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Contents

Summary

We deal with the numerical simulation of a motion of viscous compressible fluids. Neglecting their microstructure, we consider fluids as continuum and therefore, the conception of the mechanic of continuous can be applied. Taking into account the mathematical formulation of conservation laws and constitutive relations, we obtain the *compressible Navier-Stokes equations* (NSEs) which represent a nonlinear system of partial differential equations of hyperbolic-parabolic type.

Since there are fundamental obstacles to find an analytical solution of this problem, our aim is to develop a sufficiently robust, accurate and efficient *numerical method* for the solution of the NSEs. It follows from theoretical considerations, numerical experiments as well as practical measurements that a solution of the NSEs is piecewise regular but it can contain discontinuities and steep gradients as shock waves, contact discontinuities, boundary layers, wakes, etc. Therefore, we deduce that the *discontinuous Galerkin method* (DGM) seems to be a suitable approach for the numerical solution of the NSEs.

DGM is based on a piecewise polynomial but discontinuous approximation which provides robust and high-order accurate solvers, particularly in transport dominated regimes. Furthermore, there is considerable flexibility in the choice of the mesh design; indeed, DGM can easily handle non-matching and non-uniform grids, even anisotropic, and polynomial approximation degrees. This allows a simple treatment with *hp*-adaptation techniques. Additionally, orthogonal bases can easily be constructed which lead to diagonal mass matrices; this is particularly advantageous for unsteady problems. Finally, in combination with block-type preconditioners, DGMs can easily be parallelized.

We start with a numerical analysis of the DGM applied to a simplified model represented by a scalar nonlinear convection-diffusion equation. We introduce several variants of the DGM and show that a numerical solution converges to the exact one provided

that the mesh size tends to zero. Moreover, we derive a priori error estimates which give a dependence of the order of convergence of the DGM on the used degree of polynomial approximation and the regularity of the exact solution. Finally, presented classes of numerical experiments are in a good agreement with theoretical results.

In the second part, we apply the discontinuous Galerkin discretization to the system of the compressible Navier-Stokes equations. A special attention is paid to the discretization of the diffusive terms and the choice of the boundary conditions where the situation is more complicated than in the scalar case. Moreover, we deal with a higher order temporal discretization in order to solve sufficiently precisely unsteady flow regimes. We present several numerical experiments of inviscid as well as viscous flows in subsonic, transonic and supersonic flow regimes. These numerical simulations indicate a great potential of DGM for industrial applications.

Finally, we present the so-called *anisotropic mesh adaptation* method which tries to optimise computational grids in order to minimise the number of degree of freedom for a given tolerance of the computational error. We develop a mathematical background of this approach which is based on a fulfilment of a necessary condition for a grid. Then we adapt this very general technique to simulation of viscous compressible flows.

A summary of the achieved results is enclosed.

Resumé

Zabýváme se numerickou simulací proudění vazkých stlačitelných tekutin. Zanedbáním jejich miktrostruktury můžeme považovat tekutiny za kontinuum a tedy lze použít nástroje mechaniky kontinua. Z matematického popisu zákonů zachování a z konstitutivních vztahů pak odvodíme *Navierovy-Stokesovy* (NS) rovnice pro stačitelné tekutiny, které představují soustavu nelinárních parciálních diferenciálních rovnic parabolicko-hyperbolického typu.

Vzhledem k tomu, že existují pricipiální potíže k nalezení analytického řešení tohoto problému, tak naším cílem je vývoj dostatečně robustní, přesné a efektivní *numerické metody* pro řešení Navierových-Stokesových rovnic. Z teoretických úvah, numerických experimentů a rovněž i experimelntálních meření plyne, že řešení NS rovnic je po částech hladké ale může obsahovat nespojitosti či strmé gradienty jako rázové vlny, kontaktní nespojitosti, mezní vrstvy a úplavy, atd. Z tohoto důvodu se domníváme, že *nespojitá Galerkinova* (DG) metoda je vhodná k numerickému řešení NS rovnic.

DG metoda je založena na po částech polynomiální, ale nespojité aproximaci, což vede k robustním řešičům vysokého řádu přesnosti, zejména v řežimech s převládající konvekcí. Navíc je zde velká volnost ve volbě prostorových sítí, DG metoda umožňuje pracovat s neregulárními a neuniformními sítěmi (i anisotropními) a různými stupni polynomiálními aproximaci na různých elementech. To umožňuje snadnou realizaci *hp*-adaptivních metod. Dále lze snadno konstruhovat ortogonální báze a tedy diagonální matice hmotnosti, což má výhodu zejména při řešení nestacionárních problémů. Konečně, ve spojení s blokovými typy předpodmíňovačů, DG metodu lze snadno paralelizovat.

Věnujeme se numerické analýze DG metody aplikované na modelovou úlohu, která je reprezentována skalární nelineární konvektivně-difusní rovnicí. Představujeme několik variant DG metody a ukážeme, že numerické řešení konverguje k přesnému v případě, že krok sítě konverguje k nule. Dále odvozujeme a priorní

odhady chyb, které nám dávají závislost řádu konvergence DG metody na použitém stupni polynomiální aproximace a regularitě přesného řešení. Řada numerických experimentů dává velmi dobrou schodu s teoretickými výsledky.

V další části aplikujeme nespojitou Galerkinovu metodu na soustavu Navierových-Stokesových rovnic. Zvláštní pozornost je věnovýna diskretizaci difusních členů a volbě okrajových podmínek, kde je situace mnohem komplikovanější než-li pro skalární příklad. Dále se věnujeme časové diskretizaci vyššího řádu, což je třeba pro řešení nestacionárních problémů. Ukazujeme několik numerických experimentů proudění nevazké i vazké tekutiny v subsonických, transonických a supersonických režimech proudění. Tyto numerické simulace ukazují na velký potenciál použití DG metody pro průmyslové aplikace.

Nakonec se zabýváme *anizotropní adaptací sítí* (AMA), která optimalizuje výpočetní sítě ve smyslu, že minimalizuje počet stupňů volnosti pro danou toleranci na výpočetní chybu. Rozvíjíme matematickou teorii tohoto přístupu, který je založen na splňení nutné podmínky pro síť. Dále upravujeme tuto velice obecnou technologii pro případ simulací proudění vazkých stlačitelných tekutin.

Na závěr předkládáme shrnutí dosažených výsledků.

1 Introduction

The field of computational fluid dynamics (CFD) has already had a significant impact on science and engineering of fluid dynamics, ranging from a role of aircraft design to simulation of a spread pollution in the environment. The ultimate goal of the field of CFD is to understand the physical events that occur in the flow of fluids around and within designed objects. These events are related to the action and reaction of phenomena such as dissipation, diffusion, convection, shock waves, boundary layers and turbulence.

In the field of aerodynamics, all of these phenomena are governed by the compressible Navier-Stokes equations. Many of the most important aspects of these relations are nonlinear and, as a consequence, often have no analytical solution. This, of course, motivates the numerical solution of the associated partial differential equations.

The numerical solutions of the Navier-Stokes equations frequently exhibit localised structures, such as propagating discontinuities and sharp transition layers whose reliable numerical approximation presents a challenging computational task. Indeed, in order to resolve such localised phenomena in an accurate and efficient way one has to use locally refined (adapted) computational meshes.

Therefore, an efficient tool for the numerical simulation of compressible flows has to contain a sufficiently robust and accurate solver for the system of governing equations as well as a suitable adaptive method. Within this thesis we develop an efficient technique for the numerical solution of the Navier-Stokes equations which is based on a combination of the *discontinuous Galerkin method* and the *anisotropic mesh adaptation* technique.

The content of the thesis is the following. Except a general introduction to the problem of the numerical solution of the Navier-Stokes equations, the thesis is a composition of twelve articles published in international journals within last 10 years. Each articles

corresponds to one section and all these sections are thematically organised into three chapters:

- Chapter 2 – Numerical analysis of DGM,
- Chapter 3 – Application of DGM to the Navier-Stokes equations,
- Chapter 4 – Adaptive methods.

The list of articles forming the thesis with their citations is presented in Section 4.

1.1 Compressible Navier-Stokes equations

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain occupied by a viscous compressible fluid and $(0,T)$ a time interval of interest. A motion of this fluid is described by the system of the *compressible Navier-Stokes equations*, which can be written in the so-called conservation form

$$\frac{\partial \boldsymbol{w}}{\partial t} + \nabla \cdot \vec{\boldsymbol{f}}(\boldsymbol{w}) = \nabla \cdot \vec{\boldsymbol{R}}(\boldsymbol{w}, \nabla \boldsymbol{w}) \qquad \text{in } (0,T)\times\Omega, \tag{1}$$

where $\boldsymbol{w} : (0,T)\times\Omega \to \mathbb{R}^{d+2}$ is the state vector, $\vec{\boldsymbol{f}} = (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_d)$, $\boldsymbol{f}_s : \mathbb{R}^{d+2} \to \mathbb{R}^{d+2},\ s = 1,\ldots,d$ are the inviscid fluxes and $\vec{\boldsymbol{R}} = (\boldsymbol{R}_1, \ldots, \boldsymbol{R}_d),\ \boldsymbol{R}_s : \mathbb{R}^{(d+2)\times(d+1)} \to \mathbb{R}^{d+2},\ s = 1,\ldots,d$ are the viscous fluxes. The components of the state vector $\boldsymbol{w}$ are density, components of momentum and energy. The inviscid as well as viscous fluxes are nonlinear functions of their arguments. Symbols ∇ and $\nabla\cdot$ mean the gradient and divergence operators, i.e.,

$$\nabla \boldsymbol{w} \equiv \left(\frac{\partial \boldsymbol{w}}{\partial x_1}, \ldots, \frac{\partial \boldsymbol{w}}{\partial x_d} \right) \in \mathbb{R}^{d+2} \times \ldots \times \mathbb{R}^{d+2} \tag{2}$$

and

$$\nabla \cdot \vec{\boldsymbol{f}}(\boldsymbol{w}) \equiv \sum_{s=1}^{d} \frac{\partial \boldsymbol{f}_s(\boldsymbol{w})}{\partial x_s} \in \mathbb{R}^{d+2}, \tag{3}$$

respectively. The system of equations (1) follows form *conservation laws*, namely a conservation of *mass*, *momentum* and *energy*. In order to a close the problem, we consider thermodynamical relations (form of the stress tensor, state equation for perfect gas) and include a set of initial and boundary conditions. The complete derivation of the system of the Navier–Stokes equations can be found, e.g., in [21] or [43].

The compressible Navier-Stokes equations represent a nonlinear system of hyperbolic-parabolic type. During last teens years, a great progress was achieved in the question of the existence of the solution of problem (1). For a survey of theoretical results, see, e.g., monographs [35], [18], [33]. We only remark, that the existence of the solution of (1) accompanied by the state equation for perfect gas is still open. Nevertheless, we deal with a numerical solution of problem (1) in the rest of this thesis.

1.2 Numerical methods

In computational fluid dynamics, the finite volume method (FVM) is rather popular. (Cf., [15], [17], [19], [20], [21], [31] for references.) It seems that for conservation laws with discontinuous solutions the finite volume method, using piecewise constant approximations, is very suitable, because the FV approximations are discontinuous on interelement interfaces, which allows good resolution of shock waves and contact discontinuities. However, the increase of accuracy in finite volume schemes applied on unstructured and/or anisotropic meshes is problematic.

On the other hand, the most popular numerical method for a solution of partial differential equations is the finite element method (FEM). This technique is suitable for problems with sufficiently regular solutions. However, singularly perturbed problems or nonlinear conservation laws have solutions with steep gradients or discontinuities and their approximations by conforming finite elements may suffer from the Gibbs phenomenon manifested by spurious oscillations propagating from boundary or interior lay-

ers into the computational domain. One way how to avoid this drawback is to use a suitable stabilization as, e.g. the streamline diffusion method or Galerkin least squares method and shock capturing stabilization. (For a survey, see [21, Chapter 4].) If these techniques are applied to systems of partial differential equations, e.g. of compressible flow, the form of the stabilization terms is rather complex and several parameters have to be tuned.

A generalization of the FV and FE approaches is the *discontinuous Galerkin method* (DGM). This technique is based on the idea to approximate the solution of an initial-boundary value problem by piecewise polynomial functions over a FE mesh, without any requirement on interelement continuity. The original DGM method was introduced by Reed and Hill ([37]) for the solution of the neutron transport equation. The first analysis of this method was made by Le Saint and Raviart ([32]), later an improvement was achieved by Johnson and Pitkäranta ([27]). The DGM was applied to nonlinear conservation laws already in 1989 by Cockburn and Shu ([12]). Their approach uses advantages of FEM and FVM with an approximate Riemann solver. During several recent years the discontinuous Galerkin schemes have been extensively developed. For a survey, see e. g. [10], [11]. An important question is the discretization of diffusion terms in the framework of the DGMs. There exist various treatments of this problem. One possibility is to apply a mixed formulation, used, e. g., in [8]. Its disadvantage is a large number of unknowns. Another method is a direct discretization used, e. g., in [2], [36] and [26]. In [1], an excellent survey of the existing discontinuous Galerkin methods was presented and a unified analysis of DGMs for elliptic problems developed.

A great progress in the applications of DGM to the compressible flow simulation was achieved in the last 10 years, see [3], [4], [5], [6], [13], [16], [22], [23], [24], [25], [28], [29], [34], [40], [41] and the references cited therein. DGM allows on a given mesh to improve a prediction of crucial flow phenomena, such as boundary layers including transition, drag forces, wakes, vortical flows

and interaction phenomena like blade/vortex interaction. On the other hand, the disadvantage of discontinuous Galerkin techniques is a high computational complexity since DGM requires more degrees of freedom than standard FEM in order to achieve the same piecewise polynomial approximation. Nevertheless, we suppose that DGM has still an unused potential which should be used for industrial applications

2 Summary of the thesis

Within this section, we summarise this thesis dealing with an analysis of DGM applied to a convection-diffusion equation and its application to the compressible flow simulation. Let $\mathcal{T}_h,\ h > 0$ be a partition of the computational domain Ω into mutually disjoint elements K, i.e., $\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$. We call $\mathcal{T}_h$ a *triangulation* of Ω and do not require the conforming properties from the finite element method, see [9], [39]. As usually, $h = \max_{K \in \mathcal{T}_h} \mathrm{diam}(K)$.

Over the triangulation $\mathcal{T}_h$, we define the so-called *broken Sobolev space*

$$H^s(\Omega, \mathcal{T}_h) \equiv \{v; v|_K \in H(K)\ \forall\, K \in \mathcal{T}_h\} \tag{4}$$

with the norm

$$\|v\|_{H^s(\Omega,\mathcal{T}_h)} \equiv \left(\sum_{K \in \mathcal{T}_h} \|v\|^2_{H^s(K)} \right)^{1/2} \tag{5}$$

and the seminorm

$$|v|_{H^s(\Omega,\mathcal{T}_h)} \equiv \left(\sum_{K \in \mathcal{T}_h} |v|^2_{H^s(K)} \right)^{1/2}, \tag{6}$$

where $s \geq 1$ and $\|\cdot\|_{H^s(K)}$ and $|\cdot|_{H^s(K)}$ denotes norm and seminorm of the Sobolev space $H^s(K) \equiv W^{s,2}(K)$, respectively.

Furthermore, we define the space of discontinuous piecewise polynomial by

$$S_{hp} \equiv \{v;\ v \in L^2(\Omega),\ v|_K \in P_p(K)\ \forall K \in \mathcal{T}_h\}, \tag{7}$$

where $P_p(K)$ denotes the space of all polynomials on $K \in \mathcal{T}_h$ of degree $\leq p$.

2.1 Chapter 2 – Numerical analysis

In order to simplified the numerical analysis, we consider the scalar nonstationary convection-diffusion equation

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{f}(u) = \varepsilon \Delta u + g, \tag{8}$$

where $u : (0,T) \times \Omega \to I\!R$, $\vec{f} : I\!R \to I\!R^d$ is a nonlinear function of its argument, $\varepsilon > 0$ play role of the viscosity and $g \in C([0,T]; L^2(\Omega))$ is a source term. Moreover, we prescribe a Dirichlet and/or Neumann boundary conditions and an initial condition. The problem (8) represents a model problem of the Navier-Stokes equations. With the aid of techniques from [38] and [33], it is possible to prove that there exists a unique weak solution of (8).

In **Section 2.1** we develop the so-called *finite volume discontinuous Galerkin* (FVDG) method, where the approximate solution is sought in the space of piecewise linear discontinuous finite elements. Moreover, the convective terms are approximated with the aid of a *numerical flux* well-known from FVM, where a piecewise constant projection is employed. Then the FVDG method reads: Find $u_h : (0,T) \to S_{h1}$ such that

$$\begin{aligned} \text{i)} \quad & \frac{\mathrm{d}}{\mathrm{d}t}(u_h, v_h) + a_h^N(u_h, v_h) + b_h(\pi_h^0 u_h, v_h) + J_h(u_h, v_h) \\ & \qquad = \ell(v_h) \qquad \forall v_h \in S_{hp},\ t \in (0,T), \\ \text{ii)} \quad & (u_h(0), v_h) = (u_0, v_h), \end{aligned} \tag{9}$$

where $(\cdot,\cdot)$ denotes the L^2-scalar product, the linear non-symmetric form $a_h^N(\cdot,\cdot)$ represents the discretization of the diffusive term

by the nonsymmetric variant of the DG method, the nonlinear form $b_h(\cdot,\cdot)$ represents the discretization of convective terms, the linear form $J_h(\cdot,\cdot)$ represents the interior and boundary penalties and $\ell(\cdot)$ is the right-hand-side containing source terms and terms arising from the boundary conditions. Moreover, π_h^0 is the L^2-projection into the space of piecewise constant functions and $u_0 : \Omega \to I\!R$ is the initial condition. Problem (9) represents a system of ordinary differential equations which should be solved by suitable solver.

Within this section we analyse the FVDG scheme. Let u be the weak solution of problem (8) satisfying assumptions

$$u \in L^2(0,T;H^2(\Omega)), \qquad \frac{\partial u}{\partial t} \in L^2(0,T;H^1(\Omega)), \tag{10}$$

$u_h(t) \in S_{h1},\ t \in (0,T)$ be the approximate piecewise linear solution obtained by (9) and $e_h(t) \equiv u - u_h$. We derive a priori error estimate in the form

$$\sup_{t\in[0,T]} \|e_h(t)\|^2_{L^2(\Omega)} + \varepsilon \int_0^T |e_h(\vartheta)|^2_{H^1(\Omega,\mathcal{T}_h)} \mathrm{d}\vartheta \le C_1 h^2 |u|^2_{H^2(\Omega)}, \tag{11}$$

where $C_1 = O(\exp(1/\varepsilon)$ is independent of u and h. It is clear that estimate (11) cannot be used for $\varepsilon \to 0+$, because it blows up exponentially. The nonlinearity of the convective terms represents a serious obstacle for obtaining a uniform error estimate with respect to $\varepsilon \to 0+$.

Moreover, numerical experiments from Section 2.1 verify that the presented FVDG method has the first order of accuracy as the theoretical estimate (11). However, it is not possible to increase the order of accuracy by a higher order of polynomial approximation (i.e., S_{hp} with $p \ge 2$) since FVDG method uses the piecewise constant projection in the numerical flux approximating convective terms.

Therefore, in **Section 2.2** we develop the so-called *discontinuous Galerkin finite element* (DGFE) method. Similarly as in

FVDG method we employ the *nonsymetric* treatment of the diffusive term with an *interior* and *boundary penalties*. This technique is in literature denoted as NIPG (nonsymmetric interior penalty Galerkin) method. On the other hand, the convective terms are approximated with the aid of numerical flux but without the piecewise constant projection. Then the DGFE method reads: Find $u_h : (0,T) \to S_{h1}$ such that

$$\begin{aligned} \text{i)} \quad & \frac{\mathrm{d}}{\mathrm{d}t}(u_h, v_h) + a_h^N(u_h, v_h) + b_h(u_h, v_h) + J_h(u_h, v_h) \\ & \qquad = \ell(v_h) \qquad \forall v_h \in S_{hp},\ t \in (0,T), \\ \text{ii)} \quad & (u_h(0), v_h) = (u_0, v_h), \end{aligned} \tag{12}$$

where $(\cdot,\cdot)$, $a_h^N(\cdot,\cdot)$, $b_h(\cdot,\cdot)$, $J_h(\cdot,\cdot)$ and $\ell(\cdot)$ denotes the forms introduced in (9).

We derive the following error estimate. Let u be the weak solution of problem (8) satisfying

$$\frac{\partial u}{\partial t} \in L^2(0,T; H^{p+1}(\Omega)), \tag{13}$$

$u_h(t) \in S_{hp},\ t \in (0,T)$ be the approximate piecewise polynomial (degree $\le p$) solution obtained by (12) and $e_h(t) \equiv u - u_h$. Then

$$\sup_{t\in[0,T]} \|e_h(t)\|^2_{L^2(\Omega)} + \varepsilon \int_0^T |e_h(\vartheta)|^2_{H^1(\Omega,\mathcal{T}_h)} \mathrm{d}\vartheta \le C_2 h^{2p} |u|^2_{H^{p+1}(\Omega)}, \tag{14}$$

where $C_2 = O(\exp(1/\varepsilon)$ is independent of u and h. We observe that (14) is optimal ($O(h^p)$) with respect to the H^1-seminorm but sub-optimal ($O(h^p)$) with respect to the L^2-norm. This is caused by the use of the nonsymmetric variant (NIPG) which gives the corresponding diffusive form nonsymmetric and therefore it is not possible to use the well-known Aubin-Nitsch theorem. On the other hand, numerical experiments presented in Section 2.2 and also in [14] indicate that the odd degrees of polynomial approximations give the optimal experimental order of convergence in

the L^2-norm ($O(h^{p+1})$) whereas the even degrees of polynomial approximations only the suboptimal one ($O(h^p)$).

In order to achieve the optimal orders of convergence in the L^2-norm, we employ the *symmetric interior penalty Galerkin* (SIPG) variant of DGFE method in **Section 2.3** when the non-symmetric form $a_h^N(\cdot,\cdot)$ in (12) is replaced by its symmetric variant $a_h^S(\cdot,\cdot)$. Then with the aid of the Aubin-Nitsch theorem we obtain (provided that the weak solution satisfy (13)) a priori error estimates

$$\sup_{t\in[0,T]} \|e_h(t)\|^2_{L^2(\Omega)} \leq C_3 h^{2p+2} |u|^2_{H^{p+1}(\Omega)}, \tag{15}$$

where $C_3 = O(\exp(1/\varepsilon)$ is independent of u and h. Numerical experiments presented within the same section verify the theoretical orders of convergence. Moreover, these numerical experiments indicate the error estimate

$$\sup_{t\in[0,T]} \|e_h(t)\|^2_{L^2(\Omega)} \leq C_4 h^{2\mu} |u|^2_{H^s(\Omega)}, \tag{16}$$

where $C_4 > 0$, $\mu = \min(p+1, s)$ and $H^s(\Omega)$, $s > 0$ denotes (in general) the Sobolev-Slobodetskii space of functions with "non-integer derivatives". This error estimate should be used in case when the weak solution is not sufficiently regular.

The previous theoretical results (11) – (15) are based on the assumptions of regularity of the exact solution. Nevertheless, it is interesting to observed how the DG schemes behaves in case when the exact solution is discontinuous or contains steep gradients (e.g., interior or boundary layers). In **Section 2.4** we apply the FVDG and DGFE methods from Sections 2.1 and 2.2, respectively, to the viscous Burgers equation whose solution contains interior layers. Whereas the piecewise constant projection employed in FVDG method give a reasonable numerical approximation, the DGFE method produces a numerical solution suffering from unphysical overshoots and undershoots in vicinity of the interior layers (this phenomenon is called the Gibbs effect). Therefore, we develop a limiting of degree of approximation based on

the so-called *jump indicator*. The presented numerical examples indicate that this technique does not decrease an order of accuracy in parts of the computational domain, where the solution is smooth, and moreover, it avoids the Gibbs effect. This limiting technique is extended to the solution of the system of the Navier-Stokes equations in Section 3.1.

In Sections 2.1 – 2.4, we consider the scalar equation (8), where the diffusion is linear. In **Section 2.5**, we deal with a more general case where the diffusion is *quasilinear*, i.e., we seek $u : (0,T)\times\Omega \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{f}(u) = \nabla \cdot \vec{R}(u, \nabla u) + g, \tag{17}$$

where $\vec{f} : \mathbb{R} \to \mathbb{R}^d$, $\vec{R} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and $g \in C([0,T]; L^2(\Omega))$. The nonlinear dependence of the viscous terms $\vec{R}(u, \nabla u)$ on ∇u does not allow to employ the NIPG as well as SIPG variants of the DGFE method. Therefore, we use the so-called *incomplete interior penalty Galerkin* (IIPG) variant of the DGFE method and derive the same a priori error estimates as in (14). The IIPG technique does not allow to obtain the optimal L^2-error estimate as NIPG. On the other hand, the numerical experiments carried out is Section 2.5 give higher experimental orders of convergence than the theoretical ones.

Section 2.6 deals with the so-called *discrete Friedrichs inequality* for piecewise linear Crouziex-Raviart nonconforming finite elements. Let $\mathcal{T}_h$ be a triangular grid of a computational domain $\Omega \in \mathbb{R}^2$ and $\mathcal{F}_h$ the corresponding set of all edges of triangles $K \in \mathcal{T}_h$. We denote by x_Γ a centre of $\Gamma \in \mathcal{F}_h$. We define the following finite element spaces:

$$\begin{aligned} X_h &\equiv \{v_h \in L^2(\Omega);\ v_h|_K \in P^1(K),\ K \in \mathcal{T}_h, \\ &\qquad v_h \text{ is continuous at each } x_\Gamma,\ \Gamma \in \mathcal{F}_h\}, \\ X_{h0} &\equiv \{v_h \in X_h;\ v_h(x_\Gamma) = 0\ \forall x_\Gamma \in \partial\Omega,\ \Gamma \in \mathcal{F}_h\}. \end{aligned} \tag{18}$$

Then the discrete Friedrichs inequality claims that there exists a

constant $\tilde{c} > 0$ independent of h such that

$$\|v_h\|_{L^2(\Omega)} \leq \tilde{c}|v_h|_{H^1(\Omega,\mathcal{T}_h)} \qquad \forall v_h \in X_{h0}, \tag{19}$$

where $\|\cdot\|_{L^2(\Omega)}$ is the standard norm in the Lebesgue space $L^2(\Omega)$ and $|\cdot|_{H^1(\Omega,\mathcal{T}_h)}$ is the norm in the broken Sobolev space introduced by (6).

Within this section we prove the discrete Friedrich inequality (19) for the case when Ω is a *general polygonal nonconvex domain*. This results has applications in the numerical analysis of convection-diffusion problems approximated with the aid of nonconforming finite elements. Moreover, this inequality was generalized by some authors, see, e.g, [30], [42], [7].

2.2 Chapter 3 – Application to compressible flow simulations

In **Section 3.1**, we apply the DGFE method to the system of the Navier-Stokes equations (1). We implement a discontinuous piecewise linear approximation of the NIPG variant. A special attention is paid to the choice of the stabilization term, the boundary conditions, the limiting of the degree of approximation and a treatment of nonpolygonal parts of the boundary. We present numerical examples of subsonic, transonic and supersonic flow regimes and obtain a good comparison with reference solutions. The main drawback of this method is the explicit time discretization which is simple for implementation but the size of the time step is very limited. Therefore, it is necessary to carried out a high number of time steps in order to achieve a steady state solution.

In order to avoid the time step restriction mentioned in Section 3.1, it is suitable to use an implicit time discretization, e.g., [5], [24], [25]. However, a full implicit scheme leads to a necessity to solve a nonlinear system of algebraic equations at each time step which is rather expensive. Therefore, we propose in **Section 3.2** a *semi-implicit* method for the solution of the Euler equations. This technique is based on a suitable linearization of the

Euler fluxes. The linear terms are treated implicitly whereas the nonlinear ones explicitly which leads to a linear algebraic problem at each time step. The presented numerical examples indicate an enormous gain (from the point of view of the computational time) in comparison with an explicit discretization for steady-state computations.

In **Section 3.3**, we extend the semi-implicit time discretization to the viscous flow simulations, i.e., we carry out a linearization also of the viscous terms. Moreover, in order to increase the accuracy of the time discretization, we employ the so-called *backward difference formulae* (BDF). We call the resulting scheme the BDF-DGFE method, which is practically unconditionally stable, has a high order of accuracy with respect to the space and time coordinates and requires a solution only of one linear algebraic problem at each time step. Since we solve an evolution problem, it is suitable to employ an iterative solver for the solution of these linear algebraic problems. It is possible to use a solution from the previous time step as an approximation of the solution on the new time level and, moreover, a suitable preconditioner can be evaluated after several time steps. This strategy highly decrease the computational time. We present a set of numerical experiments of steady as well as unsteady flows using piecewise linear, quadratic and cubic polynomial approximations for the space semi-discretization and the two- and three-steps BDF for the time discretization. These numerical examples show a good agreement with reference results.

2.3 Chapter 4 – Adaptive methods

Most of the computations presented in Sections 3.1 – 3.3 were carried out on adaptively refined grids, which were obtained by the *anisotropic mesh adaptation* (AMA). This method exhibits a very efficient tool for compressible flow simulations and it is described in **Section 4.1**. AMA technique minimises a mesh quality parameter $Q_{\mathcal{T}_h}$ by an iterative process. The mesh quality

parameters $Q_{\mathcal{T}_h}$ is defined by

$$Q_{\mathcal{T}_h} \equiv \frac{1}{\#\mathcal{T}_h} \sum_{K \in \mathcal{T}_h} \sum_{\Gamma \in \partial K} \left(|\Gamma|_{\mathcal{H}} - \omega \right)^2, \tag{20}$$

where the sum is taken over all edges Γ of elements $K \in \mathcal{T}_h$, $\#\mathcal{T}_h$ denotes the number of elements of $\mathcal{T}_h$, $\omega > 0$ is a given constant and $|\Gamma|_{\mathcal{H}}$ is a size of edge Γ measured in the Riemann metric generated by the Hessian matrix $\mathcal{H} = \mathcal{H}(u_h)$. The matrix $\mathcal{H}(u_h)$ is evaluated for each Γ from a "smoothed" numerical solution u_h. The AMA technique is completely method-independent and problem-independent approach and, therefore, it can be applied to various finite element/volume solutions of partial differential equations. A drawback of this technique is that we have no bound for the computational error.

A theoretical background of AMA approach is developed in **Section 4.2**. Let us considered a general boundary value problem (BVP) defined on a computational domain Ω. Let $\mathcal{T}_h$ be a triangular grid of Ω, V and V_h be functional spaces where an exact and an approximate solutions u and u_h are sought, respectively.

We require that the computational error $e_h \equiv u - u_h$ is bounded by a given tolerance $\omega > 0$, i.e,

$$\|e_h\|_X = \|u - u_h\|_X \leq \omega, \tag{21}$$

where $\|\cdot\|_X$ is a suitable norm. We define a mapping $\Pi_h : V \to V_h$ by

$$v \in V,\ \Pi_h v \in V_h : \quad \|v - \Pi_h v\|_X = \min_{w_h \in V_h} \|v - w_h\|_X. \tag{22}$$

Let us emphasise that the mapping Π_h depends on the exact solution u, the norm $\|\cdot\|_X$, the finite dimensional space V_h and therefore on the mesh $\mathcal{T}_h$. On the other hand, the mapping Π_h is independent of the approximate solution u_h. Obviously,

$$\|u - \Pi_h u\|_X \leq \|u - u_h\|_X \tag{23}$$

and therefore, the *necessary condition* to fulfil (21) is

$$\|u - \Pi_h u\|_X \leq \omega. \tag{24}$$

The *main idea* of the AMA method is the following: to construct a mesh $\mathcal{T}_h$ such that

i) the necessary condition (24) is satisfied,

ii) the number of elements of mesh $\mathcal{T}_h$ is minimal.

The condition ii) follows from a natural requirement to use a smallest possible number of degree of freedom in order to save a computational time and memory of a computer.

Based on several simplifications and assumptions we arrive to a definition of an *optimal mesh* $\mathcal{T}_h$ by the relation

$$Q_{\mathcal{T}_h} = 0, \tag{25}$$

where $Q_{\mathcal{T}_h}$ is the quality parameters defined by (20) with the Hessian matrices $\mathcal{H} = \mathcal{H}(u)$ of the exact solution. However, the exact solution u is not know a priori. Therefore, we apply a smoothing procedure to the approximate solution u_h and the mesh adaptation algorithm from Section 4.1. Moreover, we present several numerical experiments dealing with the Poisson equation where the efficiency of the AMA technique is demonstrated.

Section 4.3 deals with an application of the AMA method to viscous compressible flow simulations. The presented numerical experiments indicate that the AMA technique is not able to capture thin boundary layers and wakes, namely for flows with high Reynolds numbers Re. Therefore, we take into account physical properties of viscous compressible flows, namely the thickness of boundary layers ($\approx 1/\sqrt{Re}$). Then, the AMA algorithm generates triangular grids where the low viscosity flows can be resolved very well. Moreover, we introduce two variants of a smoothing procedure which improve computational results.

3 Summary of the results

We dealt with the numerical solution of the compressible Navier-Stokes equations describing viscous compressible flows. This numerical scheme is based on the discontinuous Galerkin method employing a discontinuous piecewise polynomial approximation which is suitable for capturing of piecewise regular solutions containing discontinuities. The results achieved within this thesis are summarised in the following list.

Numerical analysis:

- In Sections 2.1 and 2.2, we presented and developed the discontinuous Galerkin method for the solution of a scalar time-dependent convection-diffusion equation with a nonlinear convection and a linear diffusion ($-\varepsilon\Delta u$). The presence of the nonlinear convection leads to error estimates of order $O(\exp(1/\varepsilon))$ which blow up exponentially for $\varepsilon \to 0$.
- In Section 2.2, we derived a priori error estimates of order $O(h^p)$ in the L^2-norm and the H^1-seminorm, where h is the step of the mesh and p is the degree of polynomial approximation. The sub-optimality of the error estimates in the L^2-norm is caused by the nonsymmetric treatment of the diffusive terms which leads to a nonsymmetric bilinear form and then it is not possible to employ the Aubin-Nitsch theorem. On the other hand, numerical experiments show the optimal order of convergence $O(h^{p+1})$ in the L^2-norm for odd degrees of polynomial approximations.
- The symmetric treatment of the diffusive term from Section 2.3 leads to optimal error estimates ($O(h^{p+1})$) also in the L^2-norm. Nevertheless, a regularity assumption of the solution of the dual problem is required. This results were verified by numerical experiments.
- We showed in Section 2.4 that DGM produces numerical solution suffering from the Gibbs phenomenon in case when

the exact solution contains discontinuities and/or steep gradients. Therefore we developed a technique based on limiting of the order of polynomial approximation which avoids the Gibbs effect and does not decrease an order of accuracy in region where the exact solution is smooth.

- In Section 2.5 we apply the DGM to a scalar convection-diffusion equation with a nonlinear diffusion $\nabla \cdot \vec{R}(u, \nabla u)$, where $\vec{R}(\cdot, \cdot)$ is a nonlinear function of its arguments. The presence of the nonlinear diffusive term does not allow to use the nonsymmetric as well as symmetric variants of DGM (Sections 2.2 and 2.3) hence we employed the so-called incomplete variant of DGM. We derived hp error estimate of order $O(h^{\mu})$, where $\mu = \min(p, s)$, p is the degree of polynomial approximation and $s \geq 2$ represents a regularity of the exact solution, i.e., $u \in H^s(\Omega)$.

- The discrete Friedrichs inequality proved in Section 2.6 for non-convex domains plays an important role in numerical analysis of nonconforming finite element. It allows estimate the L^2-norm by a discrete variant of the H^1-seminorm.

Solution of the Navier-Stokes equations

- We apply the discontinuous Galerkin method analysed in Chapter 2 to the Navier-Stokes equations. In Section 3.1, we presented an original treatment of diffusive terms and boundary conditions within the DGM framework. The resulting numerical scheme gives an accurate numerical simulations of steady state flows in subsonic, transonic and supersonic flow regimes. The main drawback of this approach is a strong restriction to the size of the time step since the explicit temporal discretization was used.

- In order to avoid the time step restriction, we developed in Sections 3.2 and 3.3 an original semi-implicit scheme, where

the inviscid as well as fluxes are formally linearized and, consequently, the linear terms are treated implicitly and the nonlinear ones by an higher order explicit extrapolation. The resulting scheme, called backward difference formulae - discontinuous Galerkin finite element (BDF-DGFE) method, is practically unconditionally stable, has a high order of accuracy with respect to the space and time coordinates and requires a solution only of one linear algebraic problem at each time step. The numerical experiments are in a good agreement with reference results for steady as well as unsteady flow regimes.

Adaptive method

- The anisotropic mesh adaptation (AMA) method presented in Sections 4.1 – 4.3 represents an efficient tool for the numerical solution of partial differential equations. Although this approach does not give an error bound, the flexibility and universality of this technique allow a wide use of AMA in the computational sciences. This method is a base of the software package ANGENER which was implemented by the author and it is freely available. Till now, there are more than 50 registered ANGENER users over the world, see the enclosed list.

Although the BDF-DGFE method give promising results there is still a lot of open problems which are the subject of further research.

4 List of articles (sections)

The thesis is formed by the following list of articles published in international journals. Bolted numbers denote the number of sections, the symbol IF denotes the value of the impact factor of the corresponding journal. The list of citations of each article is accompanied in a small font.

Sec. 2.1 V. Dolejší, M. Feistauer, C. Schwab: A Finite Volume Discontinuous Galerkin Scheme for Nonlinear Convection-Diffusion Problems. *Calcolo* , **39**:1–40, 2002. IF = 0.375

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doi:10.1016/j.cam.2007.10.055, 2007. IF=0.759

Sec. 2.6 V. Dolejší, M. Feistauer and J. Felcman: On the discrete Friedrichs inequality for nonconforming finite elements, *Numer. Func. Anal. and Optimiz.*, **20**(5&6):437-447, 1999. IF = 0.405

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