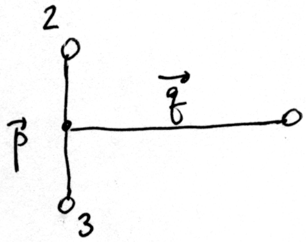
Kinematics

3 equal mass particles



$$\vec r = \vec r_2 - \vec r_3 \qquad \vec \rho = \frac{\vec r_2 + \vec r_3}{2} - \vec r_1 \qquad \vec R = \frac{1}{3}(\vec r_1 + \vec r_2 + \vec r_3)$$

$$\vec p = \frac{1}{2}(\vec p_2 - \vec p_3) \qquad \vec q = \frac{1}{3}(\vec p_1 + \vec p_3 - 2\vec p_1) \qquad \vec P = \vec p_1 + \vec p_2 + \vec p_3$$

$$\left.\begin{aligned} \vec r_1 &= \qquad\quad -\tfrac{2}{3}\vec\rho + \vec R \\ \vec r_2 &= \tfrac{1}{2}\vec r + \tfrac{1}{3}\vec \rho + \vec R \\ \vec r_3 &= -\tfrac{1}{2}\vec r + \tfrac{1}{3}\vec\rho + \vec R \end{aligned}\right\} \vec r_2 + \vec r_3 = \frac{2}{3}\vec\rho + 2\vec R$$

$$\left.\begin{aligned} \vec p_1 &= \qquad -\vec q + \tfrac{1}{3}\vec P \\ \vec p_2 &= \vec p + \tfrac{1}{2}\vec q + \tfrac{1}{3}\vec P \\ \vec p_3 &= -\vec p + \tfrac{1}{2}\vec q + \tfrac{1}{3}\vec P \end{aligned}\right\} \vec p_2 + \vec p_3 = \vec q + \frac{2}{3}\vec P$$

$$\vec p_1\cdot\vec r_1 + \vec p_2\cdot\vec r_2 + \vec p_3\cdot\vec r_3 = \vec p\cdot\vec r + \vec q\cdot\vec\rho + \vec P\cdot\vec R$$

$$\vec r_1\times\vec p_1 + \vec r_2\times\vec p_2 + \vec r_3\times\vec p_3 = \vec r\times\vec p + \vec\rho\times\vec q + \vec R\times\vec P$$

Single-nucleon m.m. operator:

$$\vec\mu = \frac{e}{2m_N c}\left[g_L\vec L + g_S\vec S\right]$$

	proton	neutron
g_L	1	0
g_S	$2\mu_p \simeq 5.587$	$2\mu_n \simeq -3.826$

where $\mu_p = 1 + \varkappa_p \simeq 2.793 \qquad \varkappa_p \simeq 1.793$

$\mu_n = \varkappa_n = -1.913 \qquad \varkappa_n \simeq -1.913$

This can be written as an operator in the isospin space:

$$\vec\mu = \frac{e}{2m_N c}\left[\frac{1+\tau_3}{2}\vec L + (\mu_s + \mu_v\tau_3)\vec S\right],$$

where $\mu_s = \mu_p + \mu_n = 1 + \varkappa_p + \varkappa_n = 1 + \varkappa_s \simeq 0.88 \rightarrow \varkappa_s \simeq -0.12$

$\mu_v = \mu_p - \mu_n = 1 + \varkappa_p - \varkappa_n = 1 + \varkappa_v \simeq 4.706 \rightarrow \varkappa_v \simeq 3.706$

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For 3N bound state and single-nucleon $\hat{\vec{\mu}}$:

$$\langle\mu\rangle = \langle\Psi|\vec{\mu}_1+\vec{\mu}_2+\vec{\mu}_3|\Psi\rangle = 3\langle\Psi|\vec{\mu}_1|\Psi\rangle$$

where

$$\vec{\mu}_1 = \mu_N\left[\frac{1+\tau_3(1)}{2}\vec{L}(1) + (\mu_s+\mu_v\tau_3(1))\vec{S}(1)\right]$$

and $\vec{L}(1) = \vec{r}_1\times\vec{p}_1 \xrightarrow[\text{c.m. frame}]{} \frac{2}{3}\vec{\rho}\times\vec{q} = \frac{2}{3}\vec{\ell}$

Thus (with the factor 3):

$$\boxed{\begin{aligned}
&\langle\mu\rangle = \mu_N\langle\Psi|\vec{\mu}_{IA}|\Psi\rangle,\\
&\vec{\mu}_{IA} = [1+\tau_3(1)]\vec{\ell} + 3[\mu_s+\mu_v\tau_3(1)]\vec{S}(1)\\
&|\Psi\rangle = \sum_\nu\int dp\,p^2\int dq\,q^2\,\langle pq\nu|\Psi\rangle\,|pq\nu\rangle;\\
&|pq\nu\rangle = |pq;[(L\ell)\mathcal{L}(Ss)\mathcal{S}]\mathcal{J}M_3;(Tt)\mathcal{T}T_3\rangle
\end{aligned}}$$

$\vec{\mu}_{IA}$ is the rank-1 operator in L/S space and rank-0 or rank-1 operator in the isospin space

The overall W-E factor:

$$(-)^{\mathcal{J}'-M_3'}\begin{pmatrix}\mathcal{J}' & 1 & \mathcal{J}\\ -M_3' & 0 & M_3\end{pmatrix}$$

For the L/S part, $\mathcal{J}=M_3=1/2$ $\mathcal{J}'=M_3'=1/2$ and we get

$$\begin{pmatrix}\frac{1}{2} & 1 & \frac{1}{2}\\ -\frac{1}{2} & 0 & \frac{1}{2}\end{pmatrix} = \frac{1}{\sqrt{6}} \simeq 0.4082 \quad \text{(wig1 in the code)}$$

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For the isospin:

$$(-)^{J'-J_3'}\begin{pmatrix} J' & k & J \\ -J_3' & 0 & J_3 \end{pmatrix} \xrightarrow[J=J_3=\frac{1}{2}]{J'=J_3'=\frac{1}{2}} \underbrace{\begin{pmatrix} \frac{1}{2} & k & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}}$$

(This gives for $k=0$ $\quad \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \simeq 0.707$ wig6

for $k=1$ $\quad \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{\sqrt{6}} \simeq 0.4082$ wig8)

All the rest follows from W-E theorem:

$$\langle (J'j')\mathcal{J}' \| [A^{(a)} \otimes B^{(b)}]^{(c)} \| (Jj)\mathcal{J}) = \hat{\mathcal{J}}'\hat{\mathcal{J}}\hat{c} \begin{Bmatrix} J' & J & a \\ j' & j & b \\ \mathcal{J}' & \mathcal{J} & c \end{Bmatrix} (J'\|A^{(a)}\|J) \cdot (j'\|B^{(b)}\|j)$$

in particular; for $A^{(a)}=1$ $\quad a=0$:

$$\langle (J'j')\mathcal{J}' \| B^{(b)} \| (Jj)\mathcal{J}) = \qquad (*)$$

$$= (-)^{\mathcal{J}'+b+J+j}\, \delta_{J'J}\, \hat{\mathcal{J}}'\hat{\mathcal{J}} \begin{Bmatrix} \mathcal{J} & \mathcal{J}' & b \\ j' & j & J \end{Bmatrix} (j'\|B^{(b)}\|j)$$

and for $B^{(b)}=1$ $\quad b=0$:

$$\langle (J'j')\mathcal{J}' \| A^{(a)} \| (Jj)\mathcal{J} \rangle = \qquad (**)$$

$$= (-)^{\mathcal{J}+a+J'+j}\, \delta_{j'j}\, \hat{\mathcal{J}}'\hat{\mathcal{J}} \begin{Bmatrix} \mathcal{J} & \mathcal{J}' & a \\ J' & J & j \end{Bmatrix} (J'\|A^{(a)}\|J)$$

Isospin factor

$$O_0^{(k)}(1) < \begin{matrix} \hat{1} & k=0 \\ \tau_0(1) & k=1 \end{matrix}$$

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From $(*)$:

$$\langle (T't')\mathcal{T}' \| O^{(k)}(1) \| (Tt)\mathcal{T} \rangle = (-)^{\mathcal{T}'+k+T+t} \delta_{T't} \hat{\mathcal{T}}'\hat{\mathcal{T}} \cdot$$

$$\cdot \begin{Bmatrix} \mathcal{T} & \mathcal{T}' & k \\ t' & t & T \end{Bmatrix} (t' \| O^{(k)}(1) \| t)$$

Since $\mathcal{T}' = \mathcal{T}_3' = \frac{1}{2} = \mathcal{T} = J_3$ $\qquad \hat{\mathcal{T}}' = \hat{\mathcal{T}} = \sqrt{2\cdot\frac{1}{2}+1} = \sqrt{2}$

$t' = t = 1/2$

we get

$$\delta_{T'T} (-)^{k+T+1} \cdot 2 \begin{Bmatrix} k & \frac{1}{2} & \frac{1}{2} \\ T & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} (t' \| O^{(k)}(1) \| t)$$

For $k=0$ $\qquad (t' \| \hat{1} \| t) = \delta_{t't} \sqrt{2t+1} = \sqrt{2}$

$k=1$ $\qquad (t' \| \tau^{(1)}(1) \| t) = 2 \underbrace{(t' \| t^{(1)}(1) \| t)}_{\sqrt{t(t+1)(2t+1)}} = \sqrt{6}$

So, together with the W-E factor from the previous page we get:

$$\delta_{t'T} (-)^{k+T+1} \; 2 \begin{pmatrix} \frac{1}{2} & k & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{Bmatrix} k & \frac{1}{2} & \frac{1}{2} \\ T & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} \left[\delta_{k0} \sqrt{2} + \delta_{k1} \sqrt{6} \right] =$$

$$\equiv \underline{\delta_{T'T} \, \text{fisos}(k,T)} \; ; \quad \begin{cases} \text{fisos}(0,T) = 1 \\ \underline{\text{fisos}(1,T) = \delta_{T0} - \frac{1}{3}\delta_{T1}} \end{cases}$$

This follows from:

$k=0$ $\qquad \text{wig6} = \begin{pmatrix} 1/2 & 0 & 1/2 \\ -1/2 & 0 & 1/2 \end{pmatrix} = \frac{1}{\sqrt{2}} \simeq 0.707 \qquad \text{wig7} = \begin{Bmatrix} T & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{Bmatrix} = \frac{(-)^{T+1}}{2}$

$k=1$ $\qquad \text{wig8} = \begin{pmatrix} 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \end{pmatrix} = \frac{1}{\sqrt{6}} \simeq 0.408$

$$\text{wig9} = \begin{Bmatrix} T & 1/2 & 1/2 \\ 1 & 1/2 & 1/2 \end{Bmatrix} \begin{matrix} \nearrow T=0 & \frac{1}{2} \\ \searrow T=1 & \frac{1}{6} \end{matrix} \rightarrow \frac{1}{2}\left[\delta_{T0} + \frac{1}{3}\delta_{T1} \right]$$

mm (5)

It is convenient to label:

isoscalar current / m.m. by $k=0$

isovector — " — by $k=1$

in particular $\mu_0 \equiv \mu_s$ (and same for $\varkappa_{s/v}$)

$\mu_1 \equiv \mu_v$

$$\vec{\mu}_{IA} = [1+\tau_3(1)]\,\vec{l} + [\mu_s + \mu_v\,\tau_3(1)]\; 3\,\vec{S}(1)$$

The isospin matrix elements are then simply

$\delta_{T'T}$ fisos (k,T) for the first term

$\delta_{T'T}\,\mu_k$ fisos (k,T) for the second term

where fisos $(0,T)=1$ (isoscalar)

fisos $(1,T) = \delta_{T0} - \frac{1}{3}\delta_{T1}$ (isovector)

L/S factors

a) Spin current $\quad 3\,\vec{s}(1)$

$$3\,\langle [(L'\ell')\mathcal{L}'\,(S's')\mathcal{S}']\,\mathcal{J}' \,\|\, \vec{s}(1) \,\|\, [(L\ell)\mathcal{L}\,(Ss)\mathcal{S}]\,\mathcal{J}\rangle \overset{(*)}{=}$$

$$= 3\,\delta_{L'L}\,\delta_{\ell'\ell}\,\delta_{\mathcal{L}'\mathcal{L}}\,(-)^{\mathcal{J}'+1+\mathcal{L}+\mathcal{S}}\,\hat{\mathcal{J}}'\hat{\mathcal{J}}\begin{Bmatrix}\mathcal{J} & \mathcal{J}' & 1\\ \mathcal{S}' & \mathcal{S} & \mathcal{L}\end{Bmatrix}\underbrace{\langle (S's')\mathcal{S}'\,\|\,\vec{s}(1)\,\|\,(Ss)\mathcal{S}\rangle}$$

$$(-)^{\mathcal{S}'+1+S+s}\,\delta_{S'S}\,\hat{\mathcal{S}}'\hat{\mathcal{S}}\begin{Bmatrix}\mathcal{S} & \mathcal{S}' & 1\\ s' & s & S\end{Bmatrix}\overset{(*)}{\underbrace{(s'\,\|\,s^{(1)}(1)\,\|\,s)}_{\delta_{s's}\sqrt{s(s+1)(2s+1)}}}$$

Altogether:

$$3\,\delta_{L'L}\,\delta_{\ell'\ell}\,\delta_{\mathcal{L}'\mathcal{L}}\,\delta_{S'S}\,(-)^{\mathcal{J}'+\mathcal{L}+\mathcal{S}+\mathcal{S}'+S+s}\,\hat{\mathcal{J}}'\hat{\mathcal{J}}\,\hat{\mathcal{S}}'\hat{\mathcal{S}}\,\sqrt{s(s+1)(2s+1)}\,\cdot$$

$$\begin{Bmatrix}\mathcal{J} & \mathcal{J}' & 1\\ \mathcal{S}' & \mathcal{S} & \mathcal{L}\end{Bmatrix}\begin{Bmatrix}\mathcal{S} & \mathcal{S}' & 1\\ s' & s & S\end{Bmatrix}$$

Since $\mathcal{J}'=\mathcal{J}=1/2=s'=s$:

$$\delta_{L'L}\,\delta_{\ell'\ell}\,\delta_{S'S}\cdot\delta_{\mathcal{L}'\mathcal{L}}\;3\sqrt{6}\,(-)^{\mathcal{L}+S+\mathcal{S}+\mathcal{S}'+1}\,\hat{\mathcal{S}}'\hat{\mathcal{S}}\begin{Bmatrix}1 & 1/2 & 1/2\\ \mathcal{L} & \mathcal{S}' & \mathcal{S}\end{Bmatrix}\begin{Bmatrix}1 & 1/2 & 1/2\\ S & \mathcal{S}' & \mathcal{S}\end{Bmatrix}$$

b) Convection current: $\vec{\ell}$

$$\langle [(L'\ell')\mathcal{L}'\,(S's')\mathcal{S}']\,\mathcal{J}' \,\|\, \vec{\ell} \,\|\, [(L\ell)\mathcal{L}\,(Ss)\mathcal{S}]\,\mathcal{J}\rangle \overset{(**)}{=} \delta_{S'S}\,\delta_{\mathcal{S}'\mathcal{S}}\,\cdot$$

$$(-)^{\mathcal{J}+1+\mathcal{L}'+\mathcal{S}}\,\hat{\mathcal{J}}'\hat{\mathcal{J}}\begin{Bmatrix}\mathcal{J} & \mathcal{J}' & 1\\ \mathcal{L}' & \mathcal{L} & \mathcal{S}\end{Bmatrix}\underbrace{\langle (L'\ell')\mathcal{L}'\,\|\,\vec{\ell}\,\|\,(L\ell)\mathcal{L}\rangle}_{(*)}$$

$$\hookrightarrow (-)^{\mathcal{L}'+1+L+\ell}\,\delta_{L'L}\,\hat{\mathcal{L}}'\hat{\mathcal{L}}\begin{Bmatrix}\mathcal{L} & \mathcal{L}' & 1\\ \ell' & \ell & L\end{Bmatrix}\cdot\delta_{\ell'\ell}\,\sqrt{\ell(\ell+1)(2\ell+1)}$$

Altogether:

$$\delta_{L'L}\,\delta_{\ell'\ell}\,\delta_{S'S}\,\delta_{\mathcal{S}'\mathcal{S}}\,(-)^{\mathcal{J}+\mathcal{S}+L+\ell}\,\sqrt{\ell(\ell+1)(2\ell+1)}\,\hat{\mathcal{J}}'\hat{\mathcal{J}}\,\hat{\mathcal{L}}'\hat{\mathcal{L}}\begin{Bmatrix}\mathcal{J} & \mathcal{J}' & 1\\ \mathcal{L}' & \mathcal{L} & \mathcal{S}\end{Bmatrix}\begin{Bmatrix}\mathcal{L} & \mathcal{L}' & 1\\ \ell & \ell & L\end{Bmatrix}$$

$\downarrow\ \mathcal{J}'=\mathcal{J}=1/2$

$$\delta_{L'L}\,\delta_{\ell'\ell}\,\delta_{S'S}\cdot\delta_{\mathcal{S}'\mathcal{S}}\,(-)^{\frac{1}{2}+\mathcal{S}+L+\ell}\,2\sqrt{\ell(\ell+1)(2\ell+1)}\,\hat{\mathcal{L}}'\hat{\mathcal{L}}\begin{Bmatrix}\mathcal{L} & \mathcal{L}' & 1\\ \ell & \ell & L\end{Bmatrix}\begin{Bmatrix}\mathcal{L} & \mathcal{L}' & 1\\ 1/2 & 1/2 & \mathcal{S}\end{Bmatrix}$$

$$\langle \mathcal{Y}\mathcal{J}, TT | \vec{\mu}^{(k)}_{IA} | \mathcal{Y}\mathcal{J}, TT \rangle =$$

$$= \sum_{\nu'\nu} \delta_{L'L}\, \delta_{\ell'\ell}\, \delta_{S'S}\, \delta_{T'T}\, \frac{1}{\sqrt{6}}\, fisos(k,T) \int dp\, p^2 \int dq\, q^2 \langle pq\nu'|\psi\rangle \langle pq\nu|\psi\rangle$$

$$\Big\{ \delta_{\mathcal{Y}'\mathcal{Y}} (-)^{L+\ell+\mathcal{Y}+\frac{1}{2}}\, 2\sqrt{\ell(\ell+1)(2\ell+1)}\, \hat{\mathcal{L}}'\hat{\mathcal{L}} \begin{Bmatrix} \mathcal{L} & \mathcal{L}' & 1 \\ \ell & \ell & L \end{Bmatrix} \begin{Bmatrix} \mathcal{L} & \mathcal{L}' & 1 \\ 1/2 & 1/2 & \mathcal{Y} \end{Bmatrix} +$$

$$+ \delta_{\mathcal{L}'\mathcal{L}} (-)^{\mathcal{L}+S+\mathcal{Y}'+\mathcal{Y}+1} \cdot \mu_k\, 3\sqrt{6}\, \hat{\mathcal{Y}}'\hat{\mathcal{Y}} \begin{Bmatrix} 1 & 1/2 & 1/2 \\ \mathcal{L} & \mathcal{Y}' & \mathcal{Y} \end{Bmatrix} \begin{Bmatrix} 1 & 1/2 & 1/2 \\ S & \mathcal{Y}' & \mathcal{Y} \end{Bmatrix} \Big\}$$

where $fisos(0,T) = 1$ $\qquad \mu_0 = \mu_s \simeq 0.88$

$fisos(1,T) = \delta_{T0} - \frac{1}{3}\delta_{T1}$ $\qquad \mu_1 = \mu_V \simeq 4.706$

$\underline{\nu = L\ \ell\ \mathcal{L}\ S\ \mathcal{Y}\ T}$

First line in $\{\ \}$ comes from the convection current; non-diag $\mathcal{L}\mathcal{L}'$

Second – " — " – " – " – the spin current, non-diag $\mathcal{Y}\mathcal{Y}'$

Selection rules (apart from those which are always satisfied by construction, e.g., $\Delta(\mathcal{L}, L, \ell)$, taking into account also that $\mathcal{J}' = \mathcal{J} = 1/2$, $s' = s = 1/2$):

Convection current $\mathcal{Y}' = \mathcal{Y}$ and $\boxed{\begin{array}{l} \Delta(\ell,\ell,1) \quad \text{i.e. } \ell \neq 0 \\ \Delta(\mathcal{L}', \mathcal{L}, 1) \end{array}}$ from $\begin{Bmatrix} \mathcal{L} & \mathcal{L}' & 1 \\ \ell & \ell & L \end{Bmatrix}$, the second 6j does not give anything more

Spin current Since $\mathcal{Y}$ or $\mathcal{Y}'$ are $1/2$ or $3/2$ $\Delta(\mathcal{Y}', \mathcal{Y}, 1)$ is always satisfied. But the first 6j implies:

$\mathcal{L}' = \mathcal{L}$ and $\boxed{\text{If } \mathcal{L}' = \mathcal{L} = 0, \text{ then } \mathcal{Y}' = \mathcal{Y} = 1/2}$