On certain geometrical aspects of surfaces associated with $\mathbb{C}P^{N-1}$ sigma models

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Abstract

In this paper, the Weierstrass technique for harmonic maps $S^2 \to \mathbb{C}P^{N-1}$ is employed in order to obtain surfaces immersed in multidimensional Euclidean spaces. It is proved that if the $\mathbb{C}P^{N-1}$ model equations are defined on the sphere S^2 and the associated action functional of this model is finite, then a specific holomorphic function (corresponding to a component of the energy-momentum tensor of a $\mathbb{C}P^{N-1}$ sigma model) vanishes. In particular it is shown that for any holomorphic or antiholomorphic solutions of this model, the Weierstrass formula for immersion X of a surface lies in the $su(N)$ algebra and can be expressed in terms of an orthogonal projector of rank $(N-1)$. The implementation of this method is presented for two-dimensional conformally parametrized surfaces immersed in the $su(3)$ algebra. The usefulness of the proposed approach is illustrated with examples, including the dilation-invariant meron-type solutions and the Veronese solutions for the $\mathbb{C}P^2$ model. Depending on the location of the critical points (zeros and poles) of the first fundamental form associated with the meron solution, it is shown that the associated surfaces are semi-infinite cylinders. It is also demonstrated that surfaces related to holomorphic and mixed Veronese solutions are immersed in $\mathbb{R}^8$ and $\mathbb{R}^3$, respectively.

Key words: Sigma models, Weierstrass formula for immersion, surfaces immersed in low-dimensional $su(N)$ algebras

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1 Introduction

The expression describing surfaces with zero mean curvature (i.e. minimal surfaces) which are immersed in three-dimensional Euclidean space was first formulated by A. Enneper [1] and K. Weierstrass [2], one and a half centuries ago. Since then this idea has been thoroughly generalized and developed (e.g. [3, 4, 5, 6]). The subject was implemented by several authors (e.g. [7, 8, 9]) who produced several variants of the Weierstrass representation. For a comprehensive review of this topic see e.g. [10, 11, 12, 13, 14, 15] and references therein.

More recently, this subject was substantially elaborated by B. Konopelchenko and I. Taimanov [16], who first established the Weierstrass formulae for any generic surface immersed in $\mathbb{R}^3$. These formulae have been used extensively to study the global properties of surfaces in $\mathbb{R}^3$, as well as their integrable deformations [17]. By simple analogy with surfaces in the $\mathbb{R}^3$ case an extension of the Weierstrass procedure to multi-dimensional Euclidean and Riemannian spaces was proposed by B. Konopelchenko and G. Landolfi [18]. Their approach was successful for certain classes of conformally parametrized surfaces immersed in these spaces. However, this procedure has some limitation due to the assumption of a specific form of the Weierstrass system of $2N$ complex-valued functions which satisfy Dirac-type equations.

It was only in the past few years that the approach to the same problem was reformulated by exploiting the connection between generalized Weierstrass representations and the $\mathbb{C}P^1$ sigma models, first established in $\mathbb{R}^3$ [19]. This idea allows one to generalize this connection for the $\mathbb{C}P^{N-1}$ case and derive in the adjoint $SU(N)$ representation the corresponding moving frame of conformally parametrized surfaces in $\mathbb{R}^{N^2-1}$ space. This modified Weierstrass representation [20, 21, 22, 23] has proven to be more general than the one proposed in [24] and to generate more diverse classes of surfaces (e.g. the Veronese surfaces). This algebraic description of surfaces on Lie groups and homogeneous spaces allows us to calculate some new expressions in closed form which determine the fundamental characteristics of these surfaces. For this purpose, using the Cartan's language of moving frames we derive the structural equations for immersion (e.g. the fundamental forms, the Gaussian curvature and the mean curvature vector) for the $\mathbb{C}P^{N-1}$ model. The $\mathbb{C}P^{N-1}$ models have found many applications in physics, to such areas as two-dimensional gravity [25], string theory [26], quantum field theory [27], statistical physics [28] and fluid mechanics [29].

This paper is a follow-up of the results obtained in [30] and is concerned with smooth, orientable two-dimensional surfaces immersed in multi-dimensional Euclidean spaces. The crux of the matter is that the equations determining the formula for immersion are formulated directly in terms of matrices which take their values in the Lie algebra $su(N)$. The main advantage of this procedure is that, using an orthogonal projector satisfying the Euler-Lagrange equations of the given sigma model, it leads to simpler formulae and allows us to write the explicit form of some expressions which previously were too involved to be presented.

The objective of this paper is to study certain geometrical aspects of surfaces associated with the $\mathbb{C}P^{N-1}$ sigma models. In particular, we discuss in detail the necessary conditions for the existence of the radius vectors of surfaces associated with the $\mathbb{C}P^{N-1}$ sigma model which consist of an orthogonal projector of rank

$N-1$. Furthermore, it is demonstrated that a parametrized surface, related to a Veronese mixed solution (i.e. an extension of the holomorphic case) is immersed in three-dimensional Euclidean space. Finally, we construct a dilation-invariant solution of the $\mathbb{C}P^2$ model and determine its geometric characteristics.

The plan of this paper is as follows. Section 2 contains a brief account of basic definitions and properties concerning the $\mathbb{C}P^{N-1}$ models and fixes the notation. We give a geometric formulation for the generalized Weierstrass formula for immersion of a surface $\mathcal{F}$ in $\mathbb{R}^{N^2-1}$. Next, we show that if the $\mathbb{C}P^{N-1}$ model is defined on the sphere S^2 and the corresponding action functional of this model is finite, then a specific holomorphic function (corresponding to a component of the energy-momentum tensor of the $\mathbb{C}P^{N-1}$ model) vanishes. In Section 3 we investigate in great detail the Veronese surfaces related to the $\mathbb{C}P^2$ model and construct their geometric characteristics. We show that the holomorphic and mixed solutions are associated with surfaces immersed in $\mathbb{R}^8$ and $\mathbb{R}^3$, respectively. In Section 4, we present examples of applications of our approach to the case of the dilation-invariant solutions of meron type. We perform the analysis using the quadratic differentials and calculate their geometric implications. Section 5 contains final remarks concerning the projector formalism, identifies some open questions on the subject and proposes some possible future developments.

2 Harmonic maps from S^2 to $\mathbb{C}P^{N-1}$ and the Weierstrass representation

This paper is devoted to the exploration of relations between the $\mathbb{C}P^{N-1}$ sigma models and the generalized Weierstrass formula for the immersion of two-dimensional surfaces in multi-dimensional Euclidean spaces. To this end we briefly review some basic notions and properties of the $\mathbb{C}P^{N-1}$ sigma models. For further details on this subject we refer the reader to e.g. [10, 11, 12, 13, 31, 32] and references therein.

In studying the $\mathbb{C}P^{N-1}$ models one is interested in maps of the form $[z] : \Omega \to \mathbb{C}P^{N-1}$ (where Ω is an open, connected subset of a complex plane $\mathbb{C}$) which are stationary points of the action functional [31]

$$S = \frac{1}{4}\int_\Omega (D_\mu z)^\dagger (D_\mu z) d\xi d\bar{\xi}\,, \qquad z^\dagger \cdot z = 1\,,$$
$$\mathbb{C} \ni \xi = \xi^1 + i\xi^2 \to z = (z_0, z_1, \ldots, z_{N-1}) \in \mathbb{C}^N, \tag{1}$$

and thus are determined as solutions of the corresponding Euler-Lagrange equations. Here, D_μ $(\mu = 1, 2)$ denote covariant derivatives acting on $z : \Omega \to \mathbb{C}^N$, defined by

$$D_\mu z = \partial_\mu z - (z^\dagger \cdot \partial_\mu z) z \ \in T_z S^{2N-1}\,, \qquad \partial_\mu = \partial_{\xi^\mu}\,, \tag{2}$$

where ξ and $\bar{\xi}$ are local coordinates in Ω and the symbol $\dagger$ denotes Hermitian conjugation. The covariant derivatives D_μ are orthogonal to the inhomogeneous coordinates z, since $z^\dagger D_\mu z = 0$ holds. They can be expressed in terms of a composite gauge field

$$A_\mu = z^\dagger \partial_\mu z\,, \qquad A_\mu^\dagger = -A_\mu\,. \tag{3}$$

Here, A_μ is a pure imaginary function of ξ^1 and ξ^2. The action functional (1) is invariant under global $U(N)$ transformations and also under the following local $U(1)$ gauge transformation

$$z \to z' = ze^{i\phi}\,, \tag{4}$$

where ϕ is a real-valued function. Note that the covariant derivatives $D_\mu z$ transform under the gauge transformation

$$D_\mu z \to D_\mu z' = (D_\mu z)e^{i\phi}\,, \tag{5}$$

so that the dependence on the phase ϕ drops out of the action functional (1) and so the model is really based on $\mathbb{C}P^{N-1}$. In the homogeneous coordinates

$$z = f(f^\dagger \cdot f)^{-\frac{1}{2}} \tag{6}$$

the equations of motion can be written in the form of a conservation law

$$\partial K - \bar{\partial} K^\dagger = 0\,, \qquad -i\partial K \in su(N)\,, \tag{7}$$

where K and $K^\dagger$ are $N \times N$ matrices of the form

$$\begin{aligned} K &= \frac{1}{f^\dagger \cdot f}\left(\bar{\partial} f \otimes f^\dagger - f \otimes \bar{\partial} f^\dagger\right) + \frac{f \otimes f^\dagger}{(f^\dagger \cdot f)^2}\left(\bar{\partial} f^\dagger \cdot f - f^\dagger \cdot \bar{\partial} f\right), \\ K^\dagger &= \frac{1}{f^\dagger \cdot f}\left(f \otimes \partial f^\dagger - \partial f \otimes f^\dagger\right) + \frac{f \otimes f^\dagger}{(f^\dagger \cdot f)^2}\left(\partial f^\dagger \cdot f - f^\dagger \cdot \partial f\right). \end{aligned} \tag{8}$$

The symbols ∂ and $\bar{\partial}$ denote the standard derivatives with respect to ξ and $\bar{\xi}$ respectively, i.e.

$$\partial = \frac{1}{2}\left(\partial_{\xi^1} - i\partial_{\xi^2}\right)\,, \qquad \bar{\partial} = \frac{1}{2}\left(\partial_{\xi^1} + i\partial_{\xi^2}\right)\,. \tag{9}$$

We shall sometimes use the index notation ∂_μ, $(\mu = 1, 2)$ for the derivatives ∂_{ξ^1}, ∂_{ξ^2}.

Since the action (1) is invariant under a global $U(N)$ transformation, without loss of generality we can set one of the components of the vector field f equal to 1. Thus, in terms of these variables $f = (1, \bar{w}_1, \ldots, \bar{w}_N)^T$ the equations of motion for the $\mathbb{C}P^{N-1}$ sigma model take the following form

$$\begin{aligned} &\partial\bar{\partial} w_i - \frac{2\bar{w}_i}{A_{N-1}}\partial w_i \bar{\partial} w_i - \frac{1}{A_{N-1}}\sum_{j\neq i}^{N-1} \bar{w}_j(\partial w_i \bar{\partial} w_j + \bar{\partial} w_i \partial w_j) = 0\,, \\ &\partial\bar{\partial} \bar{w}_i - \frac{2w_i}{A_{N-1}}\partial \bar{w}_i \bar{\partial} \bar{w}_i - \frac{1}{A_{N-1}}\sum_{j\neq i}^{N-1} w_j(\partial \bar{w}_i \bar{\partial} \bar{w}_j + \bar{\partial} \bar{w}_i \partial \bar{w}_j) = 0\,, \end{aligned} \tag{10}$$

where $i = 1, 2, ..., N-1$ and $A_{N-1} = 1 + \sum_i^{N-1} w_i \bar{w}_i$. In what follows we refer to (10) as the equations of the $\mathbb{C}P^{N-1}$ sigma model.

It is instructive to express the Euler-Lagrange equations using the $N \times N$ orthogonal projector P of rank $(N-1)$ defined on the orthogonal complement to the complex line in $\mathbb{C}^N$,

$$P = I_N - \frac{f \otimes f^\dagger}{f^\dagger \cdot f}\,, \qquad P^\dagger = P\,, \qquad P^2 = P\,, \tag{11}$$

where I_N is the $N \times N$ identity matrix. Hence, the Euler-Lagrange equation (7) takes the simpler form

$$\partial[\bar{\partial}P, P] + \bar{\partial}[\partial P, P] = 0 \,. \tag{12}$$

Note that the complex-valued functions

$$J = \frac{1}{f^\dagger \cdot f}\partial f^\dagger P \partial f \,, \qquad \bar{J} = \frac{1}{f^\dagger \cdot f}\bar{\partial} f^\dagger P \bar{\partial} f \,, \tag{13}$$

satisfy

$$\bar{\partial} J = 0 \,, \qquad \partial \bar{J} = 0 \,, \tag{14}$$

whenever f is a solution of the equations of motion (7). The quantities J and $\bar{J}$ are invariant under the global $U(N)$ transformation, i.e., $f \to af$, $a \in U(N)$. From the physical point of view, $J = (\bar{D}z)^\dagger \cdot Dz$ is related to the energy-momentum tensor [31].

Since we have expressed the Euler-Lagrange equation (12) as a conservation law, we are able to formulate the Weierstrass formula for the immersion of two-dimensional surfaces in multi-dimensional Euclidean space. Based on Poincaré's lemma, there exists a closed matrix-valued 1-form,

$$dX = i(-[\partial P, P]d\xi + [\bar{\partial}P, P]d\bar{\xi}) \,. \tag{15}$$

From the closure of the 1-form dX (i.e. $d(dX) = 0$) it follows that the integral

$$X(\xi, \bar{\xi}) = i\int_\gamma (-[\partial P, P]d\xi + [\bar{\partial}P, P]d\bar{\xi}) \,, \tag{16}$$

depends only on the end points of the curve γ (i.e. it is locally independent of the trajectory in $\mathbb{C}$). Note that (12) is invariant under the conformal transformation (i.e. the change of independent variables $\xi \to \alpha(\xi)$ and $\bar{\xi} \to \bar{\alpha}(\bar{\xi})$). Such a transformation establishes a reparametrization of the surface $\mathcal{F}$ written in terms of an integral of a 1-form (16) which remains the same geometrical object.

For the analytical description of a two-dimensional surface $\mathcal{F}$ it is convenient to use the Lie algebra isomorphism and identify the (N^2-1)-dimensional Euclidean space with the $su(N)$ algebra

$$\mathbb{R}^{N^2-1} \simeq su(N) \,. \tag{17}$$

For uniformity we use the scalar product on $su(N)$ in the form

$$< A, B >= -\frac{1}{2}\mathrm{tr}(AB) \,, \qquad A, B \in su(N) \,, \tag{18}$$

rather than the Killing form of $su(N)$ given by the formula

$$\mathcal{B}(A, B) = 2\,N\mathrm{tr}(AB) \,, \tag{19}$$

which is negative definite [33]. Consequently the first fundamental form I is given by [30]

$$I = -Jd\xi^2 + \frac{2}{f^\dagger \cdot f}\bar{\partial} f^\dagger P \partial f d\xi d\bar{\xi} - \bar{J}d\bar{\xi}^2 \,, \tag{20}$$

and the integral representation (16) defines a mapping

$$X : \Omega \ni (\xi, \bar{\xi}) \to X(\xi, \bar{\xi}) \in su(N)\,. \tag{21}$$

We treat each element of the real-valued $su(N)$ matrix function X as coordinates of a two-dimensional surface $\mathcal{F}$ immersed in $\mathbb{R}^{N^2-1}$. This map X is called the generalized Weierstrass formula for immersion. The projector P is invariant under the transformation $P \to UPU^\dagger$, where $U \in U(N)$ and thus the geometry of the surface $\mathcal{F}$ associated with a solution of (12) admits the symmetry equivalence class of solutions of (12). In this setting, our generalization lies in the realization that most of the properties of the associated surfaces with the $\mathbb{C}P^{N-1}$ sigma models can be described using an orthogonal projector. The complex tangent vectors of this immersion are

$$\partial X = iK^\dagger\,, \qquad \bar{\partial} X = iK, \tag{22}$$

where we use (8)

$$K = [\bar{\partial} P, P]\,, \qquad K^\dagger = -[\partial P, P]\,. \tag{23}$$

From the conservation law (7), it is convenient to decompose the matrix K as follows

$$K = M + L\,, \tag{24}$$

where

$$M = (I_N - P)\bar{\partial} P\,, \qquad L = -\bar{\partial} P (I_N - P)\,. \tag{25}$$

It was shown in [24] that the matrices M and L satisfy the same conservation law (7) as the matrix K

$$\partial M = \bar{\partial} M^\dagger\,, \qquad \partial L = \bar{\partial} L^\dagger\,, \tag{26}$$

and the matrices M and L differ by a total divergence

$$M = L + \bar{\partial} P\,. \tag{27}$$

It follows from (26) that ∂M and ∂L are Hermitian matrices, i.e. $-i\partial M$, $-i\partial L$ $\in su(N)$.

Let us now discuss the existence of certain classes of surfaces in the $su(N)$ algebra when the $\mathbb{C}P^{N-1}$ model is defined on the sphere S^2 and its corresponding action functional is finite. We show that in this case the considered surfaces are conformally parametrized and the first fundamental form (20) becomes

$$I = \frac{2}{f^\dagger \cdot f} \bar{\partial} f^\dagger P \partial f d\xi d\bar{\xi}\,. \tag{28}$$

Proposition. *If the complex-valued vector function* $\mathbb{C} \ni \xi \to f(\xi) \in \mathbb{C}^N \backslash \{0\}$ *satisfies the* $\mathbb{C}P^{N-1}$ *sigma model equations*

$$\left(I_N - \frac{f \otimes f^\dagger}{f^\dagger \cdot f} \right) \left[\partial \bar{\partial} f - \frac{1}{f^\dagger \cdot f} \big((f^\dagger \cdot \bar{\partial} f) \partial f + (f^\dagger \cdot \partial f) \bar{\partial} f \big) \right] = 0\,, \tag{29}$$

defined over the whole Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ *and the associated action functional*

$$S = \frac{1}{4} \int_{S^2} \frac{1}{f^\dagger \cdot f} \left(\partial f^\dagger \left(I_N - \frac{f \otimes f^\dagger}{f^\dagger \cdot f} \right) \bar{\partial} f + \bar{\partial} f^\dagger \left(I_N - \frac{f \otimes f^\dagger}{f^\dagger \cdot f} \right) \partial f \right) d\xi d\bar{\xi}\,, \tag{30}$$

is finite, then the complex-valued functions

$$J = \frac{1}{f^\dagger \cdot f}\partial f^\dagger \left(I_N - \frac{f \otimes f^\dagger}{f^\dagger \cdot f} \right) \partial f\,, \qquad \bar{J} = \frac{1}{f^\dagger \cdot f}\bar{\partial} f^\dagger \left(I_N - \frac{f \otimes f^\dagger}{f^\dagger \cdot f} \right) \bar{\partial} f\,, \tag{31}$$

are equal to zero.

Proof The procedure for constructing the general class of solutions admitting finite action (30) of the Euclidean two-dimensional $\mathbb{C}P^{N-1}$ model (29) was derived by A. Din and W. Zakrzewski [34] and R. Sasaki [35]. As a result, one gets three classes of solutions, namely (i) holomorphic (i.e. $\bar{\partial} f = 0$), (ii) antiholomorphic (i.e. $\partial f = 0$) and (iii) mixed. The mixed solutions can be determined from either the holomorphic or the antiholomorphic nonconstant functions by the following procedure. The successive application, say k times with $k \leq N-1$, of the operator P_+ defined by its action on vector-valued functions on $\mathbb{C}^N$ [31]

$$P_+ : f \in \mathbb{C}^N \to P_+ f = \partial f - f\frac{f^\dagger \partial f}{f^\dagger f}\,, \quad \bar{\partial} f = 0\,, \tag{32}$$

starting from any nonconstant holomorphic function $f \in \mathbb{C}^N$, allows one to find mixed solutions

$$f^k = P_+^k f\,, \qquad k = 0, 1, \ldots, N-1\,, \tag{33}$$

which represent harmonic maps from S^2 to the $\mathbb{C}P^{N-1}$ sigma model. Here, $P_+^0 = id$.

In order to demonstrate that the complex-valued functions J and $\bar{J}$ vanish it is sufficient to consider the orthogonality relation

$$(P_+^i f)^\dagger \cdot P_+^j f = 0\,, \qquad i \neq j\,. \tag{34}$$

for $i = k$ and $j = k+2$ with arbitrary $k = 0, 1, \ldots, N-1$. Denoting $\widetilde{f} = P_+^k f$, we get

$$\begin{aligned} 0 = \widetilde{f}^\dagger \cdot (P_+^2 \widetilde{f}) &= \widetilde{f}^\dagger \cdot \left(\partial(P_+\widetilde{f}) - (P_+\widetilde{f})\frac{(P_+\widetilde{f})^\dagger \partial(P_+\widetilde{f})}{(P_+\widetilde{f})^\dagger (P_+\widetilde{f})} \right) \\ &= \widetilde{f}^\dagger \cdot \partial(P_+\widetilde{f}) \\ &= -\partial \widetilde{f}^\dagger \cdot (P_+\widetilde{f})\,, \end{aligned} \tag{35}$$

where for the last two equalities we used the orthogonality condition $\widetilde{f}^\dagger \cdot P_+ \widetilde{f} = 0$. The right hand side of the last equality in (35) can also be written in terms of the complex-valued functions J and $\bar{J}$ given in (31)

$$0 = -\partial \widetilde{f}^\dagger \cdot \left(\partial \widetilde{f} - \frac{\widetilde{f} \otimes \widetilde{f}^\dagger}{\widetilde{f}^\dagger \cdot \widetilde{f}} \partial \widetilde{f} \right) = -(\widetilde{f}^\dagger \cdot \widetilde{f})\widetilde{J}\,. \tag{36}$$

Since $\widetilde{f}^\dagger \cdot \widetilde{f} \neq 0$, we have $\widetilde{J} = 0$.

Note that in the case of the holomorphic and antiholomorphic solutions f of the $\mathbb{C}P^{N-1}$ model equations (29) the corresponding complex-valued functions J and $\bar{J}$, given in (31), vanish identically. This completes the proof. **Q.E.D.**

Remark 1: The holomorphic function $f \in \mathbb{C}^N$, used in the proof of the Proposition, could be replaced by any nonconstant antiholomorphic function. The mixed solutions f^k are constructed in the same way, except that the derivative ∂ is replaced by $\bar{\partial}$ in the definition of the operator P_-. Thus, we have

$$P_- f = \bar{\partial} f - f \frac{f^\dagger \bar{\partial} f}{f^\dagger f} \,, \quad \partial f = 0 \,. \tag{37}$$

This yields results which are complementary to those obtained in the Proposition by the first approach.

Remark 2: In particular one can present an analogue of the Bonnet theorem. Under the hypotheses of the Proposition and in the case of holomorphic or antiholomorphic solutions f of the $\mathbb{C}P^{N-1}$ model, the Weierstrass formula for immersion X of a surface $\mathcal{F}$ lies in the $su(N)$ algebra. The position matrix X is expressed in terms of the orthogonal projector of rank $(N-1)$ by the following formula

$$X(\xi, \bar{\xi}) = \epsilon i \left(\frac{1-N}{N} I_N + P \right) , \qquad \epsilon = \pm 1 \,. \tag{38}$$

The surface $\mathcal{F}$ is determined uniquely up to Euclidean motions by its first and second fundamental forms

$$I = \mathrm{tr}(\partial P \bar{\partial} P) d\xi d\bar{\xi} \,, \tag{39}$$

and

$$II = \epsilon i \Big\{ (\partial^2 P - \Gamma^1_{11} \partial P - \Gamma^2_{11} \bar{\partial} P) d\xi^2 + 2 \partial \bar{\partial} P d\xi d\bar{\xi} + (\bar{\partial}^2 P - \Gamma^1_{22} \partial P - \Gamma^2_{22} \bar{\partial} P) d\bar{\xi}^2 \Big\} \,, \tag{40}$$

respectively, where the Christoffel symbols of the second kind are given by

$$\begin{aligned}
\Gamma^1_{11} &= \frac{\mathrm{tr}(\partial^2 P \bar{\partial} P)}{\mathrm{tr}(\partial P \bar{\partial} P)} \,, \quad \Gamma^2_{11} = \frac{\mathrm{tr}(\partial^2 P \partial P)}{\mathrm{tr}(\partial P \bar{\partial} P)} \,, \quad \Gamma^1_{12} = \Gamma^1_{21} = 0 \,, \\
\Gamma^1_{22} &= \frac{\mathrm{tr}(\bar{\partial}^2 P \bar{\partial} P)}{\mathrm{tr}(\partial P \bar{\partial} P)} \,, \quad \Gamma^2_{22} = \frac{\mathrm{tr}(\bar{\partial}^2 P \partial P)}{\mathrm{tr}(\partial P \bar{\partial} P)} \,, \quad \Gamma^2_{12} = \Gamma^2_{21} = 0 \,.
\end{aligned} \tag{41}$$

In the case of holomorphic or antiholomorphic solutions f of the $\mathbb{C}P^{N-1}$ model, according to [30], the matrix K can be expressed as

$$K = \epsilon \bar{\partial} P \,, \qquad K^\dagger = \epsilon \partial P \,, \qquad \epsilon = \pm 1 \,. \tag{42}$$

Consequently, the Weierstrass formula for immersion (16) of a surface is represented in terms of the projector P up to an overall integration constant. The real-valued $su(N)$ function X is traceless but the trace of the orthogonal projector of rank $(N-1)$ is equal to $(N-1)$. Using the freedom of the initial condition and the additivity of the trace we can choose the integration constant to be equal to $\epsilon i \frac{1-N}{N} I_N$. Hence, X lies in the $su(N)$ algebra and takes the form (38).
The corresponding moving frame on a surface $\mathcal{F}$, satisfies the Gauss-Weingarten equations

$$\begin{aligned}
\partial^2 X &= \Gamma^1_{11} \partial X + \Gamma^2_{11} \bar{\partial} X + Q_j \eta_j \,, \\
\partial \bar{\partial} X &= H_j \eta_j \,, \\
\partial \eta_j &= \frac{H_j}{\mathrm{tr}(\partial P \bar{\partial} P)} \partial X + \frac{Q_j}{\mathrm{tr}(\partial P \bar{\partial} P)} \bar{\partial} X + s_{jk} \eta_k \,,
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
\bar{\partial}^2 X &= \Gamma^1_{22}\partial X + \Gamma^2_{22}\bar{\partial} X + \bar{Q}_j \eta_j \,, \\
\partial\bar{\partial} X &= H_j \eta_j \,, \\
\bar{\partial}\eta_j &= \frac{H_j}{\mathrm{tr}(\partial P \bar{\partial} P)}\partial X + \frac{\bar{Q}_j}{\mathrm{tr}(\partial P \bar{\partial} P)}\bar{\partial} X + \bar{s}_{jk}\eta_k \,,
\end{aligned} \tag{44}$$

where

$$\begin{gathered}
Q_j = (\partial^2 X, \eta_j) = -\frac{1}{2}\mathrm{tr}(\partial^2 X \eta_j)\,, \qquad Q_j H_j = 0\,, \\
H_j = (\partial\bar{\partial} X, \eta_j) = -\frac{1}{2}\mathrm{tr}(\partial\bar{\partial} X \eta_j)\,, \qquad \bar{Q}_j H_j = 0\,, \\
s_{jk} + s_{kj} = 0\,, \qquad \bar{s}_{jk} + \bar{s}_{kj} = 0\,, \qquad j \neq k = 3, \ldots, N^2 - 1\,.
\end{gathered} \tag{45}$$

and the Christoffel symbols of the second kind are given by (41). The Gaussian curvature and the mean curvature vector take the simple form

$$\mathcal{K} = -2\frac{\partial\bar{\partial}\ln(\mathrm{tr}(\partial P \bar{\partial} P))}{\mathrm{tr}(\partial P \bar{\partial} P)}\,, \qquad \mathcal{H} = -8i\frac{[\partial P, \bar{\partial} P]}{\mathrm{tr}(\partial P \bar{\partial} P)}\,. \tag{46}$$

Consequently, the Willmore functional of a surface has the form

$$W = 32i\epsilon \int_{S^2} \frac{[\partial P, \bar{\partial} P]^2}{\mathrm{tr}(\partial P \bar{\partial} P)} d\xi d\bar{\xi} \tag{47}$$

and the topological charge of the $\mathbb{C}P^{N-1}$ model is

$$Q = -\frac{i\epsilon}{\pi}\int_{S^2} \mathrm{tr}(\partial\bar{\partial} P P) d\xi d\bar{\xi}\,. \tag{48}$$

If the integral (48) exists then it is an integer which characterizes globally the surface under consideration.

3 Veronese surfaces for the $\mathbb{C}P^2$ model

One of the simplest applications of a result concerning solutions of the $\mathbb{C}P^{N-1}$ sigma model (10) is the Veronese sequence [36]

$$f = \left(1, \sqrt{\begin{pmatrix} N-1 \\ 1 \end{pmatrix}}\,\xi, \ldots, \sqrt{\begin{pmatrix} N-1 \\ r \end{pmatrix}}\,\xi^r, \ldots, \xi^{N-1}\right)\,. \tag{49}$$

For all of the above Veronese solutions the first fundamental form is conformal and given by

$$I = (N-1)(1+|\xi|^2)^{-2} d\xi d\bar{\xi}\,. \tag{50}$$

Thus, the Gaussian curvature [30] (since $g_{11} = g_{22} = 0$)

$$\mathcal{K} = -(g_{12})^{-1}\bar{\partial}\partial \ln g_{12}\,, \tag{51}$$

for all of those solutions is found to be

$$\mathcal{K} = \frac{4}{N-1}\,. \tag{52}$$

From now on we will only be concerned with the $\mathbb{C}P^2$ model ($N=3$). The Veronese vector f for this model is given by

$$f = (1, \sqrt{2}\,\xi, \xi^2)\,. \tag{53}$$

The method used to find the radius vector $\vec{X}$ by using the generalized Weierstrass formula for immersion of $2D$ surfaces in $\mathbb{R}^8$ was proposed in [20, 30]. There, the real components of the corresponding 1-forms for any solution of the $\mathbb{C}P^2$ model are given as

$$\begin{aligned}
dX_1 &= \frac{1}{2A_2^2}\Big(\big[(w_2^2-w_1^2)(\bar{w}_1\partial\bar{w}_2-\bar{w}_2\partial\bar{w}_1)-(\bar{w}_2^2-\bar{w}_1^2)(w_1\partial w_2-w_2\partial w_1)\\
&\quad -w_2\partial\bar{w}_1+\bar{w}_2\partial w_1-w_1\partial\bar{w}_2+\bar{w}_1\partial w_2\big]d\xi+\text{c.c.}\Big),\\
dX_2 &= \frac{i}{2A_2^2}\Big(\big[(w_1^2+w_2^2)(\bar{w}_2\partial\bar{w}_1-\bar{w}_1\partial\bar{w}_2)+(\bar{w}_1^2+\bar{w}_2^2)(w_2\partial w_1-w_1\partial w_2)\\
&\quad +w_2\partial\bar{w}_1+\bar{w}_2\partial w_1-w_1\partial\bar{w}_2-\bar{w}_1\partial w_2\big]d\xi-\text{c.c.}\Big),\\
dX_3 &= \frac{1}{2A_2^2}\Big(\big[w_2\partial\bar{w}_2-w_1\partial\bar{w}_1-\bar{w}_2\partial w_2+\bar{w}_1\partial w_1\\
&\quad +2|w_1|^2(w_2\partial\bar{w}_2-\bar{w}_2\partial w_2)-2|w_2|^2(w_1\partial\bar{w}_1-\bar{w}_1\partial w_1)\big]d\xi+\text{c.c.}\Big),\\
dX_4 &= \frac{\sqrt{3}}{2A_2^2}\Big(\big[w_1\partial\bar{w}_1+w_2\partial\bar{w}_2-\bar{w}_1\partial w_1-\bar{w}_2\partial w_2\big]d\xi+\text{c.c.}\Big),\\
dX_5 &= -\frac{i}{2A_2^2}\Big(\big[(1+\bar{w}_1^2+|w_2|^2)\partial w_1+(1+w_1^2+|w_2|^2)\partial\bar{w}_1\\
&\quad +(w_2\partial\bar{w}_2-\bar{w}_2\partial w_2)(w_1-\bar{w}_1)\big]d\xi-\text{c.c.}\Big),\\
dX_6 &= -\frac{i}{2A_2^2}\Big(\big[(1+\bar{w}_2^2+|w_1|^2)\partial w_2+(1+w_2^2+|w_1|^2)\partial\bar{w}_2\\
&\quad +(w_1\partial\bar{w}_1-\bar{w}_1\partial w_1)(w_2-\bar{w}_2)\big]d\xi-\text{c.c.}\Big),\\
dX_7 &= \frac{1}{2A_2^2}\Big(\big[(1-w_1^2+|w_2|^2)\partial\bar{w}_1-(1-\bar{w}_1^2+|w_2|^2)\partial w_1\\
&\quad +(\bar{w}_2\partial w_2-w_2\partial\bar{w}_2)(w_1+\bar{w}_1)\big]d\xi+\text{c.c.}\Big),\\
dX_8 &= \frac{1}{2A_2^2}\Big(\big[(1-w_2^2+|w_1|^2)\partial\bar{w}_2-(1-\bar{w}_2^2+|w_1|^2)\partial w_2\\
&\quad +(\bar{w}_1\partial w_1-w_1\partial\bar{w}_1)(w_2+\bar{w}_2)\big]d\xi+\text{c.c.}\Big)\,.
\end{aligned} \tag{54}$$

For any holomorphic solution (w_1, w_2) of the $\mathbb{C}P^2$ model the above 8 real-valued 1-forms can easily be integrated to give the components of the radius vector

$$\vec{X}(\xi,\bar{\xi}) = \Big(X_1(\xi,\bar{\xi}),\ldots,X_8(\xi,\bar{\xi})\Big)\,, \tag{55}$$

of a two-dimensional surface in $\mathbb{R}^8$

$$X_1 = \frac{w_1\bar{w}_2+\bar{w}_1 w_2}{2\,A_2}\,,\qquad X_2 = i\frac{w_1\bar{w}_2-\bar{w}_1 w_2}{2\,A_2}\,,\qquad X_3 = \frac{|w_1|^2-|w_2|^2}{2\,A_2}\,,$$

$$X_4 = -\sqrt{3}\frac{|w_1|^2+|w_2|^2}{2\,A_2}\,, \qquad X_5 = -i\frac{w_1-\bar{w}_1}{2\,A_2}\,, \qquad X_6 = -i\frac{w_2-\bar{w}_2}{2\,A_2}\,,$$
$$X_7 = -\frac{w_1+\bar{w}_1}{2\,A_2}\,, \qquad X_8 = -\frac{w_2+\bar{w}_2}{2\,A_2}\,, \tag{56}$$

where we choose the integration constants to be zero.

Hence, using the Weierstrass formula for immersion (56) we obtain that the radius vector $\vec{X}$ of a two-dimensional parametrized surface (for the Veronese solution (53)) is immersed in $\mathbb{R}^8$. Its components are

$$\begin{aligned}
X_1 &= \frac{|\xi|^2(\xi+\bar{\xi})}{\sqrt{2}\,(1+|\xi|^2)^2} = \frac{\sqrt{2}\,x(x^2+y^2)}{(1+x^2+y^2)^2}\,,\\
X_2 &= -i\frac{|\xi|^2(\xi-\bar{\xi})}{\sqrt{2}\,(1+|\xi|^2)^2} = \frac{\sqrt{2}\,y(x^2+y^2)}{(1+x^2+y^2)^2}\,,\\
X_3 &= -\frac{|\xi|^2(|\xi|^2-2)}{2(1+|\xi|^2)^2} = -\frac{(x^2+y^2)(x^2+y^2-2)}{2(1+x^2+y^2)^2}\,,\\
X_4 &= -\sqrt{3}\frac{|\xi|^2(|\xi|^2+2)}{2(1+|\xi|^2)^2} = -\sqrt{3}\frac{(x^2+y^2)(x^2+y^2+2)}{2(1+x^2+y^2)^2}\,,\\
X_5 &= -i\frac{\xi-\bar{\xi}}{\sqrt{2}\,(1+|\xi|^2)^2} = \frac{\sqrt{2}\,y}{(1+x^2+y^2)^2}\,,\\
X_6 &= -i\frac{\xi^2-\bar{\xi}^2}{2\,(1+|\xi|^2)^2} = \frac{2\,xy}{(1+x^2+y^2)^2}\,,\\
X_7 &= -\frac{\xi+\bar{\xi}}{2\,(1+|\xi|^2)^2} = -\frac{x}{(1+x^2+y^2)^2}\,,\\
X_8 &= -\frac{\xi^2+\bar{\xi}^2}{2\,(1+|\xi|^2)^2} = \frac{-x^2+y^2}{(1+x^2+y^2)^2}\,,
\end{aligned} \tag{57}$$

where we used $\xi = x+iy$. The components X_i $(i=1,\dots,8)$ given in (57) satisfy the relation

$$4X_1^2+4X_2^2+4X_3^2+\frac{2}{\sqrt{3}}X_4+X_5^2+X_6^2+X_7^2+X_8^2 = 0\,. \tag{58}$$

We can now proceed to construct a mixed solution which, as is well-known [31], can be obtained directly from the holomorphic one. Applying the operator P_+, given by (32), to the vector field (53), we obtain the mixed solution in the form

$$P_+f = \frac{\sqrt{2}}{1+|\xi|^2}(-\sqrt{2}\,\bar{\xi}, 1-|\xi|^2, \sqrt{2}\,\xi)\,. \tag{59}$$

Let us note that for the $\mathbb{C}P^2$ model the repeated applications of the operator P_+ to a holomorphic solution f only lead to a mixed solution (59) and an antiholomorphic one $P_+^2 f$, since $P_+^3 f = 0$. Thus, the holomorphic and mixed solutions considered here indeed constitute a complete set of solutions for the $\mathbb{C}P^2$ model. Using $U(1)$ invariance of the $\mathbb{C}P^2$ model we can normalize (59) to the following

$$f_1 = (1, \widetilde{w_1}, \widetilde{w_2})\,, \tag{60}$$

where we denote

$$\widetilde{w_1} = \frac{|\xi|^2-1}{\sqrt{2}\,\bar{\xi}}\,, \qquad \widetilde{w_2} = -\frac{\xi}{\bar{\xi}}\,. \tag{61}$$

Then substituting (61) into (54) and integrating we obtain a two-dimensional parametrized surface immersed in $\mathbb{R}^3$

$$
\begin{aligned}
X_1 &= -X_7 = \frac{\xi+\bar{\xi}}{\sqrt{2}\,(1+|\xi|^2)} = \frac{\sqrt{2}\,x}{1+x^2+y^2}\,, \\
X_3 &= \frac{X_4}{\sqrt{3}} = \frac{1}{1+|\xi|^2} = \frac{1}{1+x^2+y^2}\,, \\
X_2 &= X_5 = -i\frac{\xi-\bar{\xi}}{\sqrt{2}\,(1+|\xi|^2)} = \frac{\sqrt{2}\,y}{1+x^2+y^2}\,, \\
X_6 &= X_8 = 0\,.
\end{aligned} \tag{62}
$$

Note that the components of the radius vector $\vec{X}$ in (62) satisfy the following relation

$$
X_1^2 + X_2^2 + (\sqrt{2}X_3 - \frac{1}{\sqrt{2}})^2 = \frac{1}{2}\,. \tag{63}
$$

Equation (63) represents an ellipsoid, centered at the point $(0,0,\frac{1}{2})$ in $\mathbb{R}^3$. So, this case corresponds to the immersion of the $\mathbb{C}P^2$ model into the $\mathbb{C}P^1$ model.

Let us now explore some geometrical characteristics of surfaces corresponding to two different solutions of the $\mathbb{C}P^2$ model. In the holomorphic case (53) the orthogonal projector has the following form

$$
P = \frac{1}{(1+|\xi|^2)^2}\begin{pmatrix} |\xi|^2(2+|\xi|^2) & -\sqrt{2}\,\xi & -\xi^2 \\ -\sqrt{2}\,\bar{\xi} & 1+|\xi|^4 & -\sqrt{2}\,|\xi|^2\xi \\ -\bar{\xi}^2 & -\sqrt{2}\,|\xi|^2\bar{\xi} & 1+2|\xi|^2 \end{pmatrix}, \tag{64}
$$

where $\mathrm{rank}P = 2$ and $\mathrm{tr}P = 2$. The surface is determined by (57) and its induced metric is conformal

$$
g_{11} = g_{22} = 0, \qquad g_{12} = \frac{1}{(1+|\xi|^2)^2}\,. \tag{65}
$$

The nonzero Christoffel symbols of the second kind are

$$
\Gamma^1_{11} = -\frac{2\bar{\xi}}{1+|\xi|^2}\,, \qquad \Gamma^2_{22} = -\frac{2\xi}{1+|\xi|^2}\,. \tag{66}
$$

The first fundamental form and the Gaussian curvature $\mathcal{K} = -(g_{12})^{-1}\bar{\partial}\partial \ln g_{12}$ (since g_{11} and g_{22} vanish) are given by

$$
I = \frac{2}{(1+|\xi|^2)^2}d\xi d\bar{\xi}\,, \qquad \mathcal{K} = 2, \tag{67}
$$

respectively. Making use of the expression (57) for the radius vector $\vec{X}$ we can explicitly write the second fundamental form II of the surface in the equivalent matrix form. The components of the matrix II are

$$
\begin{aligned}
II_{11} &= \frac{2i}{(1+|\xi|^2)^4}(\bar{\xi}^2 d\xi^2 + (4|\xi|^2-2)d\xi d\bar{\xi} + \xi^2 d\bar{\xi}^2)\,, \\
II_{12} &= \frac{2\sqrt{2}i}{(1+|\xi|^2)^4}(-\bar{\xi}d\xi^2 + 2\xi(|\xi|^2-2)d\xi d\bar{\xi} + \xi^3 d\bar{\xi}^2)\,,
\end{aligned}
$$

$$
\begin{aligned}
II_{13} &= \frac{2i}{(1+|\xi|^2)^4}(d\xi^2 - 6\xi^2 d\xi d\bar{\xi} + \xi^4 d\bar{\xi}^2)\,, \\
II_{21} &= \frac{2\sqrt{2}i}{(1+|\xi|^2)^4}(\bar{\xi}^3 d\xi^2 + 2\bar{\xi}(|\xi|^2-2)d\xi d\bar{\xi} - \xi d\bar{\xi}^2)\,, \\
II_{22} &= \frac{4i}{(1+|\xi|^2)^4}(-\bar{\xi}^2 d\xi^2 + (1+|\xi|^4-4|\xi|^2)d\xi d\bar{\xi} - \xi^2 d\bar{\xi}^2)\,, \\
II_{23} &= \frac{2\sqrt{2}i}{(1+|\xi|^2)^4}(\bar{\xi} d\xi^2 + 2\xi(1-2|\xi|^2)d\xi d\bar{\xi} - \xi^3 d\bar{\xi}^2)\,, \\
II_{31} &= \frac{2i}{(1+|\xi|^2)^4}(\bar{\xi}^4 d\xi^2 - 6\bar{\xi}^2 d\xi d\bar{\xi} + d\bar{\xi}^2)\,, \\
II_{32} &= \frac{2\sqrt{2}i}{(1+|\xi|^2)^4}(-\bar{\xi}^3 d\xi^2 + 2\bar{\xi}(1-2|\xi|^2)d\xi d\bar{\xi} + \xi d\bar{\xi}^2)\,, \\
II_{33} &= \frac{2i}{(1+|\xi|^2)^4}(\bar{\xi}^2 d\xi^2 + 2|\xi|^2(2-|\xi|^2)d\xi d\bar{\xi} + \xi^2 d\bar{\xi}^2)\,.
\end{aligned} \tag{68}
$$

The mean curvature $\mathcal{H} = \partial\bar{\partial}X/g_{12}$, written as a matrix, takes the form

$$
\mathcal{H} = \frac{4i}{(1+|\xi|^2)^2}\begin{pmatrix} 2|\xi|^2-1 & \sqrt{2}\,\xi(|\xi|^2-2) & -3\xi^2 \\ \sqrt{2}\,\bar{\xi}(|\xi|^2-2) & 1+|\xi|^2(|\xi|^2-4) & -\sqrt{2}\,\xi(2|\xi|^2-1) \\ -3\bar{\xi}^2 & -\sqrt{2}\,\bar{\xi}(2|\xi|^2-1) & -|\xi|^2(|\xi|^2-2) \end{pmatrix}, \tag{69}
$$

where rank $\mathcal{H} = 2$ and tr $\mathcal{H} = 0$. The total energy [31] for the holomorphic solution (53) is finite over all space

$$
u = \ln\left(\frac{|\partial w_1|^2 + |\partial w_2|^2 + |w_2\partial w_1 - w_1\partial w_2|^2}{A_2{}^2}\right) = \ln\left(\frac{2}{(1+|\xi|^2)^2}\right). \tag{70}
$$

A particular significant quantity for the solution (53) satisfying the $\mathbb{C}P^2$ model equations (76) is the topological charge

$$
Q = -\frac{1}{\pi}\int_{S^2} g_{12}\, d\xi d\bar{\xi}\,, \tag{71}
$$

defined on the whole Riemann unit sphere S^2. The integral (71) exists and is an invariant of the surface (57). It characterizes globally the surface and is an integer

$$
Q = 1\,. \tag{72}
$$

In the second case for mixed solutions (61) the corresponding orthogonal projector takes the form

$$
P_1 = \frac{1}{(1+|\xi|^2)^2}\begin{pmatrix} (1+|\xi|^4) & -\sqrt{2}\,\xi(|\xi|^2-1) & 2\xi^2 \\ -\sqrt{2}\,\bar{\xi}(|\xi|^2-1) & 4|\xi|^2 & \sqrt{2}\,\xi(|\xi|^2-1) \\ 2\bar{\xi}^2 & \sqrt{2}\,\bar{\xi}(|\xi|^2-1) & 1+|\xi|^4 \end{pmatrix}, \tag{73}
$$

where rank$P_1 = 2$ and tr$P_1 = 2$. The surface is determined by (62) and its induced metric associated with the projector (73) is also conformal

$$
g_{11} = g_{22} = 0, \qquad g_{12} = \frac{2}{(1+|\xi|^2)^2}\,. \tag{74}
$$

The first fundamental form and the Gaussian curvature are

$$I = \frac{4}{(1+|\xi|^2)^2} d\xi d\bar{\xi}\,, \qquad \mathcal{K} = 1, \tag{75}$$

respectively.

4 Examples of surfaces in the $su(3)$ algebra

The objective of this section is to construct dilation-invariant solutions of the $\mathbb{C}P^2$ model and next, using the Weierstrass formula for immersion in $\mathbb{R}^8$, to calculate some geometric properties of an associated surface to this model in a closed form.

4.1 Dilation-invariant solutions

From (10) we obtain the equations of the $\mathbb{C}P^2$ model ($N = 3$)

$$\begin{aligned}
&\partial\bar{\partial}w_1 - \frac{2\bar{w}_1}{A_2}\partial w_1\bar{\partial}w_1 - \frac{\bar{w}_2}{A_2}(\partial w_1\bar{\partial}w_2 + \bar{\partial}w_1\partial w_2) = 0\,,\\
&\partial\bar{\partial}w_2 - \frac{2\bar{w}_2}{A_2}\partial w_2\bar{\partial}w_2 - \frac{\bar{w}_1}{A_2}(\partial w_1\bar{\partial}w_2 + \bar{\partial}w_1\partial w_2) = 0\,,\\
&\partial\bar{\partial}\bar{w}_1 - \frac{2w_1}{A_2}\partial\bar{w}_1\bar{\partial}\bar{w}_1 - \frac{w_2}{A_2}(\bar{\partial}\bar{w}_1\partial\bar{w}_2 + \partial\bar{w}_1\bar{\partial}\bar{w}_2) = 0\,,\\
&\partial\bar{\partial}\bar{w}_2 - \frac{2w_2}{A_2}\partial\bar{w}_2\bar{\partial}\bar{w}_2 - \frac{w_1}{A_2}(\bar{\partial}\bar{w}_1\partial\bar{w}_2 + \partial\bar{w}_1\bar{\partial}\bar{w}_2) = 0\,,\\
&A_2 = 1 + w_1\bar{w}_1 + w_2\bar{w}_2.
\end{aligned} \tag{76}$$

Let us discuss the solutions of (76) which are invariant under the scaling symmetries

$$S = w_i\partial_{w_i} - \bar{w}_i\partial_{\bar{w}_i}\,, \qquad i = 1, 2\,. \tag{77}$$

For this purpose we determine the invariants of the vector fields (77), which imply the algebraic constraints

$$w_i\bar{w}_i = D_i \in \mathbb{R}\,, \qquad i = 1, 2\,. \tag{78}$$

Without loss of generality we may choose $D_i = 1$. Then a solution of (78) is given by

$$w_i = \frac{F_i(\xi)}{\bar{F}_i(\bar{\xi})}\,, \qquad i = 1, 2\,, \tag{79}$$

where F_i and $\bar{F}_i$ are arbitrary complex-valued functions of one complex variable ξ and $\bar{\xi}$, respectively. After substituting (79) into the $\mathbb{C}P^2$ model equations (76) it is immediately seen that the unknown functions F_i and $\bar{F}_i$ must satisfy the following differential relation

$$|F_2|^2|F_1'|^2 = |F_1|^2|F_2'|^2\,, \tag{80}$$

where prime means the differentiation with respect to the argument (*i.e.* with respect to either ξ or $\bar{\xi}$). Equation (80) implies

$$F_2'(\xi) = \frac{F_2(\xi)F_1'(\xi)}{F_1(\xi)}e^{i\psi}\,, \tag{81}$$

which has the following solution

$$F_2(\xi) = cF_1(\xi)^{e^{i\psi}}\,, \qquad c \in \mathbb{C}\,, \tag{82}$$

where ψ is an arbitrary constant. By substituting (79) and (81) into (76) it is seen that ψ must satisfy

$$\psi = \pm\frac{\pi}{3} + 2\pi m\,, \qquad m \in \mathbb{Z}\,. \tag{83}$$

Thus we obtain a class of scaling invariant solutions of the $\mathbb{C}P^2$ model equations (76) which depend on one arbitrary complex-valued function of one variable ξ and its conjugate

$$w_1 = \frac{F_1(\xi)}{\bar{F}_1(\bar{\xi})}\,, \qquad w_2 = \frac{c}{\bar{c}}\frac{F_1(\xi)^{e^{i\psi}}}{\bar{F}_1(\bar{\xi})^{e^{-i\psi}}}\,. \tag{84}$$

In the following subsection a detailed investigation of the geometric implications of the induced metric associated with a quadratic differential will be performed.

4.2 A geometric characterization

In [30] it was shown that the induced metric for the $\mathbb{C}P^2$ model equations subjected to the DCs given in (78) is conformal and the Gaussian curvature for the associated surfaces vanishes i.e. $\mathcal{K} = 0$. It was also shown that the coordinates of the radius vector $\vec{X}$ for the nonsplitting solutions of the $\mathbb{C}P^2$ model equations are given by

$$\begin{aligned}
X_1 &= \frac{i}{6\sqrt{3}\,|c|^2}|F|^{-2e^{i\psi}}(\bar{c}^2F - c^2\bar{F}|F|^{2i\sqrt{3}})\,,\\
X_2 &= -\frac{1}{6\sqrt{3}\,|c|^2}|F|^{-2e^{i\psi}}(\bar{c}^2F + c^2\bar{F}|F|^{2i\sqrt{3}})\,,\\
X_3 &= \frac{1}{6}\big((1-i\sqrt{3})\mathrm{lnF} + (1+\mathrm{i}\sqrt{3})\mathrm{ln\bar{F}}\big)\,,\\
X_4 &= -\frac{1}{6}\big((i+\sqrt{3})\mathrm{lnF} + (-\mathrm{i}+\sqrt{3})\mathrm{ln\bar{F}}\big)\,,\\
X_5 &= -\frac{F^2+\bar{F}^2}{6\sqrt{3}\,|F|^2}\,,\\
X_6 &= \frac{1}{6\sqrt{3}\,|c|^2}|F|^{-2e^{i\psi}}(\bar{c}^2\bar{F} + c^2F|F|^{2i\sqrt{3}})\,,\\
X_7 &= \frac{i(F^2-\bar{F}^2)}{6\sqrt{3}\,|F|^2}\,,\\
X_8 &= \frac{i}{6\sqrt{3}\,|c|^2}|F|^{-2e^{i\psi}}(\bar{c}^2\bar{F} - c^2F|F|^{2i\sqrt{3}})\,.
\end{aligned} \tag{85}$$

The corresponding first fundamental form is immediately given as

$$I = \frac{2}{3}\frac{|F'|^2}{|F|^2}\,d\xi\,d\bar{\xi}\,. \tag{86}$$

Note that the components of the radius vector $\vec{X}$ in (85) satisfy the following relations

$$X_1^2 + X_2^2 = X_5^2 + X_7^2 = X_6^2 + X_8^2 = \frac{1}{27}\,. \qquad (87)$$

Eliminating the functions F and $\bar{F}$ in (85) we obtain

$$\begin{aligned}
X_1 &= \frac{i}{6\sqrt{3}\,|c|^2} e^{-(v+\bar{v})e^{i\psi}} \left(\bar{c}^2 e^{v} - c^2 e^{\bar{v}} e^{i\sqrt{3}(v+\bar{v})}\right), \\
X_2 &= -\frac{1}{6\sqrt{3}\,|c|^2} e^{-(v+\bar{v})e^{i\psi}} \left(\bar{c}^2 e^{v} + c^2 e^{\bar{v}} e^{i\sqrt{3}(v+\bar{v})}\right), \\
X_5 &= -\frac{1}{3\sqrt{3}} \cos\left(\frac{3}{2}(\sqrt{3}X_3 + X_4)\right), \\
X_6 &= \frac{1}{6\sqrt{3}\,|c|^2} e^{-(v+\bar{v})e^{i\psi}} \left(\bar{c}^2 e^{\bar{v}} + c^2 e^{v} e^{i\sqrt{3}(v+\bar{v})}\right), \\
X_7 &= -\frac{1}{3\sqrt{3}} \sin\left(\frac{3}{2}(\sqrt{3}X_3 + X_4)\right), \\
X_8 &= \frac{i}{6\sqrt{3}\,|c|^2} e^{-(v+\bar{v})e^{i\psi}} \left(\bar{c}^2 e^{\bar{v}} - c^2 e^{v} e^{i\sqrt{3}(v+\bar{v})}\right), \qquad (88)
\end{aligned}$$

where $v = \frac{3}{4}(1 + i\sqrt{3})(X_3 + iX_4)$. The surface is parametrized in terms of X_3 and X_4. Now, the corresponding first fundamental form becomes

$$I = \frac{3}{2}(dX_3^2 + dX_4^2)\,. \qquad (89)$$

Note that this is just the real form of (86) where $\xi^1 = X_3$ and $\xi^2 = X_4$.

The induced metric (86) on the $(\xi, \bar{\xi})$ plane can be written as a quadratic differential

$$I = \frac{2}{3} d\left(\ln F(\xi)\right) \wedge d\left(\ln \bar{F}(\bar{\xi})\right). \qquad (90)$$

Equation (90) defines a field of line elements on a surface $\mathcal{F}$ with singularities at the critical points (i.e. the zeros and poles of the differential (90)). The geodesic trajectories of this metric are determined locally by the integral

$$\mathrm{Re}\left[e^{i\theta}\omega\right] = \text{const}, \qquad \omega = \int \frac{F'}{F} d\xi,$$

where we make use of the definitions and notations given in [37]. The simplest local trajectory structure of the quadratic differential (90) can be found by assuming that F'/F has two simple zeros and one simple pole. Then

$$F(\xi) = A\xi^n\left(1 + \mathcal{O}(\xi)\right), \quad n \in \mathbb{Z}$$

and

$$\frac{F'}{F} = \frac{n}{\xi}\left(1 + \mathcal{O}(\xi)\right) \text{ near a pole at } \xi = 0$$

$$\frac{F'}{F} = C(\xi - a)\left(1 + \mathcal{O}(\xi - a)\right) \text{ near a simple zero of } F'$$

Locally, the flat coordinates of the metric I are the real and imaginary parts of the function

$$W = \int \frac{F'}{F} d\xi = n \ln \xi + \mathcal{O}(1), \qquad (91)$$

where $\mathcal{O}(1)$ denotes some analytic function near $\xi = 0$. The critical vertical trajectory is defined to be the maximal trajectory of the ODE

$$\mathrm{Re}\left(\frac{F'}{F}d\xi\right) \qquad \text{with } \theta = 0\,.$$

The monodromy of (91) is given by $2i\pi n$. Also, the function $q = e^{\omega/n}$ is analytic in a punctured neighborhood of $\xi = 0$, since $\mathrm{Re}(\omega) = n\ln|\xi|$ and $|q| \sim |\xi|$ near $\xi = 0$. Thus, q has a removable singularity at $\xi = 0$, and hence can be extended to an analytic function

$$q(\xi) = \xi + \mathcal{O}(\xi^2)$$

Let us denote by $\mathcal{D}$ the maximal connected domain foliated by closed trajectories homotopic to a small circle around $\xi = 0$. The function q is a single valued conformal map of $\mathcal{D}$ onto the disk of radius $|n|$, since the perimeter of the disk is $2\pi|n|$. Hence, we get one semi-infinite cylinder (homeomorphic to the disk $\{0 < |\xi| \leq 1\}$) for each simple pole of F'/F. For example, for $F = \xi(\xi - 1)$ we have

$$\frac{F'}{F} = \frac{2\xi - 1}{\xi(\xi - 1)} = \frac{1}{\xi} + \frac{1}{\xi - 1}, \qquad \mathrm{Res}_{\infty}\left(\frac{F'}{F}\right) = -2. \tag{92}$$

Hence we obtain three cylinders, two for the poles at 0 and 1, each of perimeter 2π, and one of perimeter 4π for the pole at ∞. If we instead assume that F'/F has two simple zeros and three simple poles, then we get four semi-infinite cylinders glued along the based perimeter with two points of conical singularities. It should be noted that for the $\mathbb{C}P^1$ model, the surfaces corresponding to the dilation invariant solutions are cylinders [38].

Note that the singular solutions (84) are in fact the *meron-like* solutions (i.e. with logarithmically divergent action at isolated points) of the $\mathbb{C}P^2$ model equations (76). The meron solutions, obtained from the dilation invariance of the solutions of the $\mathbb{C}P^2$ model, are located at $\bar{F}(\bar{\xi}) = 0$. Merons are more durable than instantons in the sense that they can exist in a constant (not necessarily zero) Higgs field. This property is shared by both the $\mathbb{C}P^{N-1}$ and Yang-Mills models [39, 40, 41].

5 Conclusions

In this paper we have elaborated on certain geometrical aspects of surfaces associated with the $\mathbb{C}P^{N-1}$ sigma models based on the generalized Weierstrass representation. This idea has been undertaken recently by W. J. Zakrzewski in [42] using a modification of the approach presented in [30]. In the context of [42] a sequence of rank-one projectors of the form

$$\mathcal{P}_k := \mathcal{P}(V_k) = \frac{V_k \otimes V_k^\dagger}{V_k^\dagger \cdot V_k}\,, \quad \text{where } \; V_k = P_+^k f\,, \qquad k \in \mathbb{Z}^+\,, \tag{93}$$

were used to construct a family of surfaces associated with a given solution of the $\mathbb{C}P^{N-1}$ model. More specifically the successive application, say k times, of the operator P_+, given in (32), starting with any nonconstant holomorphic solution of the $\mathbb{C}P^{N-1}$ model allows one to find a new solution $P_k = P_+^k f$, which represents a harmonic map $S^2 \to \mathbb{C}P^{N-1}$. So, for every $k \leq N - 1$

the quantity P_+^k is a building element for the construction of a set of rank-one projectors $\{\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_k\}$ according to the relation (93). Note that this set of projectors determines new conservation laws of the form (7). These conservation laws can be considered new in the sense that only the first one is related to the holomorphic (or antiholomorphic) solutions and the rest are related to the mixed solutions obtained from the nonconstant holomorphic ones by applying the operator P_+ successively. Consequently, according to this procedure one can obtain new surfaces for each projector. However, there are some questions concerning this procedure.

Using the following properties [31]

$$\bar{\partial}(P_+^k f) = -P_+^{k-1} f \frac{|P_+^k f|^2}{|P_+^{k-1} f|^2} \,, \qquad \partial\left(\frac{P_+^{k-1} f}{|P_+^{k-1} f|^2}\right) = \frac{P_+^k f}{|P_+^{k-1} f|^2} \,, \tag{94}$$

and the orthogonality relation (34), it is straightforward to compute

$$\partial\mathcal{P}_k = \frac{(P_+^{k+1} f) \otimes (P_+^k f)^\dagger}{|P_+^k f|^2} - \frac{(P_+^k f) \otimes (P_+^{k-1} f)^\dagger}{|P_+^{k-1} f|^2} \tag{95}$$

and

$$[\partial\mathcal{P}_k, \mathcal{P}_k] = \frac{(P_+^{k+1} f) \otimes (P_+^k f)^\dagger}{|P_+^k f|^2} + \frac{(P_+^k f) \otimes (P_+^{k-1} f)^\dagger}{|P_+^{k-1} f|^2} \tag{96}$$

which can also be written as

$$[\partial\mathcal{P}_k, \mathcal{P}_k] = \partial\mathcal{P}_k + 2\frac{(P_+^k f) \otimes (P_+^{k-1} f)^\dagger}{|P_+^{k-1} f|^2} \,. \tag{97}$$

Similarly, we can write $[\bar{\partial}\mathcal{P}_k, \mathcal{P}_k]$ as

$$[\bar{\partial}\mathcal{P}_k, \mathcal{P}_k] = -\bar{\partial}\mathcal{P}_k - 2\frac{(P_+^{k-1} f) \otimes (P_+^k f)^\dagger}{|P_+^{k-1} f|^2} \,. \tag{98}$$

As a consequence of the commutators given in (97) and (98), the Weierstrass data (15) becomes

$$dX = -i\left[(\partial\mathcal{P}_k + 2\frac{(P_+^k f) \otimes (P_+^{k-1} f)^\dagger}{|P_+^{k-1} f|^2})d\xi + (\bar{\partial}\mathcal{P}_k + 2\frac{(P_+^{k-1} f) \otimes (P_+^k f)^\dagger}{|P_+^{k-1} f|^2})d\bar{\xi}\right] \,. \tag{99}$$

It is easily seen that for $k = 0$ (e.g. for the holomorphic solutions, or the antiholomorphic ones had we started with them) equation (99) reduces to

$$dX = -i\left[\partial\mathcal{P}_0 d\xi + \bar{\partial}\mathcal{P}_0 d\bar{\xi}\,\right] \,, \tag{100}$$

since the other terms in (99) do not appear for $k = 0$. Hence, it is concluded that X is proportional to the projector $\mathcal{P}_0$. This point has been fully discussed both in this paper and in [30, 43]. However, for $k \neq 0$ (i.e. for the nonholomorphic solutions) it does not lead to the same conclusion.

This point can be further discussed on an example which is fully analyzed in this paper and in [42] by different approaches. In [42] it is stated that the mixed solution, obtained from the Veronese vector $f=(1,\sqrt{2}\,\xi,\xi^2)$, for the $\mathbb{C}P^2$ model associated with the projector

$$\mathcal{P}_1=\frac{1}{(1+|\xi|^2)^2}\begin{pmatrix} 2|\xi|^2 & \sqrt{2}\,\bar{\xi}(|\xi|^2-1) & -2\bar{\xi}^2 \\ \sqrt{2}\,\xi(|\xi|^2-1) & (1-|\xi|^2)^2 & -\sqrt{2}\,\bar{\xi}(|\xi|^2-1) \\ -2\xi^2 & -\sqrt{2}\,\xi(|\xi|^2-1) & 2|\xi|^2 \end{pmatrix}, \tag{101}$$

leads to a radius vector $\vec{Y}$ which lies in a 5-dimensional subspace of $\mathbb{R}^8$. Moreover, the components of the radius vector $\vec{Y}$ are given as

$$\begin{aligned} &Y_1=\frac{2x(1-x^2-y^2)}{(1+x^2+y^2)^2}\,, \quad Y_2=\frac{2y(1-x^2-y^2)}{(1+x^2+y^2)^2}\,, \quad Y_3=\frac{2(x^2-y^2)}{(1+x^2+y^2)^2}\,,\\ &Y_4=\frac{4xy}{(1+x^2+y^2)^2}\,, \quad Y_5=\sqrt{3}\frac{(1-x^2-y^2)^2}{(1+x^2+y^2)^2}\,, \end{aligned} \tag{102}$$

which satisfy the following surface

$$Y_1^2+Y_2^2+4Y_3^2+4Y_4^2+\frac{1}{\sqrt{3}}Y_5=1\,. \tag{103}$$

However, using the same solution through the use of our projector (73) and procedure summarized in Section 2, we obtain an associated surface with the radius vector $\vec{X}$ immersed in a 3-dimensional subspace of $\mathbb{R}^8$. The components of the radius vector $\vec{X}$ are given in (62) and they satisfy equation (63). Since the two surfaces are obtained from the same mixed solutions of the $\mathbb{C}P^2$ model, constructed by the same procedure from the Veronese vector $f=(1,\sqrt{2}\,\xi,\xi^2)$, we expect them to be the same geometrical object in accordance with the Bonnet theorem. However, it can easily be checked that the two surfaces cannot be transformed into each other by rotations and translations.

It is also worth mentioning that from the Veronese sequences we can obtain associated surfaces with constant Gaussian curvature as stated in [42]. However, the converse statement does not apply in general. In this paper and in [30] we give examples of surfaces with constant Gaussian curvature which are not associated with Veronese sequences. Such surfaces correspond to the dilation-invariant solutions or mixed soliton solutions of the $\mathbb{C}P^2$ model.

Finally, the following relevant questions could be asked. What is the meaning of the new conservation laws in the context of surfaces immersed in multi-dimensional spaces? Are they really independent? Can they differ from each other by a total divergence (as it was drawn up in Section 2)? Other natural questions which could also be asked involve families of solutions obtained recursively and whether they can be related through an auto-Bäcklund transformation (note that a Bäcklund parameter is not present in formula (32), but it can be introduced by a gauge transformation)? In addition to these it is also important to ask if the symmetry operator (32) is expressible in terms of some combination of the known infinitesimal generators (as given in [30]) of the Lie-point symmetry algebra of the $\mathbb{C}P^{N-1}$ model. These and other questions will be addressed in our future work.

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