INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

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Preprint No. 52-2016

PRAHA 2016

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Abstract

The sharp removability conditions for single point singularities of solutions of double-phase elliptic equations are established.

1991 Mathematics Subject Classification. 35B45, 35J60.

Key words. Isolated singularity, double-phase elliptic equation

1 Statement of the problem

In this paper we study solutions of elliptic equations with non-standard growth conditions

$$-\mathrm{div}\mathbb{A}\,(x, \nabla\, u) = 0, \quad \forall x \in \Omega \setminus \{x_0\}, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, $x_0 \in \Omega$ and $\mathbb{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$.
We are assuming that the function $\mathbb{A}(\cdot, \xi)$ is Lebesgue measurable for all $\xi \in R^n$ and $\mathbb{A}(x, \cdot)$ is continuous for almost all $x \in \Omega$. The structural inequalities for $\mathbb{A}$ will represent two possible different elliptic behaviours of the operator. Namely,

*The work of Yu.N. was supported by the General Research Programm of the Czech Academy of Sciences, RVO: 67985840.

we assume that with some positive constants ν_1, ν_2 the following conditions are satisfied:

$$\mathbb{A}\,(x,\xi)\,\xi \geqslant \nu_1\, g\,(a(x),|\xi|)\,|\xi|\,,\ \ \xi\in\mathbb{R}^n,$$

$$|\mathbb{A}(x,\xi)| \leqslant \nu_2\, g(a(x),|\xi|), \tag{1.2}$$

where

$$g(a(x),t) = t^{p-1} + a(x)t^{q-1},\ t>0, \tag{1.3}$$

with a nonegative function $a(x)$ such that $a\in C^{0,\alpha}(\Omega)$ for some $\alpha\in(0,1]$.

According to the coefficient $a(x)$, the operator in (1.1) represents two different growth with respect to the gradient. On the set where $a(x)=0$ the equation (1.1) is of p-Laplacian type, while in case of $a(x)>0$ the growth of the coefficients by gradient is at rate q. So that, we will distinguish two cases of degenerate behaviour: $a(x_0)=0$ (so-called "p-phase") and $a(x_0)>0$ (so called "(p,q)-phase"). We suppose that the exponents p,q satisfy

$$1<p\leqslant q\leqslant \min\left(p+\alpha,\,\frac{np}{n-p}\right),\quad p\leqslant n,\quad \text{if}\quad a(x_0)=0, \tag{1.4}$$

$$1<p\leqslant q\leqslant n,\quad \text{if}\quad a(x_0)>0. \tag{1.5}$$

Functionals with nonstandard growth conditions appeared from the investigation of properties of composite materials in theory of homogenization and elasticity theory. In the farther, the qualitative theory of such quasilinear equations was extensively studied because of the various applications in mathematical physics and reach mathematical structures (see e.g. [1] –[6], [8]–[15], [24], [25]).

Starting from the seminal papers of J. Serrin ([19], [20]), the qualitative behaviour of solutions to quasilinear equations with the standard growth ($p=q$) was investigated by many authors in case of a point singularity. We refer to [23], [18] for an account of these results.

The questions of removability of isolated singularities or singularities on some manifolds for solutions of elliptic equation with (p,q)–growth conditions were studied recently in [13], [16]. Namely, there were considered equations of type

$$\operatorname{div}\left(g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right)=0,$$

where $g(t)$ satisfies the following conditions

$$\left(\frac{t}{\tau}\right)^{p-1}\leqslant\frac{g(t)}{g(\tau)}\leqslant\left(\frac{t}{\tau}\right)^{q-1},\quad t\geqslant\tau>0.$$

We are interested here in getting sharp pointwise conditions for solutions of equation (1.1), which guarantee that a point singularity at $\{x_0\}$ is removable. That is, the solution can be extended to the whole domain Ω.

2 Formulation of the main results

To start with, we give a definition of a basic functional space appropriate for a weak formulation of problem (1.1). Namely, by $W^{1,G}(\Omega)$ we denote the class of functions having weak derivatives in Ω and such that

$$\int_\Omega G(a(x), |\nabla u|)\, dx < \infty,$$

where $G(a(x),t) = t\, g(a(x),t)$, $t > 0$. Functions g and a were defined in (1.2), (1.3).

We can define now a weak solutions with singularity at a single point and formulate how we understand removability of this singularity of a weak solution.

Definition 2.1. A function $u(x)$ is said to be a weak solution of equation (1.1) in $\Omega \setminus \{x_0\}$, if for an arbitrary function $\psi \in C^1(\overline{\Omega})$ vanishing in a neighborhood of $\{x_0\}$, there is an inclusion $u\psi \in W^{1,G}(\Omega)$ and the integral identity

$$\int\limits_\Omega \mathbb{A}\,(x, \nabla u)\, \nabla(\psi\varphi)\, dx = 0 \tag{2.1}$$

holds for any $\varphi \in W_0^{1,G}(\Omega)$.

Definition 2.2. We say that a weak solution u of (1.1) has a removable singularity at some point $\{x_0\}$ if $u(x)$ can be extended to $\{x_0\}$ so that its extension $\tilde{u}$ belongs to $W^{1,G}(\Omega)$ and satisfies the equation (1.1).

To formulate our main results, we need to set up a characteristic describing a local behaviour of the weak solution $u(x)$ in some neighborhood of the singular point $\{x_0\}$. That is, for R, $0 < R < \min\{1, \mathit{dist}\,(x_0, \partial\,\Omega)\}$, and any r, $0 < r < R$ we define a number $M(r)$ such that

$$M(r) := \operatorname{ess\,sup}\, \{|u(x)| : \ x \in K(r,R)\}, \tag{2.2}$$

where

$$K(r,R) := B_R(x_0) \setminus B_r(x_0), \quad B_r(x_0) := \{x : \ |x - x_0| < r\}.$$

The regularity result of [5], [8] yields that $M(r) < \infty$ for any $r > 0$.

Our first main theorem is about the removability result in case of p–phase.

Theorem 2.1. *Let u be a weak solution to the equation (1.1) in $\Omega \setminus \{x_0\}$. Let the conditions (1.2), (1.4) be fulfilled and $a(x_0) = 0$. Assume also that:*

$$\lim_{r\to 0} M(r)\, r^{\frac{n-p}{p-1}} = 0, \quad \text{if} \ \ p \leqslant q \leqslant p + \alpha\, \frac{p-1}{n-1}, \quad p < n, \tag{2.3}$$

$$\lim_{r\to 0} M(r)\, r^{\frac{n-q+\alpha}{q-1}} = 0, \quad \text{if} \ \ p + \alpha\, \frac{p-1}{n-1} \leqslant q < p + \alpha, \quad p < n, \tag{2.4}$$

$$\lim_{r\to 0} M(r)\, \ln^{-1} \frac{1}{r} = 0, \quad \text{if} \ \ p = n, \quad q = n + \alpha. \tag{2.5}$$

Then the singularity of u at $\{x_0\}$ is removable.

The next theorem gives sharp removability condition in the case of (p,q)–phase.

Theorem 2.2. *Let u be a weak solution to (1.1) in $\Omega \setminus \{x_0\}$. Let the conditions (1.2), (1.5) be fulfilled and $a(x_0) > 0$. Assume also that*

$$\lim_{r\to 0} M(r)\, r^{\frac{n-q}{q-1}} = 0, \quad \text{if } \ p \leqslant q < n, \tag{2.6}$$

$$\lim_{r\to 0} M(r)\, \ln^{-1}\frac{1}{r} = 0, \quad \text{if } \ q = n. \tag{2.7}$$

Then the singularity of $u(x)$ at $\{x_0\}$ is removable.

We point out that our approach continues the studies of [17], [21], [22] and is based on the method of pointwise and integral estimates of nonlinear potentials. The rest of the paper contains proofs of the above theorems.

3 Proof of Theorems 2.1, 2.2

3.1 Integral estimates for gradient of solutions in the case of p-phase

In this Subsection we derive the auxiliary integral estimates of weak solutions to equation (1.1). First, we recall some technical tools and build a sequence of appropriate cut-off functions.

In what follows we will frequently use the following Lemma ([7], Chapter II, Lemma 4.7).

Lemma 3.1. *Let $\{y_j\}$ be a sequence of nonnegative numbers such that for any $j = 0, 1, 2, \ldots$ the inequality*

$$y_{j+1} \leq C\, b^j y_j^{1+\varepsilon}$$

holds with positive constants $\varepsilon, C > 0, b > 1$. Then the following estimate is true

$$y_j \leq C^{\frac{(1+\varepsilon)^j - 1}{\varepsilon}}\, b^{\frac{(1+\varepsilon)^j - 1}{\varepsilon^2} - \frac{j}{\varepsilon}}\, y_0^{(1+\varepsilon)^j}.$$

Particularly, if

$$y_0 \leq C^{-\frac{1}{\varepsilon}} b^{-\frac{1}{\varepsilon^2}}$$

then

$$\lim_{j\to\infty} y_j = 0.$$

For $r > 0$, $p < n$ we set a nonnegative cut-off function $\psi_r(x) \in C^1(B_R(x_0))$, $0 \leqslant \psi_r(x) \leqslant 1$, satisfying the following conditions

$$\psi_r(x) \equiv 0 \ \text{ if } \ |x - x_0| \leqslant r, \quad \psi_r(x) \equiv 1 \ \text{ if } \ |x - x_0| \geqslant 2r, \quad |\nabla\psi_r(x)| \leqslant 2\, r^{-1}.$$

For every $r>0$ and $p=n$ we set $\widetilde{\psi}_r(x)\in C^1(B_R(x_0))$, $0\leqslant\widetilde{\psi}_r(x)\leqslant 1$, such that

$$\widetilde{\psi}_r(x)\equiv 0 \text{ if } |x-x_0|\leqslant r,\ \ \widetilde{\psi}_r(x)\equiv 1 \text{ if } |x-x_0|\geqslant\sqrt{r},\ \text{ and } |\nabla\widetilde{\psi}_r(x)|\leqslant\frac{2}{|x-x_0|\,\ln\frac{1}{r}}.$$

We will use the following notations

$$u_r(x):=(u(x)-M(r))_+,\quad E(r):=\{x\in B_R(x_0):\ u(x)>M(r)\}.$$

By the know parameters we understand the numbers $\nu_1,\nu_2,n,\ p,\ q,R$ and $[a]_{C^{0,\alpha}(\Omega)}$, where

$$[a]_{C^{0,\alpha}(\Omega)}:=\sup_{x,y\in\Omega,x\neq y}\frac{|a(x)-a(y)|}{|x-y|^{\alpha}}.$$

In what follows γ stands for a generic constant that depends on the known parameters only and may vary from line to line.

Lemma 3.2. *Let $u(x)$ be a weak solution of equation (1.1) and all conditions of Theorem 2.1 are fulfilled. Then the following inequalities hold*

$$\int_{E(R)}|\nabla u|^p\,\psi_r^q\,dx+\int_{E(R)}a(x)|\nabla u|^q\,\psi_r^q\,dx\leqslant\gamma M(r)\,\mu_1(r),\ \ \text{if } p<n, \tag{3.1}$$

and

$$\int_{E(R)}|\nabla u|^n\,\widetilde{\psi}_r^{n+\alpha}\,dx+\int_{E(R)}a(x)|\nabla u|^{n+\alpha}\,\widetilde{\psi}_r^{n+\alpha}\,dx\leqslant\gamma M(r)\,\mu_2(r), \tag{3.2}$$

if $p=n$ and $q=n+\alpha$. Here

$$\mu_1(r):=\left[\begin{array}{ll} M^{p-1}(r)r^{n-p}+\left(M(r)r^{\frac{n-p}{p-1}}\right)^{q-1}, & \text{if }\ p\leqslant q\leqslant p+\alpha\,\dfrac{p-1}{n-1}\ \text{ and }\ p<n,\\ \left(M(r)r^{\frac{n-q+\alpha}{q-1}}\right)^{p-1}+M^{q-1}(r)r^{n-q+\alpha}, & \text{if }\ p+\alpha\,\dfrac{p-1}{n-1}\leqslant q<p+\alpha\ \text{ and }\ p<n;\end{array}\right.$$

$$\mu_2(r):=\left(M(r)\ln^{-1}\frac{1}{r}\right)^{n-1}+\left(M(r)\ln^{-1}\frac{1}{r}\right)^{n-1+\alpha}.$$

Proof First, note that $0\leqslant a(x)\leqslant\gamma|x-x_0|^{\alpha}$ for $x\in B_R(x_0)$. Testing (2.1) by $\varphi=u_R\psi_r^{q-1}$, $\psi=\psi_r$ if $p<n$, using the condition (1.2) and the Young inequality we have

$$\int_{E(R)}|\nabla u|^p\,\psi_r^q\,dx+\int_{E(R)}a(x)|\nabla u|^q\,\psi_r^q\,dx\leqslant\gamma r^{-p}\int\limits_{K(r,2r)}u_R^p\,dx+\gamma r^{\alpha-q}\int\limits_{K(r,2r)}u_R^q\,dx,$$

from this, applying the definition of $M(r)$ we arrive at the required inequality (3.1).

To show the second estimate in the Lemma, we test (2.1) by $\varphi = u_R \widetilde{\psi}_r^{n+\alpha-1}$, $\psi = \widetilde{\psi}_r$ if $p = n$ and $q = n+\alpha$. Using the condition (1.2) and the Young inequality, we have

$$\int_{E(R)} |\nabla u|^n \, \widetilde{\psi}_r^{n+\alpha} \, dx + \int_{E(R)} a(x) |\nabla u|^{n+\alpha} \, \widetilde{\psi}_r^{n+\alpha} \, dx$$
$$\leqslant \gamma \ln^{-n} \frac{1}{r} \int\limits_{K(r,\sqrt{r})} u_R^n |x-x_0|^{-n} \, dx + \gamma \ln^{-n-\alpha} \frac{1}{r} \int\limits_{K(r,\sqrt{r})} u_R^{n+\alpha} |x-x_0|^{-n} \, dx.$$

From this, using the definition of $M(r)$, we arrive at the required (3.2).

For any $t \geqslant M(R)$ we set

$$E_t(R) := \{x \in E(R) : \; u(x) < t\}, \quad u^{(t)}(x) := \min\{u_R(x), \, t - M(R)\}.$$

Lemma 3.3. *Let $u(x)$ be a weak solution of equation (1.1) and all the conditions of Theorem 2.1 are fulfilled. Then the following inequalities hold*

$$\int_{E_t(R)} |\nabla u|^p \, \psi_r^q \, dx \leqslant \gamma (t - M(R)) \, \mu_3(r), \;\; \text{if } p < n, \tag{3.3}$$

and

$$\int_{E_t(R)} |\nabla u|^n \, \widetilde{\psi}_r^{n+\alpha} \, dx \leqslant \gamma (t - M(R)) \, \mu_4(r), \quad \text{if } p = n \text{ and } q = n+\alpha. \tag{3.4}$$

Here

$$\mu_3(r) := \left[\begin{array}{l} \left(M(r) r^{\frac{n-p}{p-1}} \mu_1(r) \right)^{\frac{p-1}{p}} + \left(M(r) r^{\frac{n-p}{p-1}} \mu_1(r) \right)^{\frac{q-1}{q}}, \\ \qquad\qquad \text{if } \; p \leqslant q \leqslant p + \alpha \dfrac{p-1}{n-1} \;\; \text{and} \;\; p < n, \\ \left(M(r) r^{\frac{n-q+\alpha}{q-1}} \mu_1(r) \right)^{\frac{p-1}{p}} + \left(M(r) r^{\frac{n-q+\alpha}{q-1}} \mu_1(r) \right)^{\frac{q-1}{q}}, \\ \qquad\qquad \text{if } \; p + \alpha \dfrac{p-1}{n-1} \leqslant q < p + \alpha, \end{array} \right.$$

$$\mu_4(r) := \left(M(r) \ln^{-1} \frac{1}{r} \mu_2(r) \right)^{\frac{n-1}{n}} + \left(M(r) \ln^{-1} \frac{1}{r} \mu_2(r) \right)^{\frac{n-1+\alpha}{n+\alpha}}.$$

Proof Testing (2.1) by $\varphi = u^{(t)} \psi_r^{q-1}$, $\psi = \psi_r$ if $p < n$, using the condition (1.2) we have

$$\int_{E_t(R)} |\nabla u|^p \, \psi_r^q \, dx \leqslant \gamma r^{-1} (t - M(R)) \int\limits_{K(r,2r)} |\nabla u_R|^{p-1} \psi_r^{q-1} \, dx$$
$$+ \gamma r^{-1} (t - M(R)) \int\limits_{K(r,2r)} a(x) |\nabla u_R|^{q-1} \psi_r^{q-1} \, dx. \tag{3.5}$$

By the Hölder inequality and Lemma 3.2, the terms in the right-hand side of (3.5) are estimated as follows

$$\gamma r^{-1}\int_{K(r,2r)}|\nabla u_R|^{p-1}\,\psi_r^{q-1}\,dx+\gamma r^{-1}\int_{K(r,2r)}a(x)|\nabla u_R|^{q-1}\,\psi_r^{q-1}\,dx$$
$$\leqslant \gamma r^{\frac{n-p}{p}}\Bigg(\int\limits_{E(R)}|\nabla u|^p\psi_r^q\,dx\Bigg)^{\frac{p-1}{p}}+\gamma r^{\frac{n-q+\alpha}{q}}\Bigg(\int\limits_{E(R)}a(x)|\nabla u|^q\psi_r^q\,dx\Bigg)^{\frac{q-1}{q}}$$
$$\leqslant \gamma r^{\frac{n-p}{p}}\,(M(r)\mu_1(r))^{\frac{p-1}{p}}+\gamma r^{\frac{n-q+\alpha}{q}}\,(M(r)\mu_1(r))^{\frac{q-1}{q}}\,,$$

from this we arrived at the required (3.3).

Similarly, testing (2.1) by $\varphi=u^{(t)}\widetilde{\psi}_r^{n+\alpha-1}$, $\psi=\widetilde{\psi}_r$ if $p=n$ and $q=n+\alpha$, using the condition (1.2), Young's and Hölder's inequalities, and Lemma 3.2, we arrive at (3.4). This proves Lemma 3.3.

3.2 Boundedness of solutions in the case of p-phase

In this section we introduce the proof of Theorem 2.1 applying the Moser's iteration technique.

We prove the boundedness of solution to equation (1.1) only in the case of $p<n$. The proof of the boundedness of solutions in the case $p=n$ and $q=n+\alpha$ is completely similar.

We fix $\rho:\ 0<\rho\leqslant\frac{R}{2}$ and for any $j=1,2,\dots$, define the sequence of numbers

$$\rho_j^{(1)}=\frac{\rho}{2}(1+2^{-j}),\ \ \rho_j^{(2)}=\frac{\rho}{2}(3-2^{-j}),\ \ \overline{\rho}_j^{(1)}=\frac{1}{2}(\rho_j^{(1)}+\rho_{j+1}^{(1)}),\ \ \overline{\rho}_j^{(2)}=\frac{1}{2}(\rho_j^{(2)}+\rho_{j+1}^{(2)}).$$

$$D_j:=\left\{x:\ \rho_j^{(1)}\leqslant|x-x_0|\leqslant\rho_j^{(2)}\right\},\ \overline{D}_j:=\left\{x:\ \overline{\rho}_j^{(1)}\leqslant|x-x_0|\leqslant\overline{\rho}_j^{(2)}\right\},\ k_j=2k-\frac{k}{2^j},$$

here k is a positive number depending on the known parameters only, which will be specified later.

Now we introduce the sequence of nonnegative cut-off functions $\xi_j\in C_0^\infty(\overline{D}_j)$ such that $\xi_j(x)\equiv 1$ in D_j, $|\nabla\xi_j|\leqslant\gamma\,2^j\,\rho^{-1}$.

We test the integral identity (2.1) by the functions $\varphi(x)=(u_R-k_{j+1})_+\,\xi_j^{q-1}$, $\psi(x)=\xi_j(x)$, After some easy computations, using structural conditions (1.2) and the Young inequality, we deduce

$$\int\limits_{\overline{D}_j}|\nabla(u_R-k_{j+1})_+|^p\,|\xi_j^q\,dx$$
$$\leqslant\gamma\,2^{\gamma\,j}\rho^{-p}\int\limits_{\overline{D}_j}(u_R-k_{j+1})_+^p\,dx+\gamma\,2^{\gamma\,j}\rho^{\alpha-q}\int\limits_{\overline{D}_j}(u_R-k_{j+1})_+^q\,dx.$$

Using conditions (2.3), (2.4) and the definition (2.2) of $M(\rho)$, from the last inequality we have

$$\int\limits_{\overline{D}_j} |\nabla (u_R - k_{j+1})_+|^p \, \xi_j^q \, dx \leqslant \gamma \, 2^{\gamma\, j} H(\rho) \int\limits_{\overline{D}_j} (u_R - k_{j+1})_+^p \, dx, \tag{3.6}$$

where

$$H(\rho) := \left[\begin{array}{l} \rho^{-p}, \quad \text{if} \quad p \leqslant q \leqslant p + \alpha \, \dfrac{p-1}{n-1}, \\ \rho^{-p + \frac{\alpha(p-1)-(n-1)(q-p)}{q-1}}, \quad \text{if} \quad p + \alpha \, \dfrac{p-1}{n-1} \leqslant q < p + \alpha. \end{array} \right.$$

Using the Hölder inequality, Sobolev's embedding theorem and the evident inequality

$$\left| \overline{D}_j \bigcap \{u_R > k_{j+1}\} \right| \leqslant \gamma 2^{\gamma\, j} k^{-p} \int\limits_{\overline{D}_j} (u_R - k_j)_+^p \, dx$$

From (3.6) we get

$$\begin{aligned} \int\limits_{D_{j+1}} (u_R - k_{j+1})_+^p \, dx &\leqslant \int\limits_{\overline{D}_j} (u_R - k_{j+1})_+^p \xi_j^q \, dx \\ &\leqslant \gamma \int\limits_{\overline{D}_j} \left| \nabla \left((u_R - k_{j+1})_+ \xi_j^{\frac{q}{p}} \right) \right|^p \, dx \left| \overline{D}_j \bigcap \{u_R > k_{j+1}\} \right|^{\frac{p}{n}} \\ &\leqslant \gamma \, 2^{\gamma\, j} k^{-\frac{p^2}{n}} H(\rho) \left(\int\limits_{\overline{D}_j} (u_R - k_j)_+^p \, dx \right)^{1+\frac{p}{n}}. \end{aligned}$$

Setting

$$y_j := \int\limits_{\overline{D}_j} (u_R - k_j)_+^p \, dx$$

we obtain

$$y_{j+1} \leqslant \gamma 2^{j\gamma} k^{-\frac{p^2}{n}} H(\rho) y_j^{1+\frac{p}{n}}, \quad j = 0, 1, 2, \ldots .$$

Due to Lemma 3.1 this inequality implies that $y_j \to 0$ as $j \to \infty$ if k satisfies the following condition

$$y_0 = \gamma H^{-\frac{n}{p}}(\rho) k^p,$$

from this we obtain

$$(M(\rho) - M(R))^p \leqslant \gamma H^{\frac{n}{p}}(\rho) \int\limits_{D_0} u_R^p \, dx. \tag{3.7}$$

Using that $D_0 \subset K\left(\frac{\rho}{2}, R\right)$ and $u_R(x) = u^{\left(M\left(\frac{\rho}{2}\right)\right)}(x)$ for $x \in K\left(\frac{\rho}{2}, R\right)$, applying Poincaré inequality and Lemma 3.3, finally, from (3.7) we have

$$(M(\rho)-M(R))_+^p \leqslant \gamma\, \rho^p H^{\frac{n}{p}}(\rho) \int\limits_{E_{M\left(\frac{\rho}{2}\right)}(R)} |\nabla u_R|^p\, dx \leqslant \gamma\, \rho^p H^{\frac{n}{p}}(\rho)(M(\rho)-M(R))_+ \mu_3(r).$$

Iterating the last inequality, we obtain

$$(M(\rho) - M(R))_+ \leqslant \gamma \left(\rho^p H^{\frac{n}{p}}(\rho) \mu_3(r)\right)^{\frac{1}{p-1}}. \tag{3.8}$$

Passing to the limit as $r \to 0$ and using conditions (2.3), (2.4), from (3.8) we get that $M(\rho) \leqslant M(R)$ for every $\rho \leqslant \frac{R}{2}$. The boundedness of solution is proved.

3.3 End of proof of Theorem 2.1

Let K be a compact set in Ω. We take a function $\xi \in C_0^\infty(\Omega)$ such that $\xi = 1$ for $x \in K$. Testing (2.1) by $\varphi = u\,\xi\,\psi_r^{q-1}$, $\psi = \psi_r$, using (1.2), Young's inequality, boundedness of u and passing to the limit as $r \to 0$, we get

$$\int\limits_K G(a(x), |\nabla u|)\, dx \leqslant \gamma. \tag{3.9}$$

The next step is to test integral identity (2.1) by $\varphi = u\,\psi_r$, where φ is an arbitrary function belonging to $W_0^{1,G}(\Omega)$. Using boundedness of solution, after passing to the limit as $r \to 0$, we see that the integral identity (2.1) is valid with an arbitrary $\varphi \in W_0^{1,G}(\Omega)$ and $\psi \equiv 1$. This proves Theorem 2.1 in case $p \leqslant q < p + \alpha,\ p < n$.

The proof of Theorem 2.1 in the case of $p = n$, $q = n + \alpha$ is completely similar.

3.4 Proof of Theorem 2.2

We set $R_0 = \left(\frac{a(x_0)}{2[a]_{C^{0,\alpha}(\Omega)}}\right)^{\frac{1}{\alpha}}$ and let $R_1 < \min\{R_0, R\}$. We note that

$$\frac{a(x_0)}{2} \leqslant a(x) \leqslant \frac{3}{2} a(x_0),$$

for an arbitrary $x \in B_{R_1}(x_0)$. Therefore,

$$\gamma(a(x_0))\, g(t) \leqslant g(a(x), t) \leqslant \gamma(a(x_0))\, g(t), \quad t > 0,$$

where $g(t) = t^{p-1} + t^{q-1}$. Now Theorem 2.2 is an immediate consequence of Theorem 1 from [13]. This completes the proof of Theorem 2.2.

4 Acknowledgement

The author thanks Igor I. Skrypnik, this paper could not appear without his help.

References

[1] Baroni, P., Colombo, M., Mingione, G., *Harnack inequalities for double phase functionals*, Nonlinear Anal., **121**, (2015), 206–222.

[2] Colombo, M., Mingione, G., *Bounded minimisers of double phase variational integrals*, Arch. Ration. Mech. Anal., **218**, (2015), no.1, 219–273.

[3] Colombo, M., Mingione, G., *Regularity for double phase variational problems*, Arch. Ration. Mech. Anal., **215**, (2015), no.2, 443–496.

[4] Esposito, L., Leonetti, F., Mingione, G., *Sharp regularity for functionals with (p, q)-growth*, J. Differential Equations, **204**, (2004), no.1, 5–55.

[5] Fusco, N., Sbordone, C., *S*ome remarks on the regularity of minima of anisotropic integrals, Comm. Partial Differential Equations, **18**, (1993), no.1–2, 153–167.

[6] Kolodij, I.M., *On boundedness of generalized solutions of elliptic differential equations*, Vestnik Moscow Gos. Univ., **5**, (1970), 44–52.

[7] O.A. Ladyzenskaya, N.N. Ural'tseva, Linear and quasilinear elliptic equations, Academic Press, New York – London, XVIII, 1968, pp. 495.

[8] Lieberman, G. M., *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Partial Differential Equations, **16**, no.2-3, (1991), 311–361.

[9] Lieberman, G. M., *Gradient estimates for a new class of degenerate elliptic and parabolic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci.(IV), **21**, (1994), no.4, 497–522.

[10] Marcellini, P., *Regularity and existence of solutions of elliptic equations with p, q-growth conditions*, J. Differential Equations, **90**, no.1, (1991), 1–30.

[11] Marcellini, P., *Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions*, Arch. Rat. Mech. Anal., **105**, (1989), no. 3, 267–284.

[12] Mascolo, E., Papi, G., *Harnack inequality for minimizers of integral functionals with general growth*, Nonlinear Differential Equations and Appl., **3**, (1996), no. 2, 231–244.

[13] Mihăilescu, M., *Classification of isolated singularities for nonhomogeneous operators in divergence form*, J. Func. Anal., **268**, (2015), no. 8, 2336–2355.

[14] Moscariello, G., *Regularity result for quasiminima of functionals with non-polynomial growth*, J. Math. Anal. Appl., **168**, (1992), no.2, 500–510.

[15] Moscariello, G., Nania, L., *Hölder continuity of minimizers of functionals with non-standard growth conditions*, Ricerche Mat., **40**, (1991), no.2, 259–273.

[16] Namlyeyeva, Yu.V., Skrypnik, I.I., *Removable singularities for elliptic equations with (p, q)-growth conditions*, preprint IM CAS, (2015).

[17] Namlyeyeva, Yu.V., Shishkov, A.E., Skrypnik, I.I., *Isolated singularities of solutions of quasilinear anisotropic elliptic equations*, Adv. Nonlinear Stud., **6** (2006), no. 4, 617–641.

[18] Nicolosi, F., Skrypnik, I.V., Skrypnik, I.I., *Precise point-wise growth conditions for removable isolated singularities*, Commun. Partial Differential Equations, **28** (2003), no. 3–4, 677–696.

[19] Serrin, J., *Removable singularities of solutions of elliptic equations*, Arch. Rat. Mech. Anal., **20(3)** (1965), 163–169.

[20] Serrin, J., *Isolated singularities of solutions of quasi-linear equations*, Acta Math., **113** (1965), no.3, 219–240.

[21] Skrypnik, I.V., *On pointwise estimates of certain capacity potentials*, Translations of A.M.S., Providence, RI, 15–22, (1989).

[22] Skrypnik, I.V., *Methods for analysis of nonlinear elliptic boundary value problems*, Translations of Mathematical Monographs, 139, AMS, Providence, RI, 1994, 348 pp.

[23] Veron, L., *Singularities of solution of second order quasilinear equations*, Pitman. Res. Notes in Math. Series, 353, Longman, Harlow, 1996, 377 pp.

[24] Zhikov, V.V., *Averaging of functionals of the calculus of variations and elasticity theory*, (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **50**, (1986), no. 4, 675–710.

[25] Zhikov, V.V., *On Lavrentiev's phenomenon*, Russian J. Math. Phys., **3** (1995), no.2, 249–269.