



INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

**First steps in combinatorial optimization
on graphons: Matchings**

Martin Doležal

Jan Hladký

Ping Hu

Diana Piguet

Preprint No. 17-2016

PRAHA 2016

FIRST STEPS IN COMBINATORIAL OPTIMIZATION ON GRAPHONS: MATCHINGS (EXTENDED ABSTRACT)

MARTIN DOLEŽAL, JAN HLADKÝ, PING HU, AND DIANA PIGUET

ABSTRACT. Much of discrete optimization concerns problems whose underlying structures are graphs. Here, we translate the theory around the maximum matching problem to the setting of graphons, which are limit versions of finite graphs introduced by Borgs, Chayes, Lovász, Sós, Szegedy, and Vesztegombi. We study continuity properties of the thus defined matching ratio, limit versions of matching polytopes and vertex cover polytopes, and deduce a version of the LP duality for the problem of maximum fractional matching in the graphon setting. To the best of our knowledge, this is the first time the LP duality has been formulated in the setting of functional analysis (rather than finite-dimensional vector spaces).

We show an application of these results in property testing.

1. INTRODUCTION

The study of matchings is central both in graph theory and in theoretical computer science. It has three sides: structural, polyhedral, and algorithmic. The structural part of the theory includes results such as the Gallai–Edmonds matching theorem. The study of polyhedral aspects — which include the geometry of the matching polytope, the vertex cover polytope and related — is much motivated by linear programming. Finally, algorithmic questions include, e.g., the study of fast algorithms for finding the maximum matching, or are motivated by theory related to property testing and parameter estimation. These three sides are very much intertwined. For example, integrality of the fractional matching polytope and the fraction vertex cover polytope is equivalent to König’s matching theorem. Also, most of the algorithms for finding a maximum matching (such as the Ford–Fulkerson algorithm) use non-trivial properties of matchings.

Graphons are analytic object which capture properties of large graphs. They were introduced in [3, 4, 9] as limit representation of large dense graphs. Since then they have played a key role in extremal graph theory, theory of random graphs, and other parts of mathematics. Their most important contribution to computer science is the area of property testing, see e.g., [10] and more in Section 2.2. Here, we study concepts related to matchings in the setting of graphons. While our primary motivation comes from extremal graph theory, we think that the theory provides an interesting contribution to combinatorial optimization as well.

Doležal was supported by the Czech Science Foundation, grant number 16-07378S. *Hladký*: The research leading to these results has received funding from the People Programme (Marie Curie Actions) of the European Union’s Seventh Framework Programme (FP7/2007-2013) under REA grant agreement number 628974. *Hu*: This work has received funding from the European Research Council (ERC) under the European Union’s Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement No 259385 and from the Philip Leverhulme Prize PLP-2014-401 awarded by the Leverhulme Trust. *Piguet* was supported by the Czech Science Foundation, grant number GJ16-07822Y.

We shall assume the reader's familiarity with the concept of graphons. Throughout the paper we shall assume that Ω is an atomless Borel probability space.

2. MATCHINGS IN GRAPHONS

Let us give our definition of matchings in graphons. In the graphon world, there is no distinction between integral and fractional matchings. This is why our definition is actually inspired by the notion of fractional matchings in finite graphs. To see the analogy, let us recall that for a finite graph G , a function $f : V(G)^2 \rightarrow \mathbb{R}$ represents a *fractional matching* if

- (a) $f \geq 0$,
- (b) if $f(x, y) > 0$ then $xy \in E(G)$, and
- (c) for every $x \in V(G)$, we have $\sum_y f(x, y) + \sum_y f(y, x) \leq 1$.

Note that usually fractional matchings are represented using symmetric functions (i.e., typically one works with a weight function g defined on unordered pairs xy , $g(xy) = f(x, y) + f(y, x)$). This is however only a matter of notation.¹

Recall that a *support* of a function $g : X \rightarrow \mathbb{R}$ is the set $\text{supp } g = \{x \in X : g(x) \neq 0\}$.

Definition 1. Suppose that $W : \Omega^2 \rightarrow [0, 1]$ is a graphon. We say that a function $\mathbf{m} \in \mathcal{L}^1(\Omega^2)$ is a *matching* in W if

- (a) $\mathbf{m} \geq 0$ almost everywhere,
- (b) $\text{supp } \mathbf{m} \subset \text{supp } W$ up to a null-set, and
- (c) for almost every $x \in \Omega$, we have $\int_y \mathbf{m}(x, y) + \int_y \mathbf{m}(y, x) \leq 1$.

As said already, even though Definition 1 is inspired by fractional matchings in finite graphs, the resulting graphon concept is referred to as “matchings”. Note also that in Definition 1 the values of W are immaterial, only the support of W matters.

Figure 2.1 shows how to translate a fractional matching on a finite graph G to a matching on a graphon that represents G .

Given a matching \mathbf{m} in a graphon W we define its *size*, $\|\mathbf{m}\| = \int_x \int_y \mathbf{m}(x, y)$. We write $\text{MATCH}(W) \subset \mathcal{L}^1(\Omega^2)$ for the set of all matchings in W . Following the *-on* word ending already used for graphons and permutons, we call $\text{MATCH}(W)$ the *matching polyton* of W . Last, we define the *matching ratio* of a graphon W as

$$(2.1) \quad \text{match}(W) = \sup_{\mathbf{m} \in \text{MATCH}(W)} \|\mathbf{m}\|.$$

There are two connected areas of questions regarding matchings in graphons. One of them concerns their continuity properties. That is, we have a sequence of graphs (or graphons) converging to a graphon W in the cut-distance, and we want to know how their matching ratios/polytons relate to that of W . The other circle of questions

¹The current choice for these functions being not-necessarily symmetric is adopted from [8]. In [8], we have worked out a more general concept of *F-tilings*, which is a collection of vertex-disjoint copies of a fixed graph F . When we take $F = K_2$, we get the notion of matchings. If F is on the vertex set $V(F) = [k]$ then a fractional F -tiling in a graph G is a function $f : V(G)^k \rightarrow \mathbb{R}$ satisfying (1) $f \geq 0$, (2) if $f(x_1, \dots, x_k) > 0$ then for each $ij \in E(F)$ we have that $x_i x_j \in E(G)$, and (3) for every $x \in V(G)$, we have $\sum_{i=1}^k \sum_{\mathbf{y} \in V(G)^k, \mathbf{y}_i = x} f(\mathbf{y}) \leq 1$. Observe that if F is not a complete graph then fractional F -tilings are generally not symmetric under permuting their coordinates, and no symmetrization may be possible.

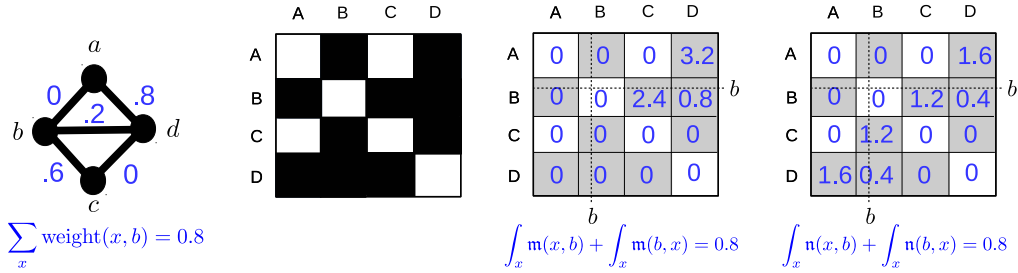


FIGURE 2.1. A graph with a sample fractional matching on the left. Two sample matchings in its graphon representation are shown on the two figures on the right. The “antisymmetric” matching m and the “symmetric” matching n . The formulas below the figures show the correspondence between the total weight at the vertex b , and the corresponding integrals in the representations of the fractional matchings.

lies in investigating the properties of a single matching polyton. The latter is a direct counterpart to investigating the properties of the fractional matching polytope.

Given a finite graph G , we write $\text{match}(G)$ for the matching number of G and $\text{fmatch}(G)$ for the fractional matching number of G . If G has n vertices then we have $0 \leq \text{match}(G) \leq \text{fmatch}(G) \leq \frac{n}{2}$. Our first (easy) result tells us that the normalized fractional matching number of G equals to the matching ratio of its graphon representation.

Proposition 2. *Suppose that $W_G : \Omega^2 \rightarrow [0, 1]$ is a graphon representation of a finite graph G of order n . Then we have $n \cdot \text{match}(W_G) = \text{fmatch}(G)$.*

2.1. Continuity properties. The bad news is that the matching ratio is not continuous with respect to the cut-distance. For example, take Z_n to be constant $\frac{1}{n}$. We have $\text{match}(Z_n) = \frac{1}{2}$, but the sequence $(Z_n)_n$ converges to the zero graphon O for which we have $\text{match}(O) = 0$. Another example is to take Z_n^* to be a graphon representation of a perfect matching of order $2n$ (i.e., the n -th graph has $2n$ vertices and n edges that form a perfect matching). Again, we have $\text{match}(Z_n^*) = \frac{1}{2}$, but the sequence $(Z_n^*)_n$ converges to the zero graphon O . However, the matching ratio is lower-semicontinuous. We state this result in two versions: for sequences of graphs and for sequences of graphons. The graph version reads as follows.

Theorem 3. *Suppose that $(G_n)_n$ is a sequence of graphs of growing orders converging to a graphon $W : \Omega^2 \rightarrow [0, 1]$ in the cut-distance. Then we have that $\liminf_n \frac{\text{fmatch}(G_n)}{v(G_n)} \geq \liminf_n \frac{\text{match}(G_n)}{v(G_n)} \geq \text{match}(W)$.*

Note that the requirement that $v(G_n) \rightarrow \infty$ is necessary. Indeed, for example taking all the graphs G_n to be a copy of a single triangle, we have $\frac{\text{match}(G_n)}{v(G_n)} = \frac{1}{3}$, but for the limit graphon W , which is just a graphon representation of a triangle, we have by Proposition 2 that $\text{match}(W) = \frac{1}{3} \text{fmatch}(K_3) = \frac{1}{2}$. We shall see that Theorem 3 follows from Theorem 4, which we state now.

Theorem 4. *Suppose that $(W_n)_n$ is a sequence of graphons $W_n : \Omega^2 \rightarrow [0, 1]$ converging to a graphon $W : \Omega^2 \rightarrow [0, 1]$ in the cut-distance. Then we have that $\liminf_n \text{match}(W_n) \geq \text{match}(W)$.*

There are two possible paths to proving Theorem 4: one from Theorem 5, and one using the machinery we work out in Section 3. Let us now show how we can use Theorem 4 to prove Theorem 3.

Sketch of proof of Theorem 3. Suppose that $(G_n)_n$ converges to W . Let us consider a sequence of numbers $\epsilon_n > 0$ which tends to zero sufficiently slowly. Let us consider an ϵ_n -regularization (in the sense of the Szemerédi regularity lemma) of each graph G_n . Let R_n be the corresponding cluster graph. Let h_n be the order of R_n . By choosing (ϵ_n) to be decreasing slowly enough, we can achieve that h_n grows arbitrarily slowly. In particular, we can achieve that $h_n \ll v(G_n)$. Let c_n be the size of the clusters in R_n . We have that $c_n h_n \approx v(G_n)$.

Suppose that $\epsilon > 0$ is fixed. Then for n sufficiently large, by Theorem 4 we can find a (graphon-) matching in the graphon representation of R_n (viewed as a finite graph whose edges represent regular pairs of positive density) whose size is at least $\text{match}(W) - \epsilon$. By Proposition 2, we therefore have a fractional matching $f : E(R_n) \rightarrow [0, 1]$ in R_n whose size is at least $(\text{match}(W) - \epsilon) h_n$. We partition each cluster $C \in V(R_n)$ into subsets $(C_D : CD \in E(R_n))$ so that the sizes of these sets are $f(CD) \cdot c_n$, and a possible leftover part C_0 . For each regular pair (X, Y) ($XY \in E(R_n)$), we thus get a subpair (X_Y, Y_X) of two sets, each of size $s_{XY} \approx f(XY) \cdot c_n$. Since a subpair of a regular pair of positive density is a regular pair of positive density,² a (non-spanning version of) Blow-up lemma allows us to find a matching M_{XY} in the graph $G_n[X_Y, Y_X]$ of size approximately s_{XY} . Taking the union of all such matchings, we obtain a matching $M = \bigcup_{XY \in E(R_n)} M_{XY}$ in G_n of size approximately

$$\sum_{XY \in E(R_n)} s_{XY} \approx c_n \sum_{XY \in E(R_n)} f(XY) \geq c_n (\text{match}(W) - \epsilon) h_n \approx (\text{match}(W) - \epsilon) v(G_n),$$

as was needed.

The oversimplification above was in the statement that a subpair (X_Y, Y_X) of an ϵ_n -regular pair (X, Y) is regular. Indeed, in our setting, $\frac{|X_Y|}{|X|} \approx f(XY)$, and it may well happen that this ratio $f(XY)$ is below ϵ_n . In other words, the sets X_Y and Y_X need not be substantial (in the sense of the Szemerédi regularity lemma). Note that this can happen even for all regular pairs represented by edges of R_n . However, we can rescue the situation by using the main result of [7] which asserts that *randomly* selected subpairs of regular pairs are with high probability regular, as long as these subpairs are bigger than a constant (as opposed to size linear in the size of the clusters which is the usual regime when considering ϵ -regular pairs). \square

Our next theorem tells that the matching polyton of a limit is smaller than the “limit” of matching polytons of graphons converging to that limit.

Theorem 5. *Suppose that $(W_n)_n$ is a sequence of graphons, $W_n : \Omega^2 \rightarrow [0, 1]$, converging to a graphon $W : \Omega^2 \rightarrow [0, 1]$ in the cut-norm. Then for every $\mathbf{m} \in \text{MATCH}(W)$*

²Here we oversimplify, see below.

there exists a sequence $(\mathbf{m}_n \in \text{MATCH}(W_n))_n$ which converges to \mathbf{m} in the cut-norm. In particular, $(\mathbf{m}_n)_n$ converges weakly to \mathbf{m} .

Proof sketch. Let $\mathbf{m} \in \text{MATCH}(W)$ be arbitrary. Let $\epsilon > 0$ be given. Let us find a partition $\Omega = \Omega_1 \sqcup \dots \sqcup \Omega_k$ (for a suitable number k) into sets of measure $1/k$ such that the step-functions defined as the averages of \mathbf{m} and W on the partition $\Omega^2 = \bigsqcup_{i,j} \Omega_i \times \Omega_j$ approximate \mathbf{m} and of W in $\mathcal{L}^1(\Omega^2)$, up to an error at most ϵ^2 . Such an approximation is possible since squares generate the sigma-algebra Ω^2 . We shall write \mathbf{m}^{ij} for the average of \mathbf{m} on $\Omega_i \times \Omega_j$. Similarly, we write W^{ij} for the average of W on $\Omega_i \times \Omega_j$. It is easy to see that for all but at most ϵk^2 many pairs (i, j) , we have that

$$(2.2) \quad \int_{\Omega_i \times \Omega_j} |\mathbf{m}(x, y) - \mathbf{m}^{ij}| \leq \frac{\epsilon}{k^2} \quad \text{and} \quad \int_{\Omega_i \times \Omega_j} |W(x, y) - W^{ij}| \leq \frac{\epsilon}{k^2}.$$

In this simplified sketch, let us assume that all the pairs (i, j) satisfy (2.2). The informal interpretation of the second half (2.2) is that W is “quasirandom with density W^{ij} ” on $\Omega_i \times \Omega_j$.

Suppose now that n is large. We want to come up with a matching $\mathbf{m}_n \in \text{MATCH}(W_n)$ which is close to \mathbf{m} in the cut-norm. Since $\|W_n - W\|_{\square}$ is small, for each pair (i, j) we have that $\int_{\Omega_i \times \Omega_j} W_n(x, y) \approx \frac{W^{ij}}{k^2}$. Furthermore, for all $x \in \Omega_i$ (for some small $\delta > 0$) but a set of measure at most δ/k we have

$$(2.3) \quad \int_{y \in \Omega_j} W_n(x, y) \approx \frac{W^{ij}}{k},$$

and that for all $y \in \Omega_j$ but a set of measure at most δ/k we have

$$(2.4) \quad \int_{x \in \Omega_i} W_n(x, y) \approx \frac{W^{ij}}{k}.$$

This follows from the above quasirandomness of W on $\Omega_i \times \Omega_j$ and from the fact that $\|W_n - W\|_{\square}$ is small (which gives that W_n must also be quasirandom on $\Omega_i \times \Omega_j$). Simplifying again, let us assume that (2.3) and (2.4) hold for all x and y with an exact equality.

Let us now define \mathbf{m}_n on the rectangle $\Omega_i \times \Omega_j$. For $(x, y) \in \Omega_i \times \Omega_j$, we set $\mathbf{m}_n(x, y) = \frac{W_n(x, y)}{W^{ij}} \cdot \mathbf{m}^{ij}$. We claim that \mathbf{m}_n is a matching in W_n . Condition (a) of Definition 1 is obvious. Condition (b) follows from the fact that one of the factors in the defining formula for $\mathbf{m}_n(x, y)$ is $W_n(x, y)$. For Condition (c), let us first observe that for $i \in [k]$ and each $x \in \Omega_i$ (except a set of a small measure) we have

$$\begin{aligned} \int_{y \in \Omega} \mathbf{m}_n(x, y) &= \sum_j \int_{y \in \Omega_j} \mathbf{m}_n(x, y) = \sum_j \int_{y \in \Omega_j} \frac{W_n(x, y)}{W^{ij}} \cdot \mathbf{m}^{ij} \\ &\stackrel{(2.3)}{\approx} \sum_j \mathbf{m}^{ij} \int_{y \in \Omega_j} 1 \stackrel{(2.2)}{\approx} \sum_j \int_{y \in \Omega_j} \mathbf{m}(x, y) = \int_{y \in \Omega} \mathbf{m}(x, y), \end{aligned}$$

and that we also have for each $j \in [k]$ and each $y \in \Omega_j$ that $\int_{x \in \Omega} \mathbf{m}_n(x, y) \approx \int_{x \in \Omega} \mathbf{m}(x, y)$. Thus, the fact that Condition (c) is satisfied for the matching \mathbf{m} implies that it is approximately satisfied for \mathbf{m}_n (again, the issue of slight imprecisions can be dealt with in a pedestrian way).

So, it remains to show that $\|\mathbf{m}_n - \mathbf{m}\|_{\square}$ is small for large n . This is a routine calculation which relies on (2.2) and the fact that $\|W_n - W\|_{\square}$ is small. \square

The example of the sequences $(Z_n)_n$ and $(Z_n^*)_n$ from the beginning of Section 2.1 shows that the matching ratio is not upper-semicontinuous. However, it is “upper-semicontinuous in the cut-distance after suitable \mathcal{L}^1 -perturbations”. This is stated below.

Proposition 6. *Suppose that $W : \Omega^2 \rightarrow [0, 1]$ is a graphon. Then for an arbitrary $\epsilon > 0$ there exists a number $\delta > 0$ such that each graphon U with $\|W - U\|_{\square} < \delta$ can be decreased in the $\mathcal{L}^1(\Omega^2)$ -distance by at most ϵ (in a suitable way) so that we obtain a graphon U^* for which $\text{match}(U^*) \leq \text{match}(W) + \epsilon$.*

2.2. Application in parameter estimation. The two main models for estimating parameters (which is the parameter counterpart to testing properties) of graphs and functions are the dense and the bounded-degree models. We first describe the *bounded-degree model*. This is for comparison only, and our results do not have any applications in this model. Suppose that D is an absolute constant, and let \mathcal{G}_D be the class of all finite graphs (modulo isomorphism) with maximum degrees at most D . Let us also write \mathcal{B}_D for the class of all rooted graphs of maximum degree at most D . We say that a parameter $f : \mathcal{G}_D \rightarrow \mathbb{R}$ is *estimable* if for each $\epsilon > 0$ there exists a number $r = r(\epsilon)$ and a function $g : (\mathcal{B}_D)^r \rightarrow \mathbb{R}$ such that for each $G \in \mathcal{G}_D$,

$$\mathbf{P} [|f(G) - g(B_1, B_2, \dots, B_r)| > \epsilon] < \epsilon ,$$

where B_1, \dots, B_r are r many r -balls in G (in the metric induced by the graph structure) rooted at randomly selected vertices. A result of Nguyen and Onak [11] asserts that for each D , the matching ratio is estimable in the class \mathcal{G}_D . This result was reproven by different methods by Elek and Lippner [6] and by Bordanave, Lelarge, and Salez [2].

Let us now turn to the *dense model*. This model is used for testing properties and estimating parameters of dense graphs. Formally, let \mathcal{G} be the class of all graphs (modulo isomorphism). We say that a parameter $f : \mathcal{G} \rightarrow \mathbb{R}$ is *estimable* if for each $\epsilon > 0$ there exists a number $r = r(\epsilon)$ and a function $g : \mathcal{G} \rightarrow \mathbb{R}$ such that for each $G \in \mathcal{G}$

$$(2.5) \quad \mathbf{P} [|f(G) - g(H)| > \epsilon] < \epsilon ,$$

where $H = G[X]$ is the subgraph of G induced by a randomly selected set X of r vertices.

The study of property testing and parameter estimation culminated in [1] where they showed that related questions are equivalent to questions about Szemerédi regularity partitions of graphs. An even more concise characterization of estimability of parameters was provided in [4] in the language of graph limits: *A graph parameter f is estimable if and only if it is continuous in the cut-distance*. Note that in that case, we may extend continuously f to all graphons, and the function g can be chosen as $g = f$.

From the examples of sequences $(Z_n)_n$ and $(Z_n^*)_n$ from the beginning of Section 2.1 we see that the matching ratio is not estimable. However, Theorem 4 and the proof method from Proposition 6 can be used to obtain that the following “robust version” of the matching ratio is estimable. Given an n -vertex graph G and $\epsilon > 0$ we define

$$\text{match}_{\epsilon}(G) = \min \text{match}(G') ,$$

where G' ranges over all subgraphs of G where at most ϵn^2 edges are deleted. Then we have the following.

Theorem 7. For each $\epsilon > 0$, the parameter $G \mapsto \frac{\text{match}_\epsilon(G)}{v(G)}$ is estimable.

The parameter match_ϵ has the following practical interpretation: we want to get the best possible guarantee on the size of a matching in a network, but even when a certain number of links in that network becomes dysfunctional. Related “robustness” questions were considered in [10].

2.3. Structure of matching polyton. For a finite graph G , there are two objects that capture the structure of its matchings, the *matching polytope* $\text{MATCH}(G) \subset \mathbb{R}^{E(G)}$ (which is the convex hull of the set of all matchings) and the *fractional matching polytope* $\text{FMATCH}(G) \subset \mathbb{R}^{E(G)}$ (which is the set of all fractional matchings). Obviously $\text{MATCH}(G) \subset \text{FMATCH}(G)$, but the most important result on their structure is that equality holds if and only if G is bipartite. Since for graphons, there is no distinction between MATCH and FMATCH , there is no such meaningful characterization.³ So, instead we focus on much simpler properties of $\text{MATCH}(W)$. Namely, we want to understand to which extent we have a counterpart of the fact that $\text{MATCH}(G)$ and $\text{FMATCH}(G)$ are compact convex subsets of $\mathbb{R}^{E(G)}$. It is easy to see that $\text{MATCH}(W) \subset \mathcal{L}^1(\Omega^2)$ is a convex set. However, it is not compact. Indeed, consider the graphon $U : \Omega^2 \rightarrow [0, 1]$, where $\Omega = [0, 1]$, defined as

$$U(x, y) = \begin{cases} 1 & \text{if } x + y \leq 1 \\ 0 & \text{if } x + y > 1 \end{cases}.$$

It is easy to see that each function $\mathbf{m}_n \in \mathcal{L}^1(\Omega^2)$ defined as

$$\mathbf{m}_n(x, y) = \begin{cases} n/2 & \text{if } 1 - n^{-1} \leq x + y \leq 1 \\ 0 & \text{if } x + y < 1 - n^{-1} \text{ or } x + y > 1 \end{cases}$$

is a matching in U . However, it is obvious that there is no accumulation point of the sequence $(\mathbf{m}_n)_n$ (in the space $\mathcal{L}^1(\Omega^2)$). Observe also that $\|\mathbf{m}_n\| = \frac{1}{2} - \frac{1}{4n}$, but it can be shown that there exists no matching in U of size $\frac{1}{2}$. This shows that the supremum in (2.1) need not be attained.

3. VERTEX COVERS

We proceed now with the definition of fractional vertex covers of a graphon. Recall that given a finite graph G , a function $c : V(G) \rightarrow [0, 1]$ is a fractional vertex cover if for each edge xy of G we have $c(x) + c(y) \geq 1$. If G is a finite graph then we write $\text{fcov}(G)$ for the size of the minimum fractional vertex cover of G . The graphon counterpart of these concepts is as follows.

Definition 8. Suppose that $W : \Omega^2 \rightarrow [0, 1]$ is a graphon. We say that a function $\mathbf{c} \in \mathcal{L}^\infty(\Omega)$ is a *fractional vertex cover* of W if $0 \leq \mathbf{c} \leq 1$ and the set

$$\text{supp } W \setminus \{(x, y) : \mathbf{c}(x) + \mathbf{c}(y) \geq 1\}$$

has measure 0.

³In Section 3 we shall however see that such a characterization is possible for the dual notion of vertex covers.

Given a fractional vertex cover \mathbf{c} of a graphon W we define its *size*, $\|\mathbf{c}\| = \int_x \mathbf{c}(x)$. We write $\text{FCOV}(W) \subset \mathcal{L}^\infty(\Omega)$ for the set of all fractional vertex covers which we call the *fractional vertex cover polyton*. The fractional vertex cover ratio of a graphon W is defined as

$$\text{fcov}(W) = \inf_{\mathbf{c} \in \text{FCOV}(W)} \|\mathbf{c}\|.$$

As with the concept of matchings, we shall investigate the continuity properties of these notions and the structure of a fractional vertex cover polyton of a single graphon. However, the most important result which connects fractional vertex cover ratio and (fractional) matching ratio is the linear programming duality.

3.1. LP duality. Recall that for a finite graph G , the LP duality asserts that $\text{fmatch}(G) = \text{fcov}(G)$. The graphon version has exactly the same form.

Theorem 9. *Suppose that W is a graphon. Then $\text{match}(W) = \text{fcov}(W)$.*

We shall sketch a proof of Theorem 9 in Section 3.2.

Let us remark that all the versions of the LP duality we have found in the literature were formulated in terms of matrices and vectors in finite-dimensional vector spaces. Thus, Theorem 9 seems to be the first instance of an “analytic LP duality” in which the usual “ $k \times \ell$ -matrix” becomes a measurable function with domain $\Omega \times \Lambda$ for two measurable spaces Ω and Λ . It is interesting to study such an analytic LP duality in general.

3.2. Continuity properties. Combining Theorem 4 and Theorem 9 we immediately get that if a sequence of graphons W_n converges to a graphon W in the cut-distance then $\liminf_n \text{fcov}(W_n) \geq \text{fcov}(W)$. Our next theorem is somewhat more descriptive.

Theorem 10. *Suppose that $(W_n)_n$ is a sequence of graphons $W_n : \Omega^2 \rightarrow [0, 1]$ converging to a graphon $W : \Omega^2 \rightarrow [0, 1]$ in the cut-norm. Suppose that for each n , \mathbf{c}_n is a fractional vertex cover of W_n . Then any accumulation point of the sequence $(\mathbf{c}_n)_n$ in the weak* topology on $\mathcal{L}^\infty(\Omega)$ is a fractional vertex cover of W .*

Theorem 10 has a pedestrian and self-contained proof, which we omit here.

Note that Theorem 10 indeed proves that if a sequence of graphons W_n converges to a graphon W in the cut-distance then $\liminf_n \text{fcov}(W_n) \geq \text{fcov}(W)$. To see this, let us apply on the graphons W_n measure-preserving bijections so that the modified sequence $(W_n)_n$ converges to W in the cut-norm. By selecting a suitable subsequence, let us assume that $\liminf_n \text{fcov}(W_n) = \lim_n \text{fcov}(W_n)$. For each n , consider a fractional vertex cover \mathbf{c}_n of W which is of size at most $\text{fcov}(W_n) + 1/n$. Since the unit ball in $\mathcal{L}^\infty(\Omega)$ with the weak* topology is sequentially compact, there exists at least one accumulation point \mathbf{c} of $(\mathbf{c}_n)_n$, say $\mathbf{c}_{n_i} \rightarrow \mathbf{c}$ for a suitable sequence (n_i) . By lower-semicontinuity of the norm in the weak* topology, we have that $\|\mathbf{c}\| \leq \lim_i \|\mathbf{c}_{n_i}\| = \lim_n \|\mathbf{c}_n\|$. By Theorem 10, \mathbf{c} is a fractional vertex cover of W . Therefore, $\liminf_n \text{fcov}(W_n) \geq \text{fcov}(W)$.

So, the most straightforward path to proving Theorem 4 would be to prove Theorem 10 and Theorem 9 first. Let us show that with Theorem 10, the proof of Theorem 9 is not hard.

Sketch of proof of Theorem 9. The “ \leq ”-direction is easy and its proof mimics the equally easy proof of this direction in the usual LP duality for finite graphs. So, the difficulty lies in the “ \geq ”-direction.

We approximate W by a step-function R (in the cut-norm) with steps of the form $\Omega_i \times \Omega_j$ for a suitable partition $\Omega = \Omega_1 \sqcup \dots \sqcup \Omega_k$. We select the approximation R carefully in such a way that any matching in R can be into a matching in W with only a small loss in size.⁴ R can be interpreted as a finite graph G_R (actually, it is a cluster graph of W), in which the sets Ω_i are vertices, and an edge is present if the value of R on $\Omega_i \times \Omega_j$ is positive. If this approximation was taken fine enough, we have by the above discussion that $\text{fcov}(G_R) \gtrsim k \cdot \text{fcov}(W)$. Note that the right-hand side denotes the size of the smallest fractional vertex cover of a finite graph. Thus, we can employ the usual LP duality and get that $\text{fcov}(G_R) = \text{fmatch}(G_R)$. In particular, we have a fractional matching f on the graph G_R of size $s \gtrsim k \cdot \text{fcov}(W)$. By rescaling, we can turn f into a matching \mathfrak{f} on the graphon R of size exactly $s/k \gtrsim \text{fcov}(W)$. By our careful choice of R , we can turn \mathfrak{f} into a matching whose size is still approximately at least $\text{fcov}(W)$. This concludes the proof. \square

3.3. Structure of fractional vertex cover polyton. We begin with the following basic result.

Lemma 11. *For each graphon W , the set $\text{FCOV}(W)$ is a convex compact in the weak* topology on $\mathcal{L}^\infty(\Omega)$.*

Indeed, convexity is obvious, and compactness follows from Theorem 10 when applied on the constant sequence $(W)_n$. Let us move to some more advanced properties of $\text{FCOV}(W)$. For finite graphs, the most important such property is that the vertices of the fractional vertex cover polytope $\text{FCOV}(G)$ of a graph G are all half-integral, and they are integral if and only if G is bipartite. To formulate a counterpart of this statement, we need to have a counterpart of vertices of a polytope. Suppose that \mathcal{L} is a vector space, and suppose that $X \subset \mathcal{L}$ is a convex set. Recall that a point $x \in X$ is called an *extreme point of X* if the only pair $x', x'' \in X$ for which $x = \frac{1}{2}(x' + x'')$ is the pair $x' = x, x'' = x$. We shall write $\mathcal{E}(X)$ to denote the set of all extreme points of X . The analogy between vertices of polytopes in finite-dimensional vector space and extreme points of a closed compact set $X \subset \mathcal{L}^\infty(\Omega)$ is provided by the Krein–Milman Theorem. Indeed, the Krein–Milman Theorem asserts that X equals to the closure of the convex hull of $\mathcal{E}(X)$.

A fractional vertex cover $\mathfrak{c} \in \text{FCOV}(W)$ is *half-integral* if $\mathfrak{c}(x) \in \{0, \frac{1}{2}, 1\}$ for almost every $x \in \Omega$. *Integral* vertex covers are defined analogously. Thus, our results are as follows.

Theorem 12. *Let W be a graphon. Then all the extreme points of $\text{FCOV}(W)$ are half-integral. Further, they are integral if and only if W is bipartite.*

The proof of Theorem 12 is inspired by the well-known proof of the finite graph version of this result. Let us illustrate the technique on sketching one (and the easiest) substatement, *if W is a bipartite graphon with bipartition $\Omega = A \sqcup B$ then all its extreme points are integral*. To this end, let us consider $\mathfrak{c} \in \text{FCOV}(W)$. For an arbitrary number $x \in [0, 1]$, define $\partial(x) = \min(x, 1 - x)$. It can be checked that

$$\mathfrak{c}_1(x) = \begin{cases} \mathfrak{c}(x) + \partial(\mathfrak{c}(x)) & \text{for } x \in A \\ \mathfrak{c}(x) - \partial(\mathfrak{c}(x)) & \text{for } x \in B \end{cases} \quad \text{and} \quad \mathfrak{c}_2(x) = \begin{cases} \mathfrak{c}(x) - \partial(\mathfrak{c}(x)) & \text{for } x \in A \\ \mathfrak{c}(x) + \partial(\mathfrak{c}(x)) & \text{for } x \in B \end{cases},$$

⁴The choice of the approximation R is non-trivial. Indeed, the fact that the matching ratio is not upper-semicontinuous on graphons tells us that not every choice R is good.

defines two fractional vertex covers of W . We have $\mathbf{c} = \frac{1}{2}(\mathbf{c}_1 + \mathbf{c}_2)$. If \mathbf{c} is not integral, then $\mathbf{c}_1 \neq \mathbf{c}_2$. Thus, in that case \mathbf{c} is not an extreme point of $\text{FCOV}(W)$.

4. ACKNOWLEDGMENTS

This is an extended abstract. The papers containing the full description of the results and detailed proofs can be found in [8, 5].

The contents of this publication reflects only the authors' views and not necessarily the views of the European Commission of the European Union.

REFERENCES

- [1] N. Alon, E. Fischer, I. Newman, and A. Shapira. A combinatorial characterization of the testable graph properties: it's all about regularity. *SIAM J. Comput.*, 39(1):143–167, 2009.
- [2] Ch. Bordenave, M. Lelarge, and J. Salez. Matchings on infinite graphs. *Probab. Theory Related Fields*, 157(1-2):183–208, 2013.
- [3] C. Borgs, J. Chayes, L. Lovász, V. T. Sós, B. Szegedy, and K. Vesztergombi. Graph limits and parameter testing. In *STOC'06: Proceedings of the 38th Annual ACM Symposium on Theory of Computing*, pages 261–270. ACM, New York, 2006.
- [4] C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, and K. Vesztergombi. Convergent sequences of dense graphs. I. Subgraph frequencies, metric properties and testing. *Adv. Math.*, 219(6):1801–1851, 2008.
- [5] M. Doležal and J. Hladký. Matching polytopons. Preprint.
- [6] G. Elek and G. Lippner. Borel oracles. An analytical approach to constant-time algorithms. *Proc. Amer. Math. Soc.*, 138(8):2939–2947, 2010.
- [7] S. Gerke, Y. Kohayakawa, V. Rödl, and A. Steger. Small subsets inherit sparse ϵ -regularity. *J. Combin. Theory Ser. B*, 97(1):34–56, 2007.
- [8] J. Hladký, P. Hu, and D. Piguet. Tilings in graphons. Preprint.
- [9] L. Lovász and B. Szegedy. Limits of dense graph sequences. *J. Combin. Theory Ser. B*, 96(6):933–957, 2006.
- [10] L. Lovász and B. Szegedy. Testing properties of graphs and functions. *Israel J. Math.*, 178:113–156, 2010.
- [11] H. N. Nguyen and K. Onak. Constant-Time Approximation Algorithms via Local Improvements. In *49th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2008, October 25-28, 2008, Philadelphia, PA, USA*, pages 327–336, 2008.

INSTITUTE OF MATHEMATICS, CZECH ACADEMY OF SCIENCE. ŽITNÁ 25, 110 00, PRAHA, CZECH REPUBLIC. THE INSTITUTE OF MATHEMATICS OF THE CZECH ACADEMY OF SCIENCES IS SUPPORTED BY RVO:67985840.

E-mail address: dolezal@math.cas.cz

E-mail address: honzahladky@gmail.com

DEPARTMENT OF COMPUTER SCIENCE, UNIVERSITY OF WARWICK, COVENTRY CV4 7AL, UNITED KINGDOM.

E-mail address: pinghu@dcs.warwick.ac.uk

THE CZECH ACADEMY OF SCIENCES, INSTITUTE OF COMPUTER SCIENCE, POD VODÁRENSKOU VĚŽÍ 2, 182 07 PRAGUE, CZECH REPUBLIC. WITH INSTITUTIONAL SUPPORT RVO:67985807.

E-mail address: piguet@cs.cas.cz