \mathbb{R}^{m \times m}$ with entries $\mathbf{S}_{j,k} = (\nabla u_{h,j}^*, \nabla u_{h,k}^*)$ and $\mathbf{Q}_{j,k} = (u_{h,j}^*, u_{h,k}^*)$
- $\mathbf{S}\mathbf{y}_i = \Lambda_i^* \mathbf{Q}\mathbf{y}_i, \quad i = 1, 2, \dots, m$
- $\Lambda_1^* \leq \Lambda_2^* \leq \cdots \leq \Lambda_m^*$
- $\lambda_i \leq \Lambda_i^* \quad$ for $i = 1, 2, \dots, m$

Method 3. Complementarity based

- FEM eigenpairs: $\Lambda_{h,i} \in \mathbb{R}$, $u_{h,i} \in V_h$, $i = 1, 2, \dots, m$
- Flux reconstruction: $\mathbf{q}_{h,i} = \sum_{\mathbf{z} \in \mathcal{N}_h} \mathbf{q}_{\mathbf{z},i}$
- Local mixed FEM: $\mathbf{q}_{\mathbf{z},i} \in \mathbf{W}_\mathbf{z}$, $d_{\mathbf{z},i} \in P_1^*(\mathcal{T}_\mathbf{z})$

$$
\begin{aligned}
(\mathbf{q}_{\mathbf{z},i}, \mathbf{w}_h)_{\omega_\mathbf{z}} - (d_{\mathbf{z},i}, \operatorname{div} \mathbf{w}_h)_{\omega_\mathbf{z}} &= (\psi_\mathbf{z} \nabla u_{h,i}, \mathbf{w}_h)_{\omega_\mathbf{z}} & \forall \mathbf{w}_h \in \mathbf{W}_\mathbf{z} \\
-(\operatorname{div} \mathbf{q}_{\mathbf{z},i}, \varphi_h)_{\omega_\mathbf{z}} &= (r_{\mathbf{z},i}, \varphi_h)_{\omega_\mathbf{z}} & \forall \varphi_h \in P_1^*(\mathcal{T}_\mathbf{z})
\end{aligned}
$$

where

- $\omega_\mathbf{z}$ is the patch of elements around vertex $\mathbf{z} \in \mathcal{N}_h$
- $\mathcal{T}_\mathbf{z}$ is the set of elements in $\omega_\mathbf{z}$
- $\mathbf{W}_\mathbf{z} = \{\mathbf{w}_h \in \mathbf{H}(\operatorname{div}, \omega_\mathbf{z}) : \mathbf{w}_h|_K \in \mathbf{RT}_1(K)\ \forall K \in \mathcal{T}_\mathbf{z}$ and $\mathbf{w}_h \cdot \mathbf{n}_{\omega_\mathbf{z}} = 0$ on $\Gamma_{\omega_\mathbf{z}}^{\mathrm{ext}}\}$
- $P_1^*(\mathcal{T}_\mathbf{z}) = \begin{cases} \{v_h \in P_1(\mathcal{T}_\mathbf{z}) : \int_{\omega_\mathbf{z}} v_h \,\mathrm{d}x = 0\} & \text{for } \mathbf{z} \in \mathcal{N}_h \setminus \partial\Omega \\ P_1(\mathcal{T}_\mathbf{z}) & \text{for } \mathbf{z} \in \mathcal{N}_h \cap \partial\Omega \end{cases}$
- $r_{\mathbf{z},i} = \Lambda_{h,i} \psi_\mathbf{z} u_{h,i} - \nabla \psi_\mathbf{z} \cdot \nabla u_{h,i}$

Method 3. Complementarity based

- FEM eigenpairs: $\Lambda_{h,i} \in \mathbb{R}$, $u_{h,i} \in V_h$, $i = 1, 2, \dots, m$
- Flux reconstruction: $\mathbf{q}_{h,i} = \sum_{\mathbf{z} \in \mathcal{N}_h} \mathbf{q}_{\mathbf{z},i}$
- Local mixed FEM: $\mathbf{q}_{\mathbf{z},i} \in \mathbf{W}_{\mathbf{z}}$, $d_{\mathbf{z},i} \in P_1^*(\mathcal{T}_{\mathbf{z}})$

$$
\begin{aligned}
(\mathbf{q}_{\mathbf{z},i}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} - (d_{\mathbf{z},i}, \operatorname{div} \mathbf{w}_h)_{\omega_{\mathbf{z}}} &= (\psi_{\mathbf{z}} \nabla u_{h,i}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} \quad && \forall \mathbf{w}_h \in \mathbf{W}_{\mathbf{z}} \\
-(\operatorname{div} \mathbf{q}_{\mathbf{z},i}, \varphi_h)_{\omega_{\mathbf{z}}} &= (r_{\mathbf{z},i}, \varphi_h)_{\omega_{\mathbf{z}}} \quad && \forall \varphi_h \in P_1^*(\mathcal{T}_{\mathbf{z}})
\end{aligned}
$$

- Error estimator: $\eta_i = \| \nabla u_{h,i} - \mathbf{q}_{h,i} \|_{L^2(\Omega)}$
- Lower bound: $\ell_1^{\mathrm{cmpl}} = \left(-\eta_1 + \sqrt{\eta_1^2 + 4\Lambda_{h,1}} \right)^2 / 4$

$$
\ell_i^{\mathrm{cmpl}} = \Lambda_{h,i} \left(1 + \underline{\lambda}_1^{-1/2} \eta_i \right)^{-1}, \quad i = 2, 3, \dots
$$

- Provided $\Lambda_{h,i} \leq 2 \left(\lambda_i^{-1} + \lambda_{i+1}^{-1} \right)^{-1}$

Comparison

	1. Inclusions	2. CR elements	3. Complementarity
convergence	**	***	*
generality	***	*	**
a priori info	*	***	**
DOFs needed	*	**	***
algebraic err.	**	***	***
adaptivity	*	**	***

Thank you for your attention

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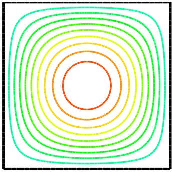
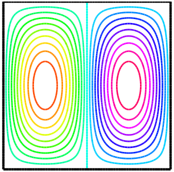
EMS School in Applied Mathematics, Kácov, May 29 – June 3, 2016

Square

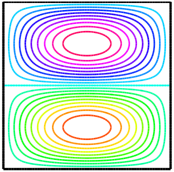
$\lambda_1 = 2, \; u_1(x, y) = \sin(x)\sin(y)$

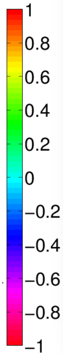


$\lambda_2 = 5, \; u_2(x, y) = \sin(2x)\sin(y)$



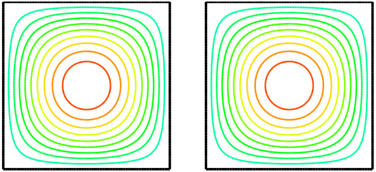
$\lambda_3 = 5, \; u_3(x, y) = \sin(x)\sin(2y)$



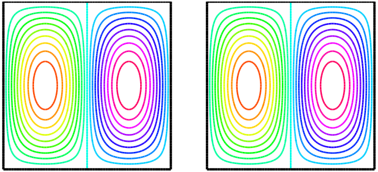


Two squares

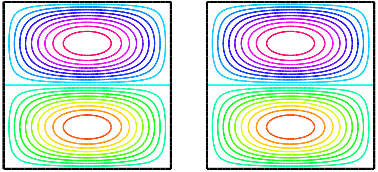
$\lambda_1 = 2$

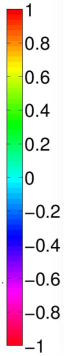


$\lambda_2 = 5$

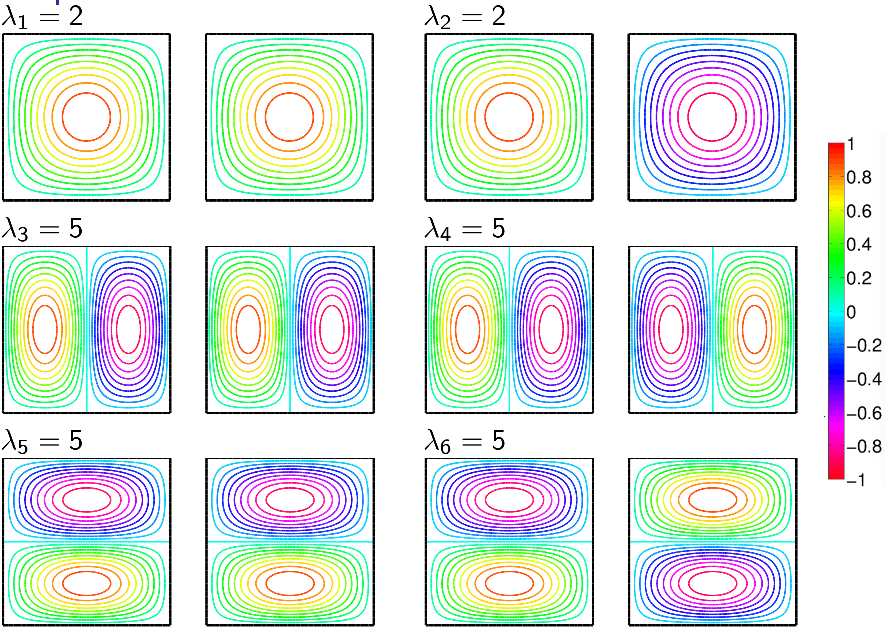


$\lambda_3 = 5$



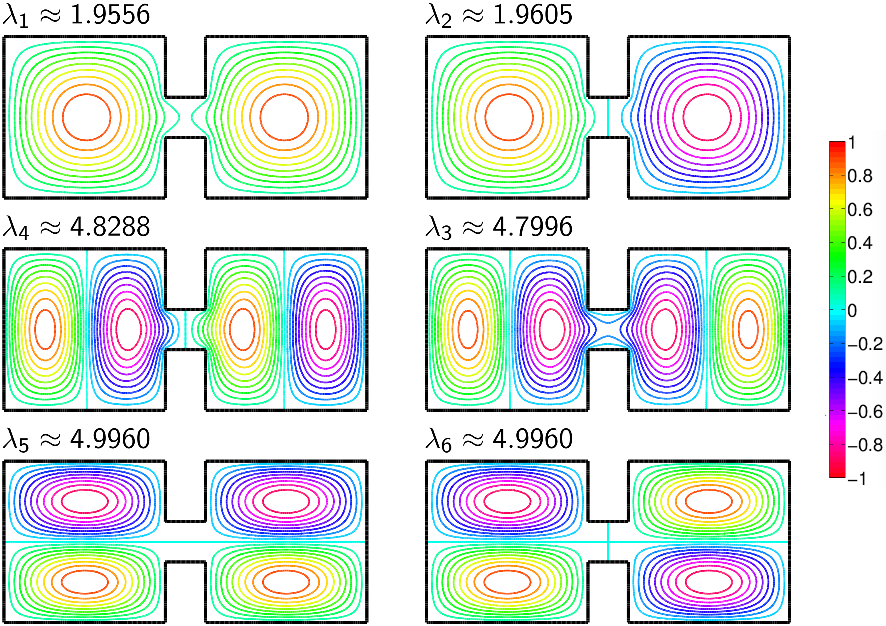


Two squares



$\lambda_1 = 2$
$\lambda_2 = 2$
$\lambda_3 = 5$
$\lambda_4 = 5$
$\lambda_5 = 5$
$\lambda_6 = 5$
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

Dumbbell


$\lambda_1 \approx 1.9556$
$\lambda_2 \approx 1.9605$
$\lambda_4 \approx 4.8288$
$\lambda_3 \approx 4.7996$
$\lambda_5 \approx 4.9960$
$\lambda_6 \approx 4.9960$
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1