

Relative energy functionals in fluid dynamics

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Compressible barotropic fluid flow

Equation of continuity - mass conservation

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equation - Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \eta \operatorname{div}_x \mathbf{u} \mathbb{I}$$

$$\mathbf{u}|_{\partial\Omega} = 0$$

Energy inequality

$$E(\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, dx \, dt = E(0)$$

$$E = \int_\Omega \frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \, dx, \quad P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \, dz$$

Weak formulation

Field equations

$$\begin{aligned} \left[\int_{\Omega} \varrho \varphi \, dx \right]_{t=t_1}^{t=t_2} &= \int_{t_1}^{t_2} \int_{\Omega} [\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi] \, dx \, dt \\ &\quad \left[\int_{\Omega} \varrho \mathbf{u} \cdot \varphi \, dx \right]_{t=t_1}^{t=t_2} \\ &= \int_{t_1}^{t_2} \int_{\Omega} [\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho) \operatorname{div}_x \varphi] \, dx \, dt \\ &\quad - \int_{t_1}^{t_2} \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi \, dx \, dt \end{aligned}$$

Energy inequality

$$\left[\int_{\Omega} \frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \, dx \right]_{t=0}^{t=\tau} + \int_0^{\tau} \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, dx \, dt \leq 0$$

Relative energy/entropy

Lyapunov function

$$E = \int_{\Omega} \left[\frac{1}{2} \varrho |\mathbf{u} - 0|^2 + P(\varrho) - P'(\bar{\varrho})(\varrho - \bar{\varrho}) - P(\bar{\varrho}) \right] dx$$

Coercivity of the pressure potential

$$\varrho \mapsto p(\varrho) \text{ non-decreasing} \Rightarrow \varrho \mapsto P(\varrho) \text{ convex}$$

Relative energy (relative entropy Dafermos [1979])

$$\mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U})$$

$$= \int_{\Omega} \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r) \right] dx$$

Relative energy inequality

Weak formulation

$$\begin{aligned} & \left[\mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U}) \right]_{t=0}^{t=\tau} \\ = & \left[\int_{\Omega} \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right] dx \right]_{t=0}^{t=\tau} - \left[\int_{\Omega} \varrho \mathbf{u} \cdot \mathbf{U} dx \right]_{t=0}^{t=\tau} \\ & + \left[\int_{\Omega} \frac{1}{2} \varrho |\mathbf{U}|^2 dx \right]_{t=0}^{t=\tau} \\ & - \left[\int_{\Omega} P'(r) \varrho dx \right]_{t=0}^{t=\tau} + \left[\int_{\Omega} [P'(r)r - P(r)] dx \right]_{t=0}^{t=\tau} \end{aligned}$$

Constraints imposed on the test functions

$$r > 0$$

$$\mathbf{U}|_{\partial\Omega} = 0$$

Dissipative solutions

Relative energy inequality

$$\begin{aligned} & \left[\mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U}) \right]_{t=0}^{t=\tau} \\ & + \int_0^\tau \int_\Omega (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \, dx \, dt \\ & \leq \int_0^\tau \mathcal{R}(\varrho, \mathbf{u} \mid r, \mathbf{U}) \, dt \end{aligned}$$

Test functions

$$r > 0, \quad \mathbf{U}|_{\partial\Omega} = 0 \text{ (or other relevant b.c.)}$$

Remainder

$$\begin{aligned} & \int_0^\tau \mathcal{R}(\varrho, \mathbf{u} \mid r, \mathbf{U}) \, dt \\ & \int_\Omega \varrho (\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U}) \cdot (\mathbf{U} - \mathbf{u}) \, dx \\ & + \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) \, dx + \int_\Omega (\rho(r) - \rho(\varrho)) \operatorname{div}_x \mathbf{U} \, dx \\ & + \int_\Omega [(r - \varrho) \partial_t P'(r) + \nabla_x P'(r) \cdot (r \mathbf{U} - \varrho \mathbf{u})] \, dx \end{aligned}$$

Results for dissipative solutions

Positive results

- Weak solutions exist for $p(\varrho) \approx a\varrho^\gamma$, $\gamma > \frac{N}{2}$ (P.L.Lions [1998] certain γ , EF, A. Novotný, H. Petzeltová [2000])
- Any weak solution is a dissipative solution (EF, B.J. Jin, A. Novotný, Y. Sun [2014])

Relative energy for complete fluids

Relative energy

$$\begin{aligned} & \mathcal{E}(\varrho, \vartheta, \mathbf{u} \mid r, \Theta, \mathbf{U}) \\ &= \int_{\Omega} \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + H_{\Theta}(\varrho, \vartheta) - \frac{\partial H_{\Theta}(r, \Theta)}{\partial \varrho} (\varrho - r) - H_{\Theta}(r, \Theta) \right] dx \end{aligned}$$

Ballistic free energy

$$H_{\Theta}(\varrho, \vartheta) = \varrho (e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta))$$

Hypothesis of thermodynamic stability

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \quad \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Weak-strong uniqueness, conditional regularity

Results (EF, B.J. Jin, A.Novotný, Y.Sun [2014])

- Weak and strong solutions of the barotropic Navier-Stokes system emanating from the same initial data coincide as long as the latter exists
- Weak solution with bounded density component emanating from smooth initial data is smooth

Singular limits

Rotating fluids

$$\begin{aligned}\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) &= 0 \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon} \varrho \mathbf{b} \times \mathbf{u} + \frac{1}{\varepsilon^{2M}} \nabla_x p(\varrho) \\ &= \varepsilon^R \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \frac{1}{\varepsilon^{2F}} \nabla_x G\end{aligned}$$

Path dependent singular limit

$\varepsilon \rightarrow 0$, certain relation between $M, R, F > 0$

- low Mach \Rightarrow compressible \rightarrow incompressible
- high Rossby \Rightarrow 3D \rightarrow 2D
- high Reynolds \Rightarrow viscous \rightarrow inviscid

Convergence to singular limit system

Target problem - Euler system

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0, \quad \operatorname{div}_x \mathbf{v} = 0, \quad x \in \mathbb{R}^2$$

Convergence results (joint work with A.Novotný, Y.Lu [2014])

■ Spatial geometry - infinite strip:

$$\Omega = \mathbb{R}^2 \times (0, \pi)$$

■ Complete slip (Navier) boundary conditions:

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad (\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0$$

Limits on domains with variable geometry

Channel like domains

$$\Omega_\varepsilon = \left\{ (\mathbf{x}, z) \mid z \in (0, 1), |\mathbf{x} - \varepsilon \mathbf{X}(z)|^2 < \varepsilon^2 R^2(z) \right\}, |\mathbf{X}(z)| < R(z)$$

Boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_\Sigma = 0, (\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{n}) \times \mathbf{n}|_\Sigma = 0$$

$$\Sigma = \partial\Omega \cap \{z \in (0, 1)\}$$

$$\mathbf{u}|_{z=0,1} = 0$$

Target systems

Inviscid limit

$$\begin{aligned}\partial_t(\rho_E A) + \partial_z(\rho_E u_E A) &= 0 \\ \partial_t(\rho_E u_E A) + \partial_z(\rho_E u_E^2 A) + A \partial_z p(\rho_E) &= 0\end{aligned}$$

Viscous limit

$$\begin{aligned}\partial_t(\rho_{NS} A) + \partial_z(\rho_{NS} u_{NS} A) &= 0 \\ \partial_t(\rho_{NS} u_{NS} A) + \partial_z(\rho_{NS} u_{NS}^2 A) + A \partial_z p(\rho_{NS}) \\ &= A \nu \partial_z^2 u_{NS} + \nu \partial_z (\partial_z \log(A) u_{NS}), \quad \nu = \frac{4}{3} \mu + \eta > 0\end{aligned}$$

$$A = R^2$$

Convergence

Korn-Poincaré inequality

$$\int_{\Omega_\varepsilon} |\mathbf{v}|^2 \, dx \leq c_{KP} \int_{\Omega_\varepsilon} |\nabla_x \mathbf{v} + \nabla_x^t \mathbf{v}|^2 \, dx$$

Convergence (joint work with P.Bella, M.Lewicka, A.Novotný [2015])

- Convergence to the target Euler system with geometric terms in the inviscid limit
- Convergence to the Navier-Stokes system in the viscous limit provided the bulk viscosity in the primitive system is positive

Equations driven by stochastic forces

Navier–Stokes system with stochastic forcing

$$d\rho + \operatorname{div}_x(\rho \mathbf{u}) dt = 0$$

$$d(\rho \mathbf{u}) + [\operatorname{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\rho)] dt = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) dt + \mathbb{G}(\rho, \rho \mathbf{u}) dW,$$

White–noise forcing

$$\mathbb{G}(\rho, \rho \mathbf{u}) dW = \sum_{k \geq 1} \mathbf{G}_k(\rho, \rho \mathbf{u}) dW_k.$$

Relative energy inequality

Relative energy inequality - joint result with D.Breit and M.Hofmanová [2015]

$$\begin{aligned} & \left[\mathcal{E}(\varrho, \mathbf{u} | r, \mathbf{U}) \right]_{t=0}^{t=\tau} \\ & + \int_0^\tau \int_{\Omega} (\mathbb{S}(\nabla \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \, dx \, dt \\ & \leq [M_{RE}]_{t=0}^{t=\tau} + \int_0^\tau \mathcal{R}(\varrho, \mathbf{u} | r, \mathbf{U}) \, dt \end{aligned}$$

Test functions

$$d r = D_t^d r \, dt + D_t^s r \, dW, \quad d \mathbf{U} = D_t^d \mathbf{U} \, dt + D_t^s \mathbf{U} \, dW$$

Stochastic remainder

Remainder

$$\begin{aligned} & \mathcal{R}(\varrho, \mathbf{u} | r, \mathbf{U}) \\ &= \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) \, dx + \int_{\Omega} \varrho \left(D_t^d \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \right) (\mathbf{U} - \mathbf{u}) \, dx \\ & \quad + \int_{\Omega} \left((r - \varrho) H''(r) D_t^d r + \nabla_x H'(r) (r \mathbf{U} - \varrho \mathbf{u}) \right) \, dx \\ & - \int_{\Omega} \operatorname{div}_x \mathbf{U} (p(\varrho) - p(r)) \, dx + \frac{1}{2} \sum_{k \geq 1} \int_{\Omega} \varrho \left| \frac{\mathbf{G}_k(\varrho, \varrho \mathbf{u})}{\varrho} - D_t^s \mathbf{U}_k \right|^2 \, dx \\ & \quad + \frac{1}{2} \sum_{k \geq 1} \int_{\Omega} \varrho H'''(r) |D_t^s r_k|^2 \, dx + \frac{1}{2} \sum_{k \geq 1} \int_{\Omega} p''(r) |D_t^s r_k|^2 \, dx \end{aligned}$$

Results for stochastic Navier-Stokes system

Weak–strong uniqueness (with D.Breit, M.Hofmanová [2015])

- Pathwise weak-strong uniqueness
- Weak-strong uniqueness in law

Inviscid–incompressible limit in the stochastic setting (with D.Breit, M.Hofmanová [2015])

Convergence to the limit stochastic Euler system for vanishing viscosity and the Mach number. Results for well-prepared data.

Possible extensions

Numerical analysis (T. Gallouet, R. Herbin, D. Maltese, A. Novotný [2014])

Relative energy inequality for the numerical scheme proposed by K. Karlsen and T. Karper. Error estimates.

Measure-valued solutions

Weak-strong uniqueness for measure-valued solutions - work in progress with P. Gwiazda, A. Swierczewska-Gwiazda, E. Wiedemann