Existence of global weak solutions for inviscid PDE models in oceanography

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- a list of interesting mathematical results,
- the main theorem on the existence (and non-uniqueness) of the global weak solutions.
- the Primitive equations as a differential inclusion and crucial parts of the proof.

What are the Primitive equations?

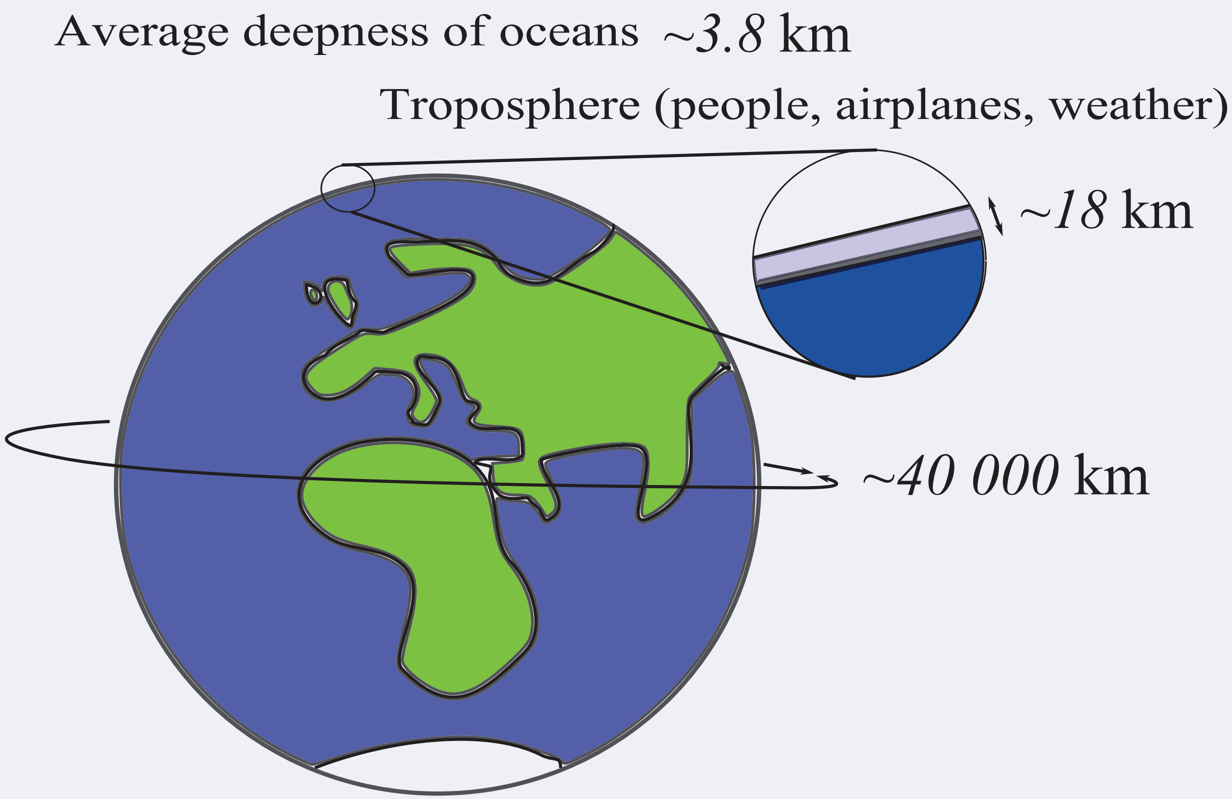

Figure: Scales in global oceanography/weather models

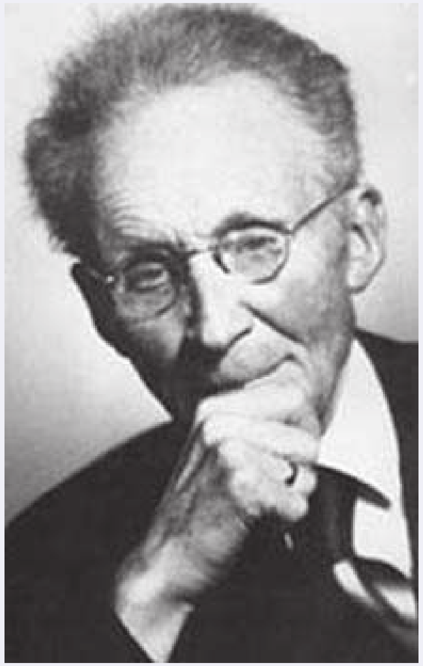

Figure: Vilhelm Bjerknes (1862-1951)

What are the Primitive equations?

We will investigate a geometrically simplified Cauchy problem[3]:
to find $\mathbf{u} = (u, v, w)$, p, $\theta\colon [0,T) \times U \to \mathbb{R}$ (U with a smooth boundary) satisfying

$$\begin{aligned} \operatorname{div} \mathbf{u} &= 0 && \text{in } (0,T) \times U, \\ u_t + uu_x + vu_y + wu_z + p_x &= 0 && \text{in } (0,T) \times U, \\ v_t + uv_x + vv_y + wv_z + p_y &= 0 && \text{in } (0,T) \times U, \\ p_z &= -\theta && \text{in } (0,T) \times U, \\ \theta_t + u\theta_x + v\theta_y + w\theta_z &= \lambda_1(\theta_{xx} + \theta_{yy}) + \lambda_2\theta_{zz} && \text{in } (0,T) \times U \end{aligned}$$

with initial conditions $u(0) = u_0$, $v(0) = v_0$ and $\theta(0) = \theta_0$.
When $\mu_1 = \mu_2 = 0$ we call the problem *the inviscid Primitive equations.*
A special feature: non-deterministic role of w.

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[3] The mathematical formulation was done by J. L. Lions, R. Temam and S. H. Wang: **New formulations of the primitive equations of atmosphere and applications.** In *Nonlinearity* (1992).

The definition of the weak solution

Definition

We call the quintet of functions (u, v, w, p, θ) a *weak solution of the inviscid Primitive equations* if

- $\mathbf{u} = (u, v, w) \in L^2(Q; \mathbb{R}^3)$, u, $v \in \mathcal{C}([0, T]; L^2_w(U))$, $p \in L^1(Q)$, $\partial_z p \in L^1(Q)$ and equations and the equalities

$$\int_0^T \int_U u \partial_t \phi_1 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t + \int_0^T \int_U u \mathbf{u} \cdot \nabla_{\mathbf{x}} \phi_1 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t - \int_U u_0(\cdot) \phi_1(0, \cdot) \, \mathrm{d}\mathbf{x} + \int_0^T \int_U p \partial_x \phi_1 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t = 0,$$

$$\int_0^T \int_U v \partial_t \phi_2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t + \int_0^T \int_U v \mathbf{u} \cdot \nabla_{\mathbf{x}} \phi_2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t - \int_U v_0(\cdot) \phi_2(0, \cdot) \, \mathrm{d}\mathbf{x} + \int_0^T \int_U p \partial_y \phi_2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t = 0$$

hold for any ϕ_1, $\phi_2 \in \mathcal{D}([0, T) \times U)$,

The definition of the weak solution

- $\mathbf{u}\chi_Q$ solves
$$\operatorname{div} \mathbf{u}\chi_Q = 0$$
in the sense of distributions on $\mathbb{R}^3$ (this includes the boundary conditions),
- θ is a strong solution of
$$\theta_t + u\theta_x + v\theta_y + w\theta_z = \lambda_1(\theta_{xx} + \theta_{yy}) + \lambda_2\theta_{zz}$$
in Q and $\theta(0,\cdot) = \theta_0(\cdot)$ in the sense of time traces,
- the equation
$$p_z = -\theta$$
holds for the weak derivative of p almost everywhere in Q.

Why are they interesting? (for mathematicians)

The **viscid case, three spatial dimensions**:

- Existence of global weak solutions (Navier-Stokes-like theory) - J. L. Lions, R. Temam and S. H. Wang: **On the equations of the large-scale ocean.** In *Nonlinearity* (1992).
- Local in time existence of smooth solutions (the same paper),
- Global in time regularity of solutions for smooth initial conditions - C. Cao, E. S. Titi: **Global well-posedness of the 3D viscous primitive equations of large scale ocean and atmosphere dynamics.** In *Ann. Math* (2007).

Why are they interesting? (for mathematicians)

The **inviscid case**:

- The term *primitive* becomes a bit misleading.
- If we erase the diffusion in the heat equation, the system is not hyperbolic - J. Oliger and A. Sundström: **Theoretical and practical aspects of some initial boundary value problems in fluid dynamics.** In *SIAM J. Appl. Math.*, (1978).
- In 3D, there are (to the best knowledge of the author) *no* a priori estimates for velocities and temperature. In 2D and $\theta \equiv 0$, there exist local in time smooth solutions - Y. Brenier: **Homogeneous hydrostatic flows with convex velocity profiles.** In *Nonlinearity*, (1999).
- Finite time blow-up for some smooth initial data - C. Cao, S. Ibrahim, K. Nakanishi and E. S. Titi: **Finite-time blowup for the inviscid primitive equations of oceanic and atmospheric dynamics.** In *Comm. Math. Phys.*, (2015).

Global existence of weak solutions for the inviscid case[4]

Theorem

Assume that $T > 0$ (arbitrary). Let u_0, $v_0 \in \mathcal{C}(\overline{U})$, $\theta_0 \in \mathcal{C}^2(\overline{U})$ and suppose that there exists $w_0 \in \mathcal{C}(\overline{U})$ such that

$$\operatorname{div}((u_0, v_0, w_0)\chi_U) = 0 \quad \textit{in the sense of distributions on } \mathbb{R}^3.$$

Then there are infinitely many weak solutions of the inviscid Primitive equations emanating from the initial conditions u_0, v_0, θ_0.

- Canonically, there will be a jump of the kinetic energy at time $t = 0$. If we denote

$$E(t) = \int_U \frac{1}{2}|u(t,x)|^2 + |v(t,x)|^2 + |w(t,x)|^2 \, \mathrm{d}x$$

then

$$\liminf_{t \to 0^+} E(t) > E(0).$$

[4] E. Chiodaroli, M. M.- **Existence and non–uniqueness of global weak solutions to inviscid primitive and Boussinesq equations.** *Preprint on arxiv*, (2016)

Infinitely many dissipative solutions

Definition

We call solutions dissipative if $E(t) \leq E(s)$ whenever $0 \leq s \leq t$.

Theorem

Assume that $T > 0$ *and* $\theta_0 \in \mathcal{C}^2(\overline{U})$*. Then there exist* u_0, $v_0 \in L^\infty(U)$ *for which we can find infinitely many weak dissipative solutions of the inviscid Primitive equations emanating from the initial data* u_0, v_0, θ_0.

- **What is the main technique which can be used to proof the theorems?**
- *Convex integration.*

What is the convex integration?

A technique, how to find a solution of a system of linear differential equations together with nonlinear constitutive relations.

$$\mathcal{L}(u(x)) = 0, \quad u(x) \in K(x) \quad \text{for } x \in \Omega.$$

- The technique has its origin in differential geometry (Nash, Gromov).
- It was used to construct surprising results about regularity of weak solutions of Euler-Lagrange equations corresponding to quasiconvex functionals - Müller, Šverák: **Convex integration for Lipschitz mappings and counterexamples to regularity.** in *Ann. Math.* (2003).
- Recently, C. De Lellis and L. Székelyhidi adapted the technique to construct paradoxical solutions of the Euler system: **On admissibility criteria for weak solutions of the Euler equations.** In *Arch. Ration. Mech. Anal.*, (2010).

The curious case of De Lellis and Székelyhidi

Theorem (De Lellis and Székelyhidi, 2011)

Let $\bar{e} \in C((0,T) \times \mathbb{T}^3) \cap C([0,T]; L^1(\mathbb{T}^3))$ *is positive in* $(0,T) \times \mathbb{T}^3$. *Then there exist infinitely many weak solutions* $\mathbf{u}$ *of the Euler equations*

$$\begin{aligned} \operatorname{div} \mathbf{u} &= 0 \quad \textit{in the sense of distributions,} \\ \partial_t \mathbf{u} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla p &= 0 \quad \textit{in the sense of distributions,} \end{aligned}$$

with pressure $p = -\frac{1}{3}|\mathbf{u}|^2$ *such that* $\mathbf{u} \in C([0,T]; L^2_{weak}(\mathbb{T}^3))$, $\mathbf{u}(0,x) = 0$ *for* $t = 0, T$ *a. e.* $x \in \mathbb{T}^3$,

$$\frac{1}{2}|\mathbf{u}(t,x)|^2 = \bar{e}(t,x) \quad \textit{for every } t \in (0,T) \textit{ a. e. } x \in \mathbb{T}^3.$$

A generalization for an abstract Euler system

Observation (E. Feireisl, 2015)

Let $\mathbb{H}\colon C([0,T]; L^2_{weak}(\mathbb{T}^3)) \to C([0,T] \times \mathbb{T}^3; \mathbb{R}^{3\times 3}_{0,sym})$,

$$\Pi \in C([0,T]; L^2_{weak}(\mathbb{T}^3)) \to C([0,T] \times \mathbb{T}^3)$$

be bounded and continuous operators satisfying some additional technical assumptions and assume that $\Pi[\mathbf{u}]$ is bounded independently on $\mathbf{u}$. Then there exist infinitely many weak solutions $\mathbf{u}$ of the following abstract version of the Euler system

$$\begin{aligned} \operatorname{div} \mathbf{u} &= 0 \quad \text{in the sense of distributions,} \\ \partial_t \mathbf{u} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u} + \mathbb{H}[\mathbf{u}]) + \nabla \Pi[\mathbf{u}] &= 0 \quad \text{in the sense of distributions,} \end{aligned}$$

such that $\mathbf{u} \in C([0,T]; L^2_{weak}(\mathbb{T}^3))$, $\mathbf{u}(0,x) = 0$ for $t = 0, T$ a. e. $x \in \mathbb{T}^3$.

The Primitive equations as a differential inclusion

- We would like to apply the machinery of convex integration. The first step is to recast the Primitive equations into the form

$$\begin{aligned} \operatorname{div} \mathbf{u} &= 0, \\ \partial_t \mathbf{u} + \operatorname{div} \left(\mathbf{u} \otimes \mathbf{u} + \mathbb{H}[\mathbf{u}] \right) + \nabla \Pi[\mathbf{u}] &= 0. \end{aligned}$$

- The main problem - the equation for the third component of the velocity is degenerated.

Inviscid primitive equations

Let us take the Primitive equations

$$
\begin{aligned}
u_x + v_y + w_z &= 0 \\
u_t + uu_x + vu_y + wu_z + p_x &= 0, \\
v_t + uv_x + vv_y + wv_z + p_y &= 0, \\
p_z &= \theta \\
\theta_t + u\theta_x + v\theta_y + w\theta_z &= \lambda_1(\theta_{xx} + \theta_{yy}) + \lambda_2\theta_{zz}
\end{aligned}
$$

Extended inviscid primitive equations

...and supplement it by an extra equation

$$
\begin{aligned}
u_x + v_y + w_z &= 0 \\
u_t + uu_x + vu_y + wu_z + p_x &= 0, \\
v_t + uv_x + vv_y + wv_z + p_y &= 0, \\
w_t + uw_x + vw_y + ww_z + p_z &= 0, \\
p_z &= \theta \\
\theta_t + u\theta_x + v\theta_y + w\theta_z &= \lambda_1(\theta_{xx} + \theta_{yy}) + \lambda_2\theta_{zz}
\end{aligned}
$$

Extended inviscid primitive equations

Let $\theta = \Theta[\mathbf{u}]$ be the solving operator for the convection diffusion equaiton, then:

$$\begin{aligned}
\operatorname{div} \mathbf{u} &= 0 \\
\mathbf{u}_t + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla p &= 0 \\
p_z &= \Theta[\mathbf{u}] \\
\Theta[\mathbf{u}]_t + \mathbf{u} \cdot \nabla \Theta[\mathbf{u}] &= \lambda_1 (\Theta[\mathbf{u}]_{xx} + \Theta[\mathbf{u}]_{yy}) + \lambda_2 \Theta[\mathbf{u}]_{zz}
\end{aligned}$$

Extended inviscid primitive equations

Then we can find a solving operator for the equation for p_z by taking

$$\Pi[u](t,x,y,z) \approx \int_{z_0}^{z} \Theta[\mathbf{u}](t,x,y,s)\,\mathrm{d}s.$$

Hence,

$$\begin{aligned} \operatorname{div}\mathbf{u} &= 0 \\ \mathbf{u}_t + \operatorname{div}(\mathbf{u}\otimes\mathbf{u}) + \nabla\Pi[\mathbf{u}] &= 0 \\ p_z &= \Theta[\mathbf{u}]. \end{aligned}$$

An example using the idea of convex integration

$$\operatorname{div} u = 0 \quad \text{in } \mathbb{T}^3, \tag{1}$$

$$|u| = 1 \quad \text{almost everywhere in } \mathbb{T}^3 \tag{2}$$

- we define a set of subsolutions

$$X_0 = \{u \in \mathcal{C}^\infty(\mathbb{T}^3; \mathbb{R}^3) \colon \text{(1) holds and } |\mathrm{u}|<1\},$$

- we define a functional on X_0 by $I(u) = \int_{\mathbb{T}^3} |u|^2 - 1 \, \mathrm{d}x$. Observe that $I(u) < 0$ in X_0.

Lemma (Effective oscillations)

Let $u \in X_0$. Then there exists $\{w_n\} \subseteq \mathcal{C}^\infty(\mathbb{T}^3; \mathbb{R}^3)$ such that

- $u + w_n \in X_0$,
- $w_n \to 0$ *weakly-$*$ in* L^∞
- $$\liminf_{n\to\infty} I(u + w_n) \geq I(u) + c\,(I(u))^2$$

where $c > 0$ does not depend on u.

A Baire cathegory argument

- Let us take $X = \overline{X_0}^{(L^\infty, weak-*)}$. Which is a completely metrizable topological space.
- With respect to the oscillatory lemma, the only possible points of continuity are such that $I(u) = 0$.
- As $I(u) = \lim_{\varepsilon\to 0} I_\varepsilon(u) = \lim_{\varepsilon\to 0} I(\eta_\varepsilon * u)$, I has to have residual set in X (very large) of points of continuity.
- Points of continuity are precisely all those points $u \in X$ which satisfy $I(u) = 0$.