Classification of the spaces of continuous functions within the Borel-Wadge hierarchy

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Introduction

X	....	a separable metrizable space
$C_p(X)$	....	continuous real functions on X
$C_p^*(X)$	....	bounded continuous real functions on X

What is the complexity of the measurable spaces $C_p(X)$ and $C_p^*(X)$?

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The Wadge Hierarchy

X, Y ..... topological spaces
$A \subseteq X$, $B \subseteq Y$
Then $A \leq_W B$ (A is Wadge reducible to B) if there exists a continuous map $f\colon X \to Y$ such that $A = f^{-1}(B)$.

Γ ..... a class of sets in Polish spaces
X ..... a Polish space
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If, moreover, $A \in \Gamma(X)$, we say that A is Γ-complete.

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Known Results

O. Okunev, 1993
T Dobrowolski and W. Marciszewski, 1995

X is σ-compact $\Rightarrow C_p(X)$ and $C_p^*(X)$ are standard Borel spaces

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X is Σ_1^1 but not σ-compact $\Rightarrow C_p(X)$ and $C_p^*(X)$ are Borel-Π_1^1-complete

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Sketch of the proof

A. Andretta and A. Marcone, 2001: $C_p^*(X)$ is in Π_n^1.
So we have to show that $C_p^*(X)$ is Borel-Π_n^1-hard.
Let D be a countable dense subset of X. Then we can consider $C_p^*(X)$ as a subspace of $\mathbb{R}^D$.
Now it remains to prove that $C_p^*(X)$ is Π_n^1-hard.

We proceed in two steps:

(A) X is nowhere locally compact
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(i) By transfinite induction, we find a nonempty closed and nowhere locally compact subspace F of X.
Then $C_p^*(F)$ is Borel-Π_n^1-hard.

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