Singular limits of compressible viscous fluids

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Scale analysis

$$X \approx \frac{x}{X_{\rm char}}, \ \partial_X \approx X_{\rm char} \partial_x$$

GEOMETRIC SCALING

Characteristic length, time, velocity, magnitude of external forces

MATERIAL SCALING

Scaling constitutive relations - pressure, viscosity, density, temperature

Scaled compressible Navier-Stokes system

Equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right), \ \mu > 0$$

Boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

$$\mathbf{u} \to 0, \ \varrho \to \overline{\varrho} > 0 \text{ as } |x| \to \infty$$

Incompressible Navier-Stokes system

Equations

$$\mathrm{div}_x \mathbf{U} = 0$$

$$\overline{\varrho}\Big(\partial_t \mathbf{U} + \mathrm{div}_x \mathbf{U} \otimes \mathbf{U} \Big) + \nabla_x \Pi = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U})$$

$$\mathbb{S}(\nabla_x \mathbf{U}) = \mu \Big(\nabla_x \mathbf{U} + \nabla_x^t \mathbf{U} \Big)$$

Boundary conditions

$$\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \ [\mathbb{S}(\nabla_x \mathbf{U})\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

$$\mathbf{U} \to 0 \text{ as } |x| \to \infty$$

Alternative choice of boundary conditions

No-slip boundary conditions

$$\mathbf{U}|_{\partial\Omega} = 0$$

Partial slip - Navier's friction

$$\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}(\nabla_x \mathbf{U})]_{\rm tan} + \beta[\mathbf{U}]_{\rm tan}|_{\partial\Omega} = 0, \ \beta \geq 0$$

Energy balance: Stability of equilibria

ENERGY INEQUALITY

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E(\varrho,\mathbf{u})\ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\ \mathrm{d}x \leq 0$$

RELATIVE ENTROPY

$$E(\varrho,\vartheta) = \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{\varepsilon^2}\Big(P(\varrho) - \partial_\varrho P(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho})\Big)$$

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z$$

PRESSURE

$$p \in C[0,\infty) \cap C^2(0,\infty),\ \ p(0) = 0,\ \ p'(\varrho) > 0 \text{ for } \varrho > 0$$

$$\lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma - 1}} = p_\infty > 0,\ \ \gamma > \frac{3}{2}$$

VISCOUS STRESS

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u}\ \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

$$\mu > 0,\ \ \eta \geq 0$$

Prepared initial data

$$\varrho(0,\cdot) = \varrho_{\varepsilon,0} = \overline{\varrho} + \varepsilon r_{\varepsilon,0},\ \mathbf{u}(0,\cdot) = \mathbf{u}_{\varepsilon,0}$$

$$\|r_{\varepsilon,0}\|_{(L^2\cap L^\infty(\Omega))} \leq c,\ \|\mathbf{u}_{0,\varepsilon}\|_{L^2(\Omega;R^3)} \leq c$$

$$\int_\Omega E(\varrho_\varepsilon, \mathbf{u}_\varepsilon)(\tau,\cdot)\mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon \ \mathrm{d}x\mathrm{d}t$$

$$\leq \int_\Omega E(\varrho_{\varepsilon,0}, \mathbf{u}_{\varepsilon,0})(\tau,\cdot)\mathrm{d}x \leq c \text{ for } \varepsilon \to 0$$

$$P(\varrho)-\partial_\varrho P(\overline{\varrho})(\varrho-\overline{\varrho})-P(\overline{\varrho}) \geq \left\{ \begin{array}{c} c(K)|\varrho - \overline{\varrho}|^2 \text{ for } \varrho \in K \subset (0,\infty) \\ \\ c(K)|\varrho - \overline{\varrho}|^\gamma \text{ for } \varrho \in [0,\infty) \setminus K \end{array} \right.$$

K compact containing an open neighbourhood of $\overline{\varrho}$

$$\mathrm{ess} \sup_{t \in (0,T)} \| \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon (t, \cdot) \|_{L^2(\Omega; R^3)} \leq c$$

Essential and residual parts

$$[h]_{\rm ess} = \chi(\varrho_\varepsilon)h,\ [h]_{\rm res} = (1-\chi(\varrho_\varepsilon))h$$

$$\chi \in C^\infty_c(0,\infty),\ 0 \leq \chi \leq 1,\ \chi \equiv 1 \text{ in a neighbourhood of } \overline{\varrho}$$

$${\rm ess} \sup_{t \in (0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right]_{\rm ess} \right\|_{L^2(\Omega)} \leq c$$

$${\rm ess} \sup_{t \in (0,T)} \int_\Omega 1_{\rm res} \ {\rm d}x \leq \varepsilon^2 c$$

$${\rm ess} \sup_{t \in (0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right]_{\rm res} \right\|_{L^q(\Omega)} \leq \varepsilon^{\frac{2-q}{q}} c,\ 1 \leq q \leq \min\{\gamma, 2\}$$

Korn's inequality

$$\int_0^T \int_\Omega \left| \nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3} \mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I} \right|^2 \ \mathrm{d}x \ \mathrm{d}t \leq c$$

$$\|\mathbf{v}\|^2_{W^{1,2}(B;R^3)}$$

$$\leq c(m,B) \left(\left\| \nabla_x \mathbf{v} + \nabla_x^t \mathbf{v} - \frac{2}{3} \mathrm{div}_x \mathbf{v} \mathbb{I} \right\|^2_{L^2(B;R^{3\times 3})} + \int_{B \cap V} |\mathbf{v}|^2 \ \mathrm{d}x \right)$$

$$|V| \geq m > 0$$

$$\|\mathbf{u}_\varepsilon\|_{W^{1,2}(\Omega;R^3)} \leq c$$

Stability

$$r_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to r \text{ weakly in } (L^2 + L^q)(\Omega)$$

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \varrho_\varepsilon - \overline{\varrho} \right\|_{(L^2 + L^q)(\Omega)} \leq \varepsilon c$$

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0, T; W^{1,2}(\Omega))$$

$$\mathrm{div}_x \mathbf{U} = 0, \ \mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

$$\mathbf{v} = \mathbf{H}[\mathbf{v}] + \nabla_x \Psi$$

$$\Delta \Psi = \mathrm{div}_x \mathbf{v} \text{ in } \Omega, \ (\nabla_x \Psi - \mathbf{v}) \cdot \mathbf{n}|_{\partial\Omega} = 0$$

$$\int_\Omega \nabla_x \Psi \cdot \nabla_x \varphi \ \mathrm{d}x = \int_\Omega \mathbf{v} \cdot \nabla_x \varphi \ \mathrm{d}x \text{ for all } \varphi \in C_c^\infty(\overline{\Omega})$$

FARWIG, KOZONO, AND SOHR [2005]

If $\Omega \subset R^3$ is a domain with uniform $C^{1,1}$ boundary, then **H** *is bounded in $L^2 \cap L^r$ for $r \geq 2$ and in $L^2 + L^q$ for $1 < q < 2$.*

Lighthill's equation

$$r_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}, \ \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon$$

$$\varepsilon \partial_t r_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = 0$$

$$\varepsilon \partial_t \mathbf{V}_\varepsilon + p'(\overline{\varrho}) \nabla_x r_\varepsilon = \varepsilon \mathrm{div}_x \mathbb{L}_\varepsilon$$

$$\mathbf{V}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Lighthill's tensor

$$\mathbb{L}_\varepsilon = \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) - (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \frac{1}{\varepsilon^2} \left(p(\varrho_\varepsilon) - p'(\overline{\varrho})(\varrho_\varepsilon - \overline{\varrho}) - p(\overline{\varrho}) \right) \mathbb{I}$$

Lighthill's equation (weak formulation)

$$\int_0^T \int_\Omega \left(\varepsilon r_\varepsilon \partial_t \varphi + \mathbf{V}_\varepsilon \cdot \nabla_x \varphi \right) \, \mathrm{d}x \, \mathrm{d}t = - \int_\Omega r_{\varepsilon,0} \varphi(0, \cdot) \, \mathrm{d}x$$

for all $\varphi \in C_c^\infty([0, T) \times \overline{\Omega})$

$$\int_0^T \int_\Omega \left(\varepsilon \mathbf{V}_\varepsilon \cdot \partial_t \varphi + p'(\overline{\varrho}) r_\varepsilon \mathrm{div}_x \varphi \right) \, \mathrm{d}x \, \mathrm{d}t$$

$$= \varepsilon \int_0^T \int_\Omega \mathbb{L}_\varepsilon : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega \mathbf{V}_{\varepsilon,0} \cdot \varphi(0, \cdot) \, \mathrm{d}x$$

for all $\varphi \in C_c^\infty([0, T) \times \overline{\Omega}; R^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$

Uniform bounds

$$r_\varepsilon = [r_\varepsilon]_{\rm ess} + [r_\varepsilon]_{\rm res}$$

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| [r_\varepsilon]_{\rm ess} \right\|_{L^2(\Omega)} \leq c$$

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| [r_\varepsilon]_{\rm res} \right\|_{L^q(\Omega)} \leq \varepsilon^{\frac{2-q}{q}} c, \ 1 \leq q \leq \min\{\gamma, 2\}$$

$$\left\| r_{\varepsilon,0} \right\|_{L^2 \cap L^\infty(\Omega)} \leq c$$

$$\mathbf{V}_\varepsilon = [\mathbf{V}_\varepsilon]_{\rm ess} + [\mathbf{V}_\varepsilon]_{\rm res}$$

$$\mathrm{ess}\sup_{t\in(0,T)} \|[\mathbf{V}_\varepsilon]_{\rm ess}\|_{L^2(\Omega;R^3)} \leq c$$

$$\mathrm{ess}\sup_{t\in(0,T)} \|[\mathbf{V}_\varepsilon]_{\rm res}\|_{L^r(\Omega;R^3)} \leq \varepsilon^{1/\gamma} c, \ \ r = \frac{2\gamma}{\gamma+1}$$

$$\|\mathbf{V}_{\varepsilon,0}\|_{L^2(\Omega;R^3)} \leq c$$

$$\mathbb{L}_\varepsilon = \mathbb{L}^1_\varepsilon + \mathbb{L}^2_\varepsilon + L^3_\varepsilon \mathbb{I}$$

$$\int_0^T \|\mathbb{L}^1_\varepsilon\|^2_{L^2(\Omega;R^{3\times 3})} \ {\rm d}t \leq c$$

$$\int_0^T \|\mathbb{L}^2_\varepsilon\|^2_{L^q(\Omega;R^{3\times 3})} \ {\rm d}t \leq c, \ \ q = \frac{6\gamma}{4\gamma + 3}$$

$$\mathrm{ess} \sup_{t \in (0,T)} \|L^3_\varepsilon\|_{L^1(\Omega)} \leq c$$

GOAL

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ in } L^2((0,T) \times K; R^3) \text{ for any compact } K \subset \Omega$$

STEP 1:

$$\int_0^T \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{u}_\varepsilon \varphi \ \mathrm{d}x \to \overline{\varrho} \int_0^T \int_\Omega |\mathbf{U}|^2 \varphi \ \mathrm{d}x \ \mathrm{d}t, \ \varphi \in C_c^\infty(\Omega)$$

STEP 2:

$$\left\{ t \mapsto \int_\Omega (\varrho_\varepsilon \mathbf{u}_\varepsilon)(t,\cdot)\cdot\varphi \ \mathrm{d}x \right\} \to \left\{ t \mapsto \overline{\varrho} \int_\Omega \mathbf{U}(t,\cdot)\cdot\varphi \ \mathrm{d}x \right\}$$

in $L^2(0,T)$

STEP 3:

COMPACTNESS OF THE SOLENOIDAL COMPONENT

$$\left\{ t \mapsto \int_\Omega (\varrho_\varepsilon \mathbf{u}_\varepsilon)(t,\cdot)\cdot\mathbf{H}[\varphi] \ \mathrm{d}x \right\} \to \left\{ t \mapsto \overline{\varrho} \int_\Omega \mathbf{U}(t,\cdot)\cdot\varphi \ \mathrm{d}x \right\}$$

in $L^2(0,T)$

Acoustic potential

$$\varrho_\varepsilon \mathbf{u}_\varepsilon = \mathbf{V}_\varepsilon = \mathbf{H}[\mathbf{V}_\varepsilon] + \nabla_x \Psi_\varepsilon$$

ULTIMATE GOAL

$$\left\{ t \mapsto \int_\Omega \Psi_\varepsilon \mathrm{div}_x \varphi \ \mathrm{d}x \right\} \to 0 \text{ in } L^2(0,T)$$

for any $\varphi \in C_c^\infty(\Omega; R^3)$

Acoustic (wave) equation

$$\varepsilon \partial_t r_\varepsilon + \Delta \Psi_\varepsilon = 0$$

$$\varepsilon \partial_t \Psi_\varepsilon + p'(\overline{\varrho}) r_\varepsilon = \Delta_N^{-1} [\mathrm{div}_x \mathrm{div}_x \mathbb{L}_\varepsilon]$$

$$\nabla_x \Psi_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Δ_N -Neumann Laplacian

Acoustic equation - weak formulation

$$\int_0^T \int_\Omega \Big(\varepsilon r_\varepsilon \partial_t \varphi + \nabla_x \Psi_\varepsilon \cdot \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t = - \int_\Omega \varepsilon r_{\varepsilon,0} \varphi(0,\cdot) \ \mathrm{d}x$$

$$\text{for all } \varphi \in C_c^\infty([0,T) \times \overline{\Omega})$$

$$\int_0^T \int_\Omega \Big(\varepsilon \Psi_\varepsilon \partial_t \varphi - p'(\overline{\varrho}) r_\varepsilon \varphi \Big) \ \mathrm{d}x$$
$$- \varepsilon \int_0^T \int_\Omega \Big((\mathbb{L}^1_\varepsilon + \mathbb{L}^2_\varepsilon) : \nabla_x^2 \Delta_N^{-1}[\varphi] + L^3_\varepsilon \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t$$
$$+ \varepsilon \int_\Omega \mathbf{V}_{\varepsilon,0} \cdot \nabla_x \Delta_N^{-1}[\varphi] \ \mathrm{d}x \text{ for all } \varphi \in C_c^\infty([0,T) \times \overline{\Omega})$$

$$\mathcal{D}(-\Delta_N)$$

$$= \left\{ v \in \ W^{1,2}(\Omega) \ \middle| \ \int_\Omega \nabla_x v \cdot \nabla_x \varphi \ \mathrm{d}x = \int_\Omega g \varphi \ \mathrm{d}x, \ g \in L^2(\Omega) \right.$$

$$\left. \text{for all } \varphi \in C_c^\infty(\overline{\Omega}) \right\}$$

$$-\Delta_N[v] = -\Delta v$$

$-\Delta_N$ is self-adjoint non-negative operator in $L^2(\Omega)$. If Ω is of class $C^{1,1}$, then

$$\mathcal{D}(-\Delta_N) = \{ v \in W^{2,2}(\Omega), \ \nabla_x \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0 \}$$

Spectral measures

$\sigma(-\Delta_N) \subset [0, \infty)$, $\{P_\lambda\}_{\lambda \geq 0}$ orthogonal projections in $L^2(\Omega)$

FUNCTIONAL CALCULUS

$$\langle G(-\Delta_N)[\psi]; \varphi \rangle = \int_0^\infty G(\lambda) \, \mathrm{d} \langle \psi; P_\lambda[\varphi] \rangle$$

SPECTRAL THEOREM

$$\langle G(-\Delta_N)[\psi]; \varphi \rangle = \int_0^\infty G(\lambda) \tilde{\psi}(\lambda) \, \mathrm{d}\mu_\varphi(\lambda)$$

$$\mu_\varphi(\lambda) = \langle P_\lambda[\varphi]; \varphi \rangle \,, \; \|\tilde{\psi}\|_{L^2_{\mu_\varphi}} \leq \|\psi\|_{L^2(\Omega)}$$

Duhamel's formula

BOUNDS ON THE INITIAL DATA

$$\|r_{\varepsilon,0}\|_{L^2(\Omega)} \leq c,$$

$$\Psi_{0,\varepsilon} = (-\Delta)_N^{-1/2}[g_\varepsilon^1], \ \|g_\varepsilon^1\|_{L^2(\Omega)} \leq c$$

BOUNDS ON THE FORCING TERMS

$$(-\Delta_N)^{-1}\mathrm{div}_x\mathrm{div}_x\mathbb{L}_\varepsilon^1 = g_\varepsilon^2$$

$$(-\Delta_N)^{-1}\mathrm{div}_x\mathrm{div}_x\mathbb{L}_\varepsilon^2 = (-\Delta_N)[g_\varepsilon^3] + g_\varepsilon^4$$

$$L_\varepsilon^3 = (-\Delta_N)[g_\varepsilon^5] + g_\varepsilon^6$$

Acoustic equation - abstract formulation

$$\varepsilon \partial_t r_\varepsilon + \Delta_N \Psi_\varepsilon = 0$$

$$\varepsilon \partial_t \Psi_\varepsilon + r_\varepsilon = \varepsilon \Big((-\Delta_N)[h^1_\varepsilon] + h^2_\varepsilon \Big)$$

$$r_\varepsilon(0) = r_{\varepsilon,0}, \ \Psi_\varepsilon(0) = (-\Delta)_N^{-1/2}[g^0_\varepsilon]$$

$$p'(\overline{\varrho}) \equiv 1, \ \|g^0_\varepsilon\|_{L^2(\Omega)} + \|h^1_\varepsilon\|_{L^2(0,T;L^2(\Omega))} + \|h^2_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq c$$

Abstract Duhamel's formula

$$\Psi_\varepsilon(t) = \frac{1}{2\sqrt{-\Delta_N}} \exp\left(\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[g^0_\varepsilon + \mathrm{i} r_{\varepsilon,0}\right]$$

$$+ \frac{1}{2\sqrt{-\Delta_N}} \exp\left(-\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[g^0_\varepsilon - \mathrm{i} r_{\varepsilon,0}\right]$$

$$\frac{1}{2}\int_0^t \left(\exp\left(\mathrm{i}\sqrt{-\Delta_N}\frac{t-s}{\varepsilon}\right) + \exp\left(-\mathrm{i}\sqrt{-\Delta_N}\frac{t-s}{\varepsilon}\right)\right)$$

$$\left[(-\Delta_N)[h^1_\varepsilon] + h^2_\varepsilon\right] \ \mathrm{d}s$$

Dispersive estimates

$$\left\langle \exp\left(\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right) G(-\Delta_N)[\psi], \varphi \right\rangle$$

$$= \int_0^\infty \exp\left(\mathrm{i}\sqrt{y}\frac{t}{\varepsilon}\right) G(y)\tilde{\psi}(y)\mathrm{d}\mu_\varphi(y)$$

$$\int_0^T \left| \left\langle \exp\left(\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right) G(-\Delta_N)[\psi], \varphi \right\rangle \right|^2 \ \mathrm{d}t$$

$$= \int_0^T \int_0^\infty \int_0^\infty \exp\left(\mathrm{i}\left(\sqrt{y} - \sqrt{x}\right) \frac{t}{\varepsilon}\right)$$

$$G(y)G(x)\tilde{\psi}(y)\overline{\tilde{\psi}(x)} \ \mathrm{d}\mu_\varphi(y) \ \mathrm{d}\mu_\varphi(x)$$

$$\leq e \int_0^\infty \int_0^\infty \left[\int_{-\infty}^\infty \exp(-(t/T)^2) \exp \left(\mathrm{i} \left(\sqrt{y} - \sqrt{x} \right) \frac{t}{\varepsilon} \right) \right] \ \mathrm{d}t$$

$$G(y) G(x) \tilde{\psi}(y) \overline{\tilde{\psi}(x)} \ \mathrm{d}\mu_\varphi(y) \ \mathrm{d}\mu_\varphi(x)$$

$$= eT\sqrt{\pi} \int_0^\infty \int_0^\infty \Big(\exp \left(- \frac{T^2 (\sqrt{x} - \sqrt{y})^2}{4\varepsilon^2} \right)$$

$$G(y) G(x) \tilde{\psi}(y) \overline{\tilde{\psi}(x)} \ \Big) \mathrm{d}\mu_\varphi(y) \ \mathrm{d}\mu_\varphi(x)$$

$$\int_0^T \left| \left\langle \exp\left(\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right) G(-\Delta_N)[\psi], \varphi \right\rangle \right|^2 \, \mathrm{d}t \leq \omega^2(\varepsilon, \varphi, G) \|\psi\|^2_{L^2(\Omega)}$$

$$\omega^2(\varepsilon, \varphi, G) = eT\sqrt{\pi} \left(\int_0^\infty \int_0^\infty \exp\left(-\frac{T^2(\sqrt{x} - \sqrt{y})^2}{2\varepsilon^2} \right) \times \right.$$

$$\left. \times G^2(x) G^2(y) \mathrm{d}\mu_\varphi(y) \, \mathrm{d}\mu_\varphi(x) \right)^{1/2}$$

- $\omega(\varepsilon, \varphi, G) \to 0$ as $\varepsilon \to 0$ uniformly for $|G| \leq 1$

$$\Longleftrightarrow$$

- The spectral measure μ_φ does not charge points in $[0, \infty)$

$$\Longleftrightarrow$$

- Neumann Laplacian Δ_N does not admit eigenvalues in $L^2(\Omega)$

Rellich's lemma

Rellich [1948]

Suppose that

$$\Delta w = \lambda w \text{ for } |x| > R, \ w \not\equiv 0.$$

Then

$$\liminf_{r \to \infty} \int_{|x|=r} \left(|\partial_r v|^2 + |v|^2 \right) > 0$$

Decay of the non-homogenous terms

$$\int_0^T \int_0^t \left| \left\langle \exp \left(-\mathrm{i}\sqrt{-\Delta_N} \frac{t-s}{\varepsilon} \right) G(-\Delta_N)[F_\varepsilon], \varphi \right\rangle \right|^2 \ \mathrm{d}s \ \mathrm{d}t$$

$$\leq \int_0^T \int_0^T \left| \left\langle \exp \left(-\mathrm{i}\sqrt{-\Delta_N} \frac{t-s}{\varepsilon} \right) G(-\Delta_N)[F_\varepsilon(s)], \varphi \right\rangle \right|^2 \ \mathrm{d}t \ \mathrm{d}s$$

$$\leq \omega^2(\varepsilon, G, \varphi) \int_0^T \left\| \exp \left(\mathrm{i}\sqrt{-\Delta_N} \frac{s}{\varepsilon} \right) [F_\varepsilon(s)] \right\|^2_{L^2(\Omega)} \ \mathrm{d}s$$

$$= \omega^2(\varepsilon, \varphi, G) \int_0^T \|F_\varepsilon(s)\|^2_{L^2(\Omega)} \ \mathrm{d}s.$$

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \frac{1}{\varepsilon^2}\varrho \nabla_x F$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right), \ \mu > 0$$

STATIC DENSITY DISTRIBUTION

$$\nabla_x p(\tilde{\varrho}) = \tilde{\varrho}\nabla_x F, \ \int_\Omega \tilde{\varrho} - \varrho \ \mathrm{d}x = 0$$

Energy balance: Stability of equilibria

ENERGY INEQUALITY

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E(\varrho,\mathbf{u})\ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\ \mathrm{d}x \leq 0$$

$$E(\varrho,\vartheta) = \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{\varepsilon^2}\Big(P(\varrho) - \partial_\varrho P(\tilde\varrho)(\varrho - \tilde\varrho) - P(\tilde\varrho)\Big)$$

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z$$

ANELASTIC NAVIER-STOKES SYSTEM

$$\mathrm{div}_x(\tilde{\varrho}\mathbf{U}) = 0$$

$$\tilde{\varrho}\Big(\partial_t \mathbf{U} + \mathrm{div}_x \mathbf{U} \otimes \mathbf{U}\Big) + \tilde{\varrho}\nabla_x \Pi = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U})$$

$$\mathbb{S}(\nabla_x \mathbf{U}) = \mu\Big(\nabla_x \mathbf{U} + \nabla_x^t \mathbf{U}\Big)$$

Acoustic equation

$$p(\varrho) = \varrho, \ \nabla_x \varrho - \varrho \nabla_x F = \tilde{\varrho} \nabla_x \left(\frac{\varrho - \tilde{\varrho}}{\tilde{\varrho}} \right)$$

$$r_\varepsilon = \frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon \tilde{\varrho}}, \ \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon$$

$$\varepsilon \partial_t r_\varepsilon + \frac{1}{\tilde{\varrho}} \mathrm{div}_x \mathbf{V}_\varepsilon = 0$$

$$\varepsilon \partial_t \mathbf{V}_\varepsilon + \tilde{\varrho} \nabla_x r_\varepsilon = \varepsilon \mathrm{div}_x \mathbb{L}_\varepsilon$$

$$\mathbf{V}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0$$

$$\mathbb{L}_\varepsilon = \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) - \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$$

Generalized Helmholtz projection

$$\mathbf{v} = \mathbf{H}_{\tilde{\varrho}}[\mathbf{v}] + \tilde{\varrho} \nabla_x \Psi$$

$$\mathrm{div}_x (\tilde{\varrho} \nabla_x \Psi) = \mathrm{div}_x \mathbf{v} \text{ in } \Omega, \ (\tilde{\varrho} \nabla_x \Psi - \mathbf{v}) \cdot \mathbf{n}|_{\partial \Omega} = 0$$

$$\varepsilon \partial_t r_\varepsilon + \frac{1}{\tilde{\varrho}} \mathrm{div}_x (\tilde{\varrho} \nabla_x \Psi_\varepsilon) = 0$$

$$\varepsilon \partial_t \Psi_\varepsilon + r_\varepsilon = \varepsilon \Delta^{-1}_{\tilde{\varrho}, N} \mathrm{div}_x \mathrm{div}_x \mathbb{L}_\varepsilon$$

$$\tilde{\varrho} \nabla_x \Psi \cdot \mathbf{n}|_{\partial \Omega} = 0$$

$$\Delta_{\tilde{\varrho}, N} \Psi \equiv \frac{1}{\tilde{\varrho}} \mathrm{div}_x (\tilde{\varrho} \nabla_x \Psi), \ \tilde{\varrho} \nabla_x \Psi \cdot \mathbf{n}|_{\partial \Omega} = 0$$

EQUATIONS

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + + \frac{1}{\varepsilon}(\omega \times \varrho \mathbf{u}) + \frac{1}{\varepsilon^{2m}} \nabla_x p(\varrho)$$

$$= \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \frac{1}{\varepsilon^2} \varrho \nabla_x G$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I}), \ \mu > 0$$

$$\omega = [0,0,1], \ G = |x_h|^2, \ x_h = [x_1, x_2, 0]$$

BOUNDARY CONDITIONS

$$\Omega = R^2 \times (-1,1), \ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = [\mathbb{S}(\nabla_x \mathbf{u})\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

Acoustic equation

$$r_\varepsilon = \frac{\varrho_\varepsilon - \tilde{\varrho}_\varepsilon}{\varepsilon^m}, \nabla_x p(\tilde{\varrho}_\varepsilon) = \varepsilon^{2(m-1)} \tilde{\varrho}_\varepsilon \nabla_x G, \ \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon$$

$$\varepsilon^m \partial_t r_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = \varepsilon^\alpha F^1_\varepsilon$$

$$\varepsilon^m \partial_t \mathbf{V}_\varepsilon + p'(\overline{\varrho}) \nabla_x r_\varepsilon = \varepsilon^{2(m-1-\alpha)} \mathbf{F}^2_\varepsilon$$

Lesky and Racke [2003], Metcalfe [2004]

For $\varphi \in C_0^\infty(R^2)$ *we have*

$$\int_{-\infty}^{\infty} \int_\Omega \left| \varphi(x_h) \exp\left({\rm i}\sqrt{-\Delta} t \right) [v] \right|^2 \ {\rm d}x \ {\rm d}t \leq c(\varphi) \|v\|^2_{L^2(\Omega)}.$$

$$\int_{-T}^{T} \int_\Omega \left| \varphi(x_h) \exp\left({\rm i}\sqrt{-\Delta} \frac{t}{\varepsilon^m} \right) [v] \right|^2 \ {\rm d}x \ {\rm d}t \leq \varepsilon^m c(\varphi) \|v\|^2_{L^2(\Omega)}$$

Limit system

2D incompressible Navier-Stokes equations

$$\varrho_\varepsilon \to \overline{\varrho},\ \mathbf{u}_\varepsilon \to \mathbf{U}_h,\ \mathbf{U}_h = \mathbf{U_h}(t, x_h)$$

$$\mathrm{div}_h \mathbf{U}_h = 0$$

$$\overline{\varrho}\Big(\partial_t \mathbf{U}_h + \mathrm{div}_h (\mathbf{U}_h \otimes \mathbf{U}_h) \Big) + \nabla_h \Pi = \mu \Delta_h \mathbf{U}_h$$

Oberbeck-Boussinesq system:

$$\mathrm{div}_x \mathbf{U} = 0$$

$$\overline{\varrho}\Big(\partial_t \mathbf{U} + \mathrm{div}_x(\mathbf{U} \otimes \mathbf{U})\Big) + \nabla_x \Pi = \mathrm{div}_x \mathbb{S} + r \nabla_x F \text{ in } (0,T) \times \Omega$$

$$\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

$$\overline{\varrho} c_p \Big(\partial_t \Theta + \mathrm{div}_x(\Theta \mathbf{U})\Big) - \mathrm{div}_x(G\mathbf{U}) - \mathrm{div}_x(\kappa \nabla_x \Theta) = 0 \text{ in } (0,T) \times \Omega$$

$$G = aF, \ \nabla_x \Theta \cdot \mathbf{n}|_{\partial\Omega} = 0$$

$$r + \alpha \Theta = 0, \ \alpha > 0$$

Scaled Navier-Stokes-Fourier system:

$$\mathrm{Sr}\ \partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\mathrm{Sr}\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\mathrm{Ma}^2} \nabla_x p = \frac{1}{\mathrm{Re}} \mathrm{div}_x \mathbb{S} + \frac{1}{\mathrm{Fr}^2} \varrho \nabla_x F$$

$$\mathrm{Sr}\ \partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \frac{1}{\mathrm{Pe}} \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

$$\sigma \geq \frac{1}{\vartheta} \left(\frac{\mathrm{Ma}^2}{\mathrm{Re}} \mathbb{S} : \nabla_x \mathbf{u} - \frac{1}{\mathrm{Pe}} \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{\mathrm{Ma}^2}{2} \varrho |\mathbf{u}|^2 + \varrho e - \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2} \varrho F \right) = 0$$

Boundary conditions:

Complete slip boundary conditions:

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}]_{\rm tan} = 0$$

Navier's slip with friction:

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \beta[\mathbf{u}]_{\rm tan} + [\mathbb{S}\mathbf{n}]_{\rm tan}|_{\partial\Omega} = 0, \ \mathbf{q} \cdot \mathbf{n} + \beta|\mathbf{u}|^2|_{\partial\Omega} = 0$$

Constitutive relations:

GIBBS' RELATION:

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right)$$

NEWTON'S RHEOLOGICAL LAW:

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

FOURIER'S LAW:

$$\mathbf{q} = -\kappa \nabla_x \vartheta$$

Characteristic numbers:

Symbol	Definition	Name
Sr	$\text{length}_{\text{ref}}/(\text{time}_{\text{ref}}\text{velocity}_{\text{ref}})$	Strouhal number
Ma	$\text{velocity}_{\text{ref}}/\sqrt{\text{pressure}_{\text{ref}}/\text{density}_{\text{ref}}}$	Mach number
Re	$\text{density}_{\text{ref}}\text{velocity}_{\text{ref}}\text{length}_{\text{ref}}/\text{viscosity}_{\text{ref}}$	Reynolds number
Fr	$\text{velocity}_{\text{ref}}/\sqrt{\text{length}_{\text{ref}}\text{force}_{\text{ref}}}$	Froude number
Pe	$\text{pressure}_{\text{ref}}\text{length}_{\text{ref}}\text{velocity}_{\text{ref}}/(\text{temperature}_{\text{ref}}\text{heat conductivity}_{\text{ref}})$	Péclet number

Low Mach number limit - weak stratification:

$$\mathrm{Sr} = \mathrm{Re} = \mathrm{Pe} = 1, \ \mathrm{Ma} = \varepsilon, \ \mathrm{Fr} = \sqrt{\varepsilon}, \ \beta = 0$$

STRATEGY:

1. Existence theory for the primitive Navier-Stokes-Fourier system
2. Uniform bounds independent of the singular parameter
3. Passage to the limit - analysis of acoustic waves
4. Identification of the limit system

Scaled Navier-Stokes-Fourier system:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \text{ in } (0,T) \times \Omega$$

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \frac{1}{\varepsilon} \varrho \nabla_x F \text{ in } (0,T) \times \Omega$$

$$[\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma \text{ in } (0,T) \times \Omega$$

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{\varepsilon^2}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varepsilon \varrho F \right) \ \mathrm{d}x = 0$$

$$\sigma \geq \frac{1}{\vartheta} \left(\varepsilon^2 \mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \geq 0$$

Total dissipation balance:

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 \right.$$

$$\left. + \frac{1}{\varepsilon^2}\left(H(\varrho,\vartheta) - \partial_\varrho H(\tilde\varrho_\varepsilon,\overline{\vartheta})(\varrho - \tilde\varrho_\varepsilon) - H(\tilde\varrho_\varepsilon,\overline{\vartheta}) \right) \right)(\tau,\cdot)\ \mathrm{d}x$$

$$+\frac{\overline{\vartheta}}{\varepsilon^2}\int_0^\tau \int_\Omega \sigma\ \mathrm{d}x\ \mathrm{d}t =$$

$$\int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \right.$$

$$\left. \frac{1}{\varepsilon^2}\left(H(\varrho_0,\vartheta_0) - \partial_\varrho H(\tilde\varrho_\varepsilon,\overline{\vartheta})(\varrho_0 - \tilde\varrho_\varepsilon) - H(\tilde\varrho_\varepsilon,\overline{\vartheta}) \right) \right)\ \mathrm{d}x$$

$$\nabla_x p(\tilde\varrho_\varepsilon,\overline{\vartheta}) = \varepsilon\tilde\varrho_\varepsilon\nabla_x F,\ \int_\Omega \tilde\varrho_\varepsilon\ \mathrm{d}x = \int_\Omega \varrho_0\ \mathrm{d}x,\ \tilde\varrho_\varepsilon \approx \overline{\varrho}$$

Ballistic free energy:

$$H(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \overline{\vartheta} \varrho s(\varrho, \vartheta)$$

$$\frac{\partial^2 H(\varrho, \overline{\vartheta})}{\partial \varrho^2} = \frac{1}{\varrho} \frac{\partial p(\varrho, \overline{\vartheta})}{\partial \varrho} > 0$$

- $\varrho \mapsto H(\varrho, \overline{\vartheta})$ is strictly convex

$$\frac{\partial H(\varrho, \vartheta)}{\partial \vartheta} = \frac{\varrho}{\vartheta} (\vartheta - \overline{\vartheta}) \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta}$$

- $\vartheta \mapsto H(\varrho, \vartheta)$ attains its strict local minimum at $\overline{\vartheta}$

Thermodynamic stability hypothesis:

- **Positive compressibility:**

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0$$

- **Positive specific heat at constant volume:**

$$\frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Coercivity of ballistic free energy:

$$H(\varrho, \vartheta) - \frac{\partial H(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho}(\varrho - \tilde{\varrho}) - H(\tilde{\varrho}, \overline{\vartheta})$$

$$\geq c(B)\Big(|\varrho - \tilde{\varrho}|^2 + |\vartheta - \overline{\vartheta}|^2 \Big)$$

provided ϱ, ϑ belong to a compact interval $B \subset (0, \infty)$

$$\geq c(B)\Big(1 + \varrho e(\varrho, \vartheta) + \varrho |s(\varrho, \vartheta)| \Big)$$

otherwise

as soon as $\tilde{\varrho}$, $\overline{\vartheta}$ belong to $\mathrm{int}[B]$

Ill-prepared initial data:

$$\varrho_0 \approx \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \ \{\varrho^{(1)}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^1 \cap L^\infty(\Omega), \ \int_\Omega \varrho^{(1)}_{0,\varepsilon} \ \mathrm{d}x = 0$$

$$\vartheta_0 \approx \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon}, \ \{\vartheta^{(1)}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^1 \cap L^\infty(\Omega), \ \int_\Omega \vartheta^{(1)}_{0,\varepsilon} \ \mathrm{d}x = 0$$

$$\mathbf{u}_0 \approx \mathbf{u}_{0,\varepsilon}, \ \{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega; R^3)$$

Uniform bounds:

$$\left\{ \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right\}_{\varepsilon>0} \ \text{bounded in } L^\infty(0,T;L^2 \oplus L^q(\Omega)),\ q<2$$

$$\left\{ \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\}_{\varepsilon>0} \ \text{bounded in } L^\infty(0,T;L^2 \oplus L^q(\Omega)),\ q<2$$

$$\left\{ \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 \right\}_{\varepsilon>0} \ \text{bounded in } L^\infty(0,T;L^1(\Omega))$$

$$\left\{ \frac{\sigma_\varepsilon}{\varepsilon^2} \right\}_{\varepsilon>0} \ \text{bounded in } \mathcal{M}^+([0,T]\times\overline{\Omega})$$

$$\left\{ \nabla_x \mathbf{u}_\varepsilon \right\}_{\varepsilon>0} \ \text{bounded in } L^2((0,T)\times\Omega;R^{3\times3})$$

$$\left\{ \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon} \right\}_{\varepsilon>0} \ \text{bounded in } L^2((0,T)\times\Omega;R^3)$$

Convergence:

$$\varrho_\varepsilon \to \overline{\varrho} \text{ in } L^\infty(0,T;L^2 \oplus L^q(\Omega))$$

$$\vartheta_\varepsilon \to \overline{\vartheta} \text{ in } L^\infty(0,T;L^2 \oplus L^q(\Omega))$$

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;R^3))$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \Theta \text{ weakly in } L^2(0,T;W^{1,2}(\Omega))$$

Target system:

$$\mathrm{div}_x \mathbf{U} = 0$$

$$\overline{\varrho}\Big(\partial_t \mathbf{U} + \mathrm{div}_x(\mathbf{U} \otimes \mathbf{U})\Big) + \nabla_x \Pi = \mathrm{div}_x \mathbb{S} + r \nabla_x F \text{ in } (0, T) \times \Omega$$

$$\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

$$\overline{\varrho} c_p \Big(\partial_t \Theta + \mathrm{div}_x(\Theta \mathbf{U})\Big) - \mathrm{div}_x(G\mathbf{U}) - \mathrm{div}_x(\kappa \nabla_x \Theta) = 0 \text{ in } (0, T) \times \Omega$$

$$G = aF, \ \nabla_x \Theta \cdot \mathbf{n}|_{\partial\Omega} = 0$$

$$r + \alpha\Theta = 0, \ \alpha > 0$$

Lighthill's acoustic equation ($F = 0$):

$$\varepsilon \partial_t Z_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = \varepsilon \mathrm{div}_x \mathbf{F}^1_\varepsilon$$

$$\varepsilon \partial_t \mathbf{V}_\varepsilon + \omega \nabla_x Z_\varepsilon = \varepsilon \left(\mathrm{div}_x \mathbb{F}^2_\varepsilon + \nabla_x F^3_\varepsilon + \frac{A}{\varepsilon^2 \omega} \nabla_x \Sigma_\varepsilon \right), \ \mathbf{V}_\varepsilon \cdot \mathbf{n}|_{\partial \Omega} = 0$$

$$Z_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + \frac{A}{\omega} \varrho_\varepsilon \left(\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right) + \frac{A}{\varepsilon \omega} \Sigma_\varepsilon, \ \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon$$

$$< \Sigma_\varepsilon ; \varphi > = < \sigma_\varepsilon ; I[\varphi] >$$

$$I[\varphi](t,x) = \int_0^t \varphi(z,x) \ \mathrm{d}z \text{ for any } \varphi \in L^1(0,T; C(\overline{\Omega}))$$