Compressible fluid flows driven by stochastic forcing

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Why compressible and/or stochastic ?

Field equations

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u})\mathrm{d}t = 0$$

$$\mathrm{d}(\varrho \mathbf{u}) + [\mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho)]\mathrm{d}t = [\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})]\mathrm{d}t + \varrho \mathrm{d}W_E + \varrho \mathbf{u} \mathrm{d}W_F$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} + \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Initial data

initial density $\varrho(0, \cdot) = \varrho_0$ - a random variable

initial velocity $\mathbf{u}(0, \cdot) = \mathbf{u}_0$ - a random variable

Stochastic forcing

$\varrho \, \mathrm{d}W_E$ -random external volumic force

$\varrho \mathbf{u} \, \mathrm{d}W_F$ -random friction force

Concepts of solutions

Strong solutions

- random variables ϱ, $\mathbf{u}$ have smooth paths
- stochastic integral defined on the original probabilistic space
- equations satisfied pathwise (a.s.)

Weak martingale solutions

- random variables with continuous paths in the weak L^p−topology
- stochastic basis $\{\Omega, \mathbb{P}, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}\}$ and the noise W_E, W_F considered as "unknowns"
- equations satisfied pathwise in the weak (distributional) sense

Dissipative (weak) martingale solution

A dissipative martingale solutions is a weak martingale solution satisfying a certain form of energy balance

Dissipative martingale solutions

Equation of continuity

$$\int_0^T \int_\Omega \partial_t \psi \varrho \varphi + \psi \varrho \mathbf{u} \cdot \nabla_x \varphi \ \mathrm{d}x\mathrm{d}t = -\psi(0) \int_\Omega \varrho_0 \varphi \ \mathrm{d}x$$

$$\psi \in C_c^\infty[0,T), \ \varphi \in C_c^\infty(\Omega)$$

Momentum equation

$$\int_0^T \int_\Omega \partial_t \psi \varrho \mathbf{u} \cdot \boldsymbol{\varphi} + \psi \varrho \mathbf{u} \otimes \mathbf{u} : \cdot \nabla_x \boldsymbol{\varphi} + \psi p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \ \mathrm{d}x\mathrm{d}t$$

$$= \int_0^T \psi \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \psi \mathrm{d}t - \psi(0) \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \boldsymbol{\varphi} \ \mathrm{d}x$$

$$- \int_0^T \psi \left(\int_\Omega \varrho \cdot \boldsymbol{\varphi} \ \mathrm{d}x \right) \mathrm{d}W_E - \int_0^T \psi \left(\int_\Omega \varrho \mathbf{u} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \right) \mathrm{d}W_F$$

$$\psi \in C_c^\infty[0,T), \ \boldsymbol{\varphi} \in C_c^\infty(\Omega; R^3)$$

Energy balance

Pressure potential, diffusion coefficients

$$P(\varrho) = \varrho \int_1^{\varrho} \frac{p(z)}{z^2} \mathrm{d}z$$

$$\varrho \ \mathrm{d}W_E + \varrho \mathbf{u} \ \mathrm{d}W_F = \varrho \sum_{k=1}^{\infty} \mathbf{G}_k(\varrho, \varrho \mathbf{u}) \ \mathrm{d}W_k$$

Energy balance (inequality)

$$- \int_0^T \partial_t \psi \left(\int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right] \ \mathrm{d}x \right) \ \mathrm{d}t$$

$$+ \int_0^T \psi \int_\Omega \mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t \le \psi(0) \int_\Omega \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0) \right] \ \mathrm{d}x$$

$$+ \frac{1}{2} \int_0^T \psi \left(\int_\Omega \sum_{k \ge 1} \frac{|\mathbf{G}_k(\varrho, \varrho \mathbf{u})|^2}{\varrho} \ \mathrm{d}x \right) \mathrm{d}t + \int_0^T \psi \mathrm{d}M_E$$

$$\psi \in C_c^\infty[0, T)$$

Existence of solutions

Dissipative martingale solutions

[**Breit-Hofmanova 2015**]
Dissipative martingale solutions exist for any finite energy initial data $[\varrho_0, \mathbf{u}_0]$ and globally Lipschitz diffusion coefficients $\mathbf{G}_k$.

Strong solutions

Strong solutions exist locally in time for any smooth initial data. Locally means there is a positive maximal stopping time.

Weak strong uniqueness

Dissipative martingale and weak solutions coincide *in law* as long as the latter exists.

Relative energy inequality

Relative energy functional

$$\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) = \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r)\right] \mathrm{d}x$$

Relative energy inequality

$$-\int_0^T \partial_t \psi \ \mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) \ \mathrm{d}t$$

$$+\int_0^T \psi \int_\Omega (\mathbb{S}(\nabla \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \, \mathrm{d}x \, \mathrm{d}t$$

$$\leq \psi(0)\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right)(0) + \int_0^T \psi \mathrm{d}M_{RE} + \int_0^T \psi \mathcal{R}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) \mathrm{d}t$$

Remainder

Test functions

$$\mathrm{d}r = D_t^d r \mathrm{d}t + D_t^s r \, \mathrm{d}W, \ \mathrm{d}\mathbf{U} = D_t^d \mathbf{U} \mathrm{d}t + D_t^s \mathbf{U} \, \mathrm{d}W$$

Remainder term

$$\mathcal{R}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) = \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) \, \mathrm{d}x$$

$$+ \int_\Omega \varrho \Big(D_t^d \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x$$

$$+ \int_\Omega \left((r - \varrho) P''(r) D_t^d r + \nabla_x P'(r) (r\mathbf{U} - \varrho \mathbf{u}) \right) \, \mathrm{d}x$$

$$- \int_\Omega \mathrm{div}_x \mathbf{U} (p(\varrho) - p(r)) \, \mathrm{d}x + \frac{1}{2} \sum_{k \geq 1} \int_\Omega \varrho \left| \frac{\mathbf{G}_k(\varrho, \varrho \mathbf{u})}{\varrho} - D_t^s \mathbf{U}_k \right|^2 \, \mathrm{d}x$$

$$+ \frac{1}{2} \sum_{k \geq 1} \int_\Omega \varrho P'''(r) |D_t^s r_k|^2 \, \mathrm{d}x + \frac{1}{2} \sum_{k \geq 1} \int_\Omega p''(r) |D_t^s r_k|^2 \, \mathrm{d}x$$

Other applications

Low Mach number limit

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u})\mathrm{d}t = 0$$

$$\mathrm{d}(\varrho \mathbf{u}) + [\mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho)]\mathrm{d}t$$

$$= [\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})]\mathrm{d}t + \varrho \mathrm{d}W_E + \varrho \mathbf{u} \mathrm{d}W_F$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} + \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u}\mathbb{I}$$

Low Mach and vanishing viscosity limit

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u})\mathrm{d}t = 0$$

$$\mathrm{d}(\varrho \mathbf{u}) + [\mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho)]\mathrm{d}t$$

$$= \varepsilon[\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})]\mathrm{d}t + \varrho \mathrm{d}W_E + \varrho \mathbf{u} \mathrm{d}W_F$$

Stationary solutions

Main result

Let $N = 3$, $M_0 > 0$. Let $p = a\varrho^\gamma$, with $\gamma > \frac{3}{2}$. Suppose that the diffusion coefficients $\mathbf{G}_k$ satisfy

$$\mathbf{G}_k(x, \varrho, \varrho\mathbf{u}) = \varrho\mathbf{g}_k(x, \varrho, \varrho\mathbf{u})$$

$$\mathbf{g}_k \in C^1(\Omega \times [0, \infty) \times R^3); R^3), \ \mathbf{g}_k \text{ globally Lipschitz in } \varrho, \ \mathbf{q} = \varrho\mathbf{u}$$

$$|\mathbf{g}_k(x, \varrho, \varrho\mathbf{u})| \leq \alpha_k, \ \sum_{k \geq 0} \alpha_k^2 = G < \infty$$

Then problem the compressible Navier-Stokes system admits a stationary martingale solution $[\varrho, \varrho\mathbf{u}]$.

Boundedness in probability

Boundedness in probability

$$\mathbb{E}\left[\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right](\tau,\cdot)\ \mathrm{d}x\right] \leq E_0$$

Energy balance

$$\mathrm{d}\left[\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right]\ \mathrm{d}x\right]$$

$$+\boxed{\int_\Omega \mathbb{S}(\nabla\mathbf{u}) : \nabla\mathbf{u}\ \mathrm{d}x\ \mathrm{d}t}$$

$$\leq +\frac{1}{2}\int_\Omega \sum_{k\geq 1} \frac{|\mathbf{G}_k(\varrho,\varrho\mathbf{u})|^2}{\varrho}\ \mathrm{d}x\Big)\mathrm{d}t + \mathrm{d}M_E$$

Energy dissipation

$$\int_\Omega \mathbb{S}(\nabla\mathbf{u}) : \nabla\mathbf{u}\ \mathrm{d}x \approx \|\mathbf{u}\|^2_{W^{1,2}}$$