Haar meager sets

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Haar null sets

Definition (Christensen, 1972)

Let $(G, \cdot)$ be a Polish group.
A set $A \subset G$ is Haar null (HN) if there are

- a Borel set $B \supset A$
- a Borel probability measure μ on G

such that for every $g, h \in G$ we have $\mu(g \cdot B \cdot h) = 0$.

Properties (Christensen)

- *Haar null sets form a σ-ideal.*
- *In locally compact Polish groups, Haar null sets coincide with sets of Haar measure zero.*

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Properties (Darji)

- *Haar meager sets form a σ-ideal.*
- *In locally compact Polish groups, Haar meager sets coincide with meager sets.*

Local compactness

In locally compact Polish groups:

- HN = of Haar measure zero
- HM = meager

In general Polish groups:

- There is no Haar measure, but the notion of HN sets still makes sense.
- Both notions of meager sets and HM sets make sense. However, HM sets provide better analog of HN sets.

Theorem (Darji)

Each HM set is meager.

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Let G be an abelian non-locally compact Polish group. Then there is a meager set $A \subset G$ which is not HM.

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Can we replace 'Borel set $B \supset A$' by

- $B = A$?
- universally measurable set $B \supset A$?
- analytic set $B \supset A$?
- F_σ set $B \supset A$?

We can ask analogous question for HN sets, too.

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Answers

	HN	HM
$B = A$	No [ES]	No [DRVV]
universally measurable	No (see below)	No (see below)
coanalytic	No [EV]	No [DV]
analytic	Yes [S]	Yes [DRVV]
Borel set $B \supset A$	original definition	original definition
G_δ (for HN)	No [EV]	
F_σ (for HM)		No [DV]

[ES] Elekes, Steprāns, 2013
[DRVV] Doležal, Rmoutil, Vejnar, Vlasák, 2016
[EV] Elekes, Vidnyánszky, 2014
[DV] Doležal, Vlasák, 2016

[S] Solecki, 1996

Example

Example (Darji, 2014)

Assume the Continuum Hypothesis.
The 'Borel set $B \supset A$' from the definition of HM sets cannot be replaced by $B = A$.

Proof:
Let $K \subset \mathbb{R}$ be a perfect nowhere dense compact set.
We construct a set $A \subset \mathbb{R}$ such that

- A is not meager (and so it is not HM),
- $(A + x) \cap K$ is at most countable for every $x \in \mathbb{R}$ (so that for the identity mapping f on K, the set $f^{-1}(A + x)$ is meager in K for every $x \in \mathbb{R}$).

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Example (continuation)

Let $(C_\alpha)_{\alpha<\omega_1}$ be an enumeration of all F_σ meager subsets of $\mathbb{R}$.
Let $(r_\alpha)_{\alpha<\omega_1}$ be an enumeration of all elements of $\mathbb{R}$.
For each $\alpha < \omega_1$, we pick $x_\alpha \in \mathbb{R} \setminus \bigcup_{\beta\leq\alpha}(C_\beta \cup (K - r_\beta))$.
We set $A := \{x_\alpha \colon \alpha < \omega_1\}$.
Then A is not meager (as it is not contained in any C_α).
For every $\alpha < \omega_1$ we have

$$|(A + r_\alpha) \cap K| = |A \cap (K - r_\alpha)| \leq \alpha < \omega_1.$$

□

Theorem (folklore)

There is a decomposition $\mathbb{R} = A \sqcup B$ such that A is of Lebesgue measure zero and B is meager.

Proof:
For every $k \in \mathbb{N}$, there is an open dense subset of $\mathbb{R}$ with Lebesgue measure at most $\frac{1}{k}$.
(Take the union of countably many open intervals with very small lengths such that each rational number is contained in one of them.)
The intersection of the countably many sets above is a comeager subset of $\mathbb{R}$ of Lebesgue measure zero. □

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Theorem (DRVV)

Let G be an abelian Polish group with an uncountable closed locally compact subgroup. Then there is a decomposition $G = A \sqcup B$ such that A is HN and B is HM.
In particular, this holds for $G = \mathbb{R}^{\mathbb{N}}$ or $G = X$ where X is a Banach space.

Theorem (DRVV)

Let G be a non-discrete abelian Polish group such that its identity element has a local basis consisting of open subgroups. Then there is a decomposition $G = A \sqcup B$ such that A is HN and B is HM.
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Problem 1

Let G be a Polish group.
Is there a decomposition $G = A \sqcup B$ such that A is HN and B is HM?

Problem 2

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Let G be a Polish group. Let B be a Borel HM subset of G. Is there a compact set $K \subset G$ such that for every $g, h \in G$, the set $g \cdot B \cdot h \cap K$ is meager in K?

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The answer to Problem 2 'is not easy':

Example (DRVV)

There exist a G_δ HM set $A \subset \mathbb{R}$, a compact metric space K and a continuous function $f: K \to \mathbb{R}$ witnessing that A is HM such that $A \cap f(K)$ is comeager in $f(K)$.

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Thank you for your attention!