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Ondřej Kreml
Šárka Nečasová
Tomasz Piasecki

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Ondřej Kreml ^{1*} Šárka Nečasová ^{1*} Tomasz Piasecki ^{2†}

¹Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Praha 1, Czech Republic

² Institute of Mathematics, Polish Academy of Sciences
Śniadeckich 8, 00-656 Warszawa, Poland

Abstract

We consider the compressible Navier-Stokes system on time-dependent domains with prescribed motion of the boundary, supplemented with slip boundary conditions for the velocity. We derive the relative entropy inequality in the spirit of [7] for the system on moving domain and use it to prove the weak-strong uniqueness property.

Keywords: compressible Navier-Stokes equations, time-dependent domain, relative entropy, weak-strong uniqueness

1 Introduction

The flow of a compressible barotropic viscous fluid is in the absence of external forces described by the following system of partial differential equations

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \quad (1.1)$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}). \quad (1.2)$$

Here ϱ denotes the density of the fluid and \mathbf{u} is the velocity. The stress tensor \mathbb{S} is determined by the standard Newton rheological law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \eta \operatorname{div}_x \mathbf{u} \mathbb{I} \quad (1.3)$$

with $\mu > 0$ and $\eta \geq 0$ being constants.

The pressure $p(\varrho)$ is a given continuously differentiable function of the density. We introduce the pressure potential as

$$H(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} dz. \quad (1.4)$$

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We are interested in properties of solutions to the system of equations (1.1)-(1.2) on a moving domain $\Omega = \Omega_t$ with prescribed movement of the boundary. More precisely, the boundary of the domain Ω_t occupied by the fluid is assumed to be described by means of a given velocity field $\mathbf{V}(t, x)$, where $t \geq 0$ and $x \in \mathbb{R}^3$. Assuming \mathbf{V} is regular, we solve the associated system of differential equations

$$\frac{d}{dt} \mathbf{X}(t, x) = \mathbf{V}(t, \mathbf{X}(t, x)), \quad t > 0, \quad \mathbf{X}(0, x) = x, \quad (1.5)$$

and set

$$\Omega_\tau = \mathbf{X}(\tau, \Omega_0), \quad \text{where } \Omega_0 \subset \mathbb{R}^3 \text{ is a given domain, } \Gamma_\tau = \partial\Omega_\tau, \text{ and } Q_\tau = \{(t, x) \mid t \in (0, \tau), x \in \Omega_\tau\}.$$

The impermeability of the boundary of the physical domain is described by the condition

$$(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}|_{\Gamma_\tau} = 0 \text{ for any } \tau \geq 0, \quad (1.6)$$

where $\mathbf{n}(t, x)$ denotes the unit outer normal vector to the boundary Γ_t .

Moreover, we prescribe the Navier slip boundary conditions, i.e.

$$([\mathbf{S}\mathbf{n}]_{\tan} + \kappa [\mathbf{u} - \mathbf{V}]_{\tan})|_{\Gamma_\tau} = 0, \quad (1.7)$$

where $\kappa \geq 0$ represents a friction coefficient. In particular the choice $\kappa = 0$ yields the complete slip boundary condition.

Finally, the system of equations (1.1)-(1.2) is supplemented by the initial conditions

$$\varrho(0, \cdot) = \varrho_0, \quad (\varrho \mathbf{u})(0, \cdot) = (\varrho \mathbf{u})_0 \quad \text{in } \Omega_0. \quad (1.8)$$

Global weak solutions to the compressible barotropic Navier–Stokes system on a fixed domain were proved to exist in a pioneering work by Lions [15]. This theory was later extended by Feireisl and collaborators ([10], [4], [5], [6]) to cover larger class of pressure laws. The existence theory in the case of moving domains was developed in [8] for no-slip boundary conditions using the so-called Brinkman penalization and in [9] for slip boundary conditions.

There is also quite large amount of literature available concerning existence of strong solutions for the system (1.1)–(1.2) (or even for more complex systems involving also heat conductivity of the fluid) on a fixed domain, such solutions are proved to exist either locally in time or globally provided the initial data are sufficiently close to a rest state, let us mention among others [17], [18], [21], [22]. All these results are proved under the no-slip boundary conditions. The only global existence result without smallness assumptions on the data has been obtained by Vaigant and Kazhikov [20] for space periodic boundary conditions in two space dimensions. The case of slip boundary condition on fixed domain was considered by Zajaczkowski [24] and Hoff [13]. Even local-in-time existence results of strong solutions on moving domains (regardless of boundary behavior) seems to be an open question at this moment and it will be a matter of our forthcoming paper [14].

The concept of relative entropies has been successfully used in the context of partial differential equations (see among others Carillo et al. [1], Masmoudi [16], Saint-Raymond [19], Wang and Jiang [23]). Germain [12] introduced a notion of solution to the system (1.1)–(1.2) based on a relative entropy inequality with respect to a hypothetical strong solution. Similar idea was adapted by Feireisl et al. [11] who defined a suitable weak solution to the barotropic Navier–Stokes system based on a general relative entropy inequality with respect to any sufficiently smooth pair of functions. In [7] the authors used relative entropy inequality to prove the weak-strong uniqueness property. Doboszczak [3] proved both the relative entropy inequality as well as the weak-strong uniqueness property in the case of moving domain and no-slip boundary condition.

In this paper we first prove that weak solutions to the system (1.1)–(1.2) on moving domains proved to exist in [9] possess also an energy inequality (Theorem 3.1). Using this energy inequality we prove the relative entropy (energy) inequality (Theorem 4.1) and finally the weak-strong uniqueness property (Theorem 5.1).

2 Preliminaries

For simplicity we assume throughout the paper that $\kappa, \eta = 0$, however all the results hold (with appropriate modifications of formulae) also with $\kappa, \eta > 0$.

We define weak solutions to the compressible Navier-Stokes system on moving domains as follows.

Definition 2.1 *We say that the couple (ϱ, \mathbf{u}) is a weak solution of problem (1.1)-(1.2) with boundary conditions (1.6)-(1.7) and initial conditions (1.8) if*

- $\varrho \in L^\infty(0, T; L^\gamma(\mathbb{R}^3))$, $\varrho \geq 0$ a.e. in Q_T .
- $\mathbf{u}, \nabla_x \mathbf{u} \in L^2(Q_T)$, $(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}(\tau, \cdot)|_{\Gamma_\tau} = 0$ for a.a. $\tau \in [0, T]$.
- The continuity equation (1.1) is satisfied in the whole \mathbb{R}^3 provided the density is extended by zero outside Ω_t , i.e.

$$\int_{\Omega_\tau} \varrho \varphi(\tau, \cdot) \, dx - \int_{\Omega_0} \varrho_0 \varphi(0, \cdot) \, dx = \int_0^\tau \int_{\Omega_t} (\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi) \, dx dt \quad (2.1)$$

for any $\tau \in [0, T]$ and any test function $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^3)$.

- Moreover, equation (1.1) is also satisfied in the sense of renormalized solutions introduced by DiPerna and Lions [2]:

$$\int_{\Omega_\tau} b(\varrho) \varphi(\tau, \cdot) \, dx - \int_{\Omega_0} b(\varrho_0) \varphi(0, \cdot) \, dx = \int_0^\tau \int_{\Omega_t} (b(\varrho) \partial_t \varphi + b(\varrho) \mathbf{u} \cdot \nabla_x \varphi + (b(\varrho) - b'(\varrho) \varrho) \operatorname{div}_x \mathbf{u} \varphi) \, dx dt \quad (2.2)$$

for any $\tau \in [0, T]$, any $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^3)$, and any $b \in C^1[0, \infty)$, $b(0) = 0$, $b'(r) = 0$ for large r .

- The momentum equation (1.2) is replaced by the family of integral identities

$$\begin{aligned} & \int_{\Omega_\tau} \varrho \mathbf{u} \cdot \boldsymbol{\varphi}(\tau, \cdot) \, dx - \int_{\Omega_0} (\varrho \mathbf{u})_0 \cdot \boldsymbol{\varphi}(0, \cdot) \, dx \\ &= \int_0^\tau \int_{\Omega_t} (\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \boldsymbol{\varphi} + p(\varrho) \operatorname{div}_x \boldsymbol{\varphi} - \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \boldsymbol{\varphi}) \, dx dt \end{aligned} \quad (2.3)$$

for any $\tau \in [0, T]$ and any test function $\boldsymbol{\varphi} \in C_c^\infty([0, T] \times \mathbb{R}^3)$ satisfying

$$\boldsymbol{\varphi} \cdot \mathbf{n}|_{\Gamma_\tau} = 0 \text{ for any } \tau \in [0, T]. \quad (2.4)$$

The existence of weak solutions to the problem (1.1)-(1.8) was proved in [9].

Theorem 2.1 *Let $\Omega_0 \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\nu}$, and let $\mathbf{V} \in C^1([0, T]; C_c^3(\mathbb{R}^3))$ be given. Assume that the pressure $p \in C[0, \infty) \cap C^1(0, \infty)$ satisfies*

$$p(0) = 0, \quad p'(\varrho) > 0 \text{ for any } \varrho > 0, \quad \lim_{\varrho \rightarrow \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0 \text{ for a certain } \gamma > 3/2.$$

Let the initial data satisfy

$$\varrho_0 \in L^\gamma(\mathbb{R}^3), \quad \varrho_0 \geq 0, \quad \varrho_0 \not\equiv 0, \quad \varrho_0|_{\mathbb{R}^3 \setminus \Omega_0} = 0, \quad (\varrho \mathbf{u})_0 = 0 \text{ a.a. on the set } \{\varrho_0 = 0\}, \quad \int_{\Omega_0} \frac{1}{\varrho_0} |(\varrho \mathbf{u})_0|^2 \, dx < \infty.$$

Then the problem (1.1)-(1.8) admits a weak solution on any time interval $(0, T)$ in the sense specified through Definition 2.1.

3 Energy inequality

Our first result is a crucial observation allowing for proving the main theorem about weak-strong uniqueness.

Theorem 3.1 *Let the assumptions of Theorem 2.1 be satisfied.*

Then the problem (1.1)-(1.8) admits a weak solution on any time interval $(0, T)$ in the sense specified through Definition 2.1, satisfying moreover the energy inequality in the following form

$$\begin{aligned}
& \int_{\Omega_\tau} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) \right) (\tau, \cdot) \, dx + \frac{1}{2} \int_0^\tau \int_{\Omega_t} \mu \left| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{1} \right|^2 \, dx dt \\
& \leq \int_{\Omega_0} \left(\frac{1}{2\varrho_0} |(\varrho \mathbf{u})_0|^2 + H(\varrho_0) \right) \, dx + \int_{\Omega_\tau} (\varrho \mathbf{u} \cdot \mathbf{V})(\tau, \cdot) \, dx - \int_{\Omega_0} (\varrho \mathbf{u})_0 \cdot \mathbf{V}(0, \cdot) \, dx \\
& + \int_0^\tau \int_{\Omega_t} \left(\mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{1} \right) : \nabla_x \mathbf{V} - \varrho \mathbf{u} \cdot \partial_t \mathbf{V} - \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \mathbf{V} - p(\varrho) \operatorname{div}_x \mathbf{V} \right) \, dx dt.
\end{aligned} \tag{3.1}$$

Proof: We follow the same series of approximations and penalizations as it is introduced in [9, Section 3] in the proof of Theorem 2.1. The starting point is thus the modified energy inequality written on the fixed domain B which is a ball large enough such that $\Omega_t \subset B$ for all $t \in [0, T]$ and $\mathbf{V} = 0$ on ∂B , see formula (3.10) in [9]

$$\begin{aligned}
& \int_B \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) + \frac{\delta}{\beta - 1} \varrho^\beta \right) (\tau, \cdot) \, dx + \frac{1}{2} \int_0^\tau \int_B \mu_\omega \left| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{1} \right|^2 \, dx dt \\
& + \frac{1}{\varepsilon} \int_0^\tau \int_{\Gamma_t} |(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}|^2 \, dS_x \, dt \leq \int_B \left(\frac{1}{2\varrho_{0,\delta}} |(\varrho \mathbf{u})_{0,\delta}|^2 + H(\varrho_{0,\delta}) + \frac{\delta}{\beta - 1} \varrho_{0,\delta}^\beta \right) \, dx \\
& + \int_B \left((\varrho \mathbf{u} \cdot \mathbf{V})(\tau, \cdot) - (\varrho \mathbf{u})_{0,\delta} \cdot \mathbf{V}(0, \cdot) \right) \, dx \\
& + \int_0^\tau \int_B \left(\mu_\omega \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{1} \right) : \nabla_x \mathbf{V} - \varrho \mathbf{u} \cdot \partial_t \mathbf{V} - \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \mathbf{V} - p(\varrho) \operatorname{div}_x \mathbf{V} - \frac{\delta}{\beta - 1} \varrho^\beta \operatorname{div}_x \mathbf{V} \right) \, dx dt.
\end{aligned} \tag{3.2}$$

Passing first with ε to zero, it is not difficult to observe that using the a priori estimates available, all the terms on the right hand side of (3.2) converge to their counterparts. On the left hand side the last term is positive and thus can be omitted. Finally, using the convexity of $\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}$ we have

$$\int_0^T \int_B \mathbb{S}_\omega(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, dx dt \leq \liminf_{\varepsilon \rightarrow 0} \int_0^T \int_B \mathbb{S}_\omega(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon \, dx dt. \tag{3.3}$$

Next, passing with ω to zero, we first observe that all the terms which include the density can be rewritten as the integrals over Ω_t instead of integrals over B using the fundamental Lemma 4.1 in [9]. The viscosity term on the right hand side can be treated easily, in particular the integral over $B \setminus \Omega_t$ vanish due to the fact that $\mu_\omega \rightarrow 0$ on this set. On the left hand side we split the integral of the term with stress tensor into two parts, the integral over $B \setminus \Omega_t$ can be omitted since it is positive and on Ω_t we use the fact that $\mu_\omega = \mu$ is constant and thus we can use again the convexity of $\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}$ to obtain similar inequality as (3.3).

Finally, we pass with δ to zero. Here we use the - nowadays already standard - results of [5] to pass to the limit on the right hand side and to adjust the initial conditions, while on the left hand side we use the weak lower semicontinuity of the energy at time τ .

Q.E.D.

4 Relative energy inequality

Having now already the energy inequality, we can deduce the relative energy inequality in the spirit of [7]. Before stating the theorem, we introduce some notation. For a weak solution (ϱ, \mathbf{u}) and a pair of test functions (r, \mathbf{U}) defined on Q_T we define the relative energy $\mathcal{E}([\varrho, \mathbf{u}][r, \mathbf{U}])$ as

$$\mathcal{E}([\varrho, \mathbf{u}][r, \mathbf{U}]) (\tau) = \int_{\Omega_\tau} \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + H(\varrho) - H'(r)(\varrho - r) - H(r) \right) (\tau, \cdot) dx. \quad (4.1)$$

We prove the following

Theorem 4.1 *Let (ϱ, \mathbf{u}) be a weak solution to the compressible Navier-Stokes system (1.1)-(1.8) constructed in Theorem 3.1. Then (ϱ, \mathbf{u}) satisfies the following relative energy inequality*

$$\begin{aligned} & \mathcal{E}([\varrho, \mathbf{u}][r, \mathbf{U}]) (\tau) + \int_0^\tau \int_{\Omega_t} (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) dx dt \\ & \leq \mathcal{E}([\varrho_0, \mathbf{u}_0][r(0, \cdot), \mathbf{U}(0, \cdot)]) + \int_0^\tau \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U})(t) dt \end{aligned} \quad (4.2)$$

for a.a. $\tau \in (0, T)$ and any pair of test functions (r, \mathbf{U}) such that $\mathbf{U} \in C_c^\infty(\overline{Q_T})$, $\mathbf{U} \cdot \mathbf{n} = \mathbf{V} \cdot \mathbf{n}$ on Γ_t for $t \in [0, T]$, $r \in C_c^\infty(\overline{Q_T})$, $r > 0$. The remainder term \mathcal{R} is given by

$$\begin{aligned} \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U})(t) &= \int_{\Omega_t} \varrho (\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U}) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) dx \\ &+ \int_{\Omega_t} \operatorname{div}_x \mathbf{U} (p(r) - p(\varrho)) + (r - \varrho) \partial_t H'(r) + (r \mathbf{U} - \varrho \mathbf{u}) \cdot \nabla_x H'(r) dx \end{aligned} \quad (4.3)$$

Proof: The idea of the proof is the same as in the original paper [7]. We will combine the energy inequality (3.1) provided by Theorem 3.1 together with weak formulations of the continuity and momentum equations with suitable test functions. Since \mathbf{U} is not a proper test function in the momentum equation due to its boundary condition, we test the momentum equation with $\boldsymbol{\varphi} = \mathbf{U} - \mathbf{V}$ to obtain

$$\begin{aligned} & \int_{\Omega_\tau} \varrho \mathbf{u} \cdot (\mathbf{U} - \mathbf{V})(\tau, \cdot) dx - \int_{\Omega_0} (\varrho \mathbf{u})_0 \cdot (\mathbf{U} - \mathbf{V})(0, \cdot) dx \\ &= \int_0^\tau \int_{\Omega_t} (\varrho \mathbf{u} \cdot \partial_t (\mathbf{U} - \mathbf{V}) + \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x (\mathbf{U} - \mathbf{V}) + p(\varrho) \operatorname{div}_x (\mathbf{U} - \mathbf{V}) - \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x (\mathbf{U} - \mathbf{V})) dx dt. \end{aligned} \quad (4.4)$$

Subtracting (4.4) from the energy inequality (3.1) we obtain

$$\begin{aligned} & \int_{\Omega_\tau} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) - \varrho \mathbf{u} \cdot \mathbf{U} \right) (\tau, \cdot) dx - \int_{\Omega_0} \frac{1}{2} \varrho_0 |\mathbf{u}|^2 + H(\varrho_0) - (\varrho \mathbf{u})_0 \cdot \mathbf{U}(0, \cdot) dx \\ &+ \int_0^\tau \int_{\Omega_t} \mathbb{S}(\nabla_x \mathbf{u}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) dx dt \leq \int_0^\tau \int_{\Omega_t} (-\varrho \mathbf{u} \cdot \partial_t \mathbf{U} - \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \mathbf{U} - p(\varrho) \operatorname{div}_x \mathbf{U}) dx dt. \end{aligned} \quad (4.5)$$

Next, we use in the continuity equation as a test function the quantities $\frac{1}{2} |\mathbf{U}|^2$ and $H'(r)$ respectively to obtain

$$\int_{\Omega_\tau} \frac{1}{2} \varrho |\mathbf{U}|^2 (\tau, \cdot) dx - \int_{\Omega_0} \frac{1}{2} \varrho_0 |\mathbf{U}|^2 (0, \cdot) dx = \int_0^\tau \int_{\Omega_t} (\varrho \mathbf{U} \cdot \partial_t \mathbf{U} + \varrho \mathbf{u} \cdot \nabla_x \mathbf{U} \cdot \mathbf{U}) dx dt \quad (4.6)$$

and

$$\int_{\Omega_\tau} \varrho H'(r)(\tau, \cdot) \, dx - \int_{\Omega_0} \varrho_0 H'(r)(0, \cdot) \, dx = \int_0^\tau \int_{\Omega_t} (\varrho \partial_t H'(r) + \varrho \mathbf{u} \cdot \nabla_x H'(r)) \, dx dt \quad (4.7)$$

Adding (4.6) and subtracting (4.7) from (4.5) we obtain

$$\begin{aligned} & \int_{\Omega_\tau} \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + H(\varrho) - H'(r)\varrho \right) (\tau, \cdot) \, dx - \int_{\Omega_0} \frac{1}{2\varrho_0} |(\varrho \mathbf{u})_0 - \varrho_0 \mathbf{U}(0, \cdot)|^2 + H(\varrho_0) - H'(r(0, \cdot))\varrho_0 \, dx \\ & + \int_0^\tau \int_{\Omega_t} \mathbb{S}(\nabla_x \mathbf{u}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \, dx dt \leq \int_0^\tau \int_{\Omega_t} ((\varrho \partial_t \mathbf{U} + \varrho \mathbf{u} \cdot \nabla_x \mathbf{U}) \cdot (\mathbf{U} - \mathbf{u}) - p(\varrho) \operatorname{div}_x \mathbf{U}) \, dx dt \\ & - \int_0^\tau \int_{\Omega_t} (\varrho \partial_t H'(r) + \varrho \mathbf{u} \cdot \nabla_x H'(r)) \, dx dt. \end{aligned} \quad (4.8)$$

Observing that the definition (1.4) implies

$$p(r) = rH'(r) - H(r), \quad (4.9)$$

we immediately achieve

$$\partial_t p(r) = r \partial_t H'(r). \quad (4.10)$$

Hence, the inequality (4.8) can be further rewritten as

$$\begin{aligned} & \int_{\Omega_\tau} \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + H(\varrho) - H'(r)\varrho \right) (\tau, \cdot) \, dx - \int_{\Omega_0} \frac{1}{2\varrho_0} |(\varrho \mathbf{u})_0 - \varrho_0 \mathbf{U}(0, \cdot)|^2 + H(\varrho_0) - H'(r(0, \cdot))\varrho_0 \, dx \\ & + \int_0^\tau \int_{\Omega_t} (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \, dx dt + \int_0^\tau \int_{\Omega_t} \partial_t p(r) \, dx dt \\ & \leq \int_0^\tau \int_{\Omega_t} ((\varrho \partial_t \mathbf{U} + \varrho \mathbf{u} \cdot \nabla_x \mathbf{U}) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U})) \, dx dt \\ & + \int_0^\tau \int_{\Omega_t} ((r - \varrho) \partial_t H'(r) - p(\varrho) \operatorname{div}_x \mathbf{U} - \varrho \mathbf{u} \cdot \nabla_x H'(r)) \, dx dt. \end{aligned} \quad (4.11)$$

Now we claim that the following identity holds

$$\int_{\Omega_t} p(r) \operatorname{div}_x \mathbf{U} + r \mathbf{U} \cdot \nabla_x H'(r) \, dx = \int_{\Omega_t} \operatorname{div}_x (\mathbf{V} p(r)) \, dx. \quad (4.12)$$

Indeed, using the boundary condition $\mathbf{U} \cdot \mathbf{n} = \mathbf{V} \cdot \mathbf{n}$ we write

$$\begin{aligned} \int_{\Omega_t} p(r) \operatorname{div}_x \mathbf{U} \, dx & = \int_{\Omega_t} p(r) \operatorname{div}_x (\mathbf{U} - \mathbf{V}) \, dx + \int_{\Omega_t} p(r) \operatorname{div}_x \mathbf{V} \, dx \\ & = - \int_{\Omega_t} \mathbf{U} \cdot \nabla_x p(r) \, dx + \int_{\Omega_t} \operatorname{div}_x (\mathbf{V} p(r)) \, dx = - \int_{\Omega_t} r \mathbf{U} \cdot \nabla_x H'(r) \, dx + \int_{\Omega_t} \operatorname{div}_x (\mathbf{V} p(r)) \, dx, \end{aligned} \quad (4.13)$$

where we used (4.9) as well. Adding (4.12) to (4.11) we obtain

$$\begin{aligned}
& \int_{\Omega_\tau} \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + H(\varrho) - H'(r)\varrho \right) (\tau, \cdot) \, dx - \int_{\Omega_0} \frac{1}{2\varrho_0} |(\varrho \mathbf{u})_0 - \varrho_0 \mathbf{U}(0, \cdot)|^2 + H(\varrho_0) - H'(r(0, \cdot))\varrho_0 \, dx \quad (4.14) \\
& + \int_0^\tau \int_{\Omega_t} (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \, dx dt + \int_0^\tau \int_{\Omega_t} (\partial_t p(r) + \operatorname{div}_x(\mathbf{V}p(r))) \, dx dt \\
& \leq \int_0^\tau \int_{\Omega_t} ((\varrho \partial_t \mathbf{U} + \varrho \mathbf{u} \cdot \nabla_x \mathbf{U}) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U})) \, dx dt \\
& + \int_0^\tau \int_{\Omega_t} ((r - \varrho) \partial_t H'(r) + (p(r) - p(\varrho)) \operatorname{div}_x \mathbf{U} + (r\mathbf{U} - \varrho \mathbf{u}) \cdot \nabla_x H'(r)) \, dx dt.
\end{aligned}$$

The proof of Theorem 4.1 is finished observing that standard transport theorem yields the identity

$$\begin{aligned}
& \int_0^\tau \int_{\Omega_t} (\partial_t p(r) + \operatorname{div}_x(\mathbf{V}p(r))) \, dx dt = \int_0^\tau \frac{d}{dt} \int_{\Omega_t} p(r) \, dx dt \quad (4.15) \\
& = \int_{\Omega_\tau} p(r)(\tau, \cdot) \, dx - \int_{\Omega_0} p(r)(0, \cdot) \, dx = \int_{\Omega_\tau} (rH'(r) - H(r))(\tau, \cdot) \, dx - \int_{\Omega_0} (rH'(r) - H(r))(0, \cdot) \, dx.
\end{aligned}$$

Q.E.D.

Note that the class of admissible test functions (r, \mathbf{U}) can be extended by density arguments in a similar manner as in [7, Section 3.2.2].

5 Weak-strong uniqueness

Finally, we prove the weak-strong uniqueness principle.

Theorem 5.1 *Let (ϱ, \mathbf{u}) be a weak solution to the compressible Navier-Stokes system (1.1)-(1.8) constructed in Theorem 3.1. Let $(\tilde{\varrho}, \tilde{\mathbf{u}})$ be a strong solution to the same problem satisfying*

$$0 < \inf_{Q_T} \tilde{\varrho} \leq \sup_{Q_T} \tilde{\varrho} < \infty \quad (5.1)$$

$$\nabla_x \tilde{\varrho} \in L^2(0, T; L^q(\Omega_t)), \quad \nabla_x^2 \tilde{\mathbf{u}} \in L^2(0, T, L^q(\Omega_t)) \quad (5.2)$$

with $q > \max\{3; \frac{6\gamma}{5\gamma-6}\}$, and emanating from the same initial data. Then

$$\varrho = \tilde{\varrho}, \quad \mathbf{u} = \tilde{\mathbf{u}} \quad \text{in } Q_T. \quad (5.3)$$

Proof: The proof follows the same ideas as in [7], however we present it here for completeness of presentation. Plugging in $(r, \mathbf{U}) = (\tilde{\varrho}, \tilde{\mathbf{u}})$ in the relative energy inequality (4.2) we obtain

$$\begin{aligned}
& \mathcal{E}([\varrho, \mathbf{u}][\tilde{\varrho}, \tilde{\mathbf{u}}])(\tau) + \int_0^\tau \int_{\Omega_t} (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \tilde{\mathbf{u}})) : (\nabla_x \mathbf{u} - \nabla_x \tilde{\mathbf{u}}) \, dx dt \quad (5.4) \\
& \leq \int_0^\tau \int_{\Omega_t} \varrho (\partial_t \tilde{\mathbf{u}} + \mathbf{u} \cdot \nabla_x \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) + \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) : (\nabla_x \tilde{\mathbf{u}} - \nabla_x \mathbf{u}) \, dx dt \\
& + \int_0^\tau \int_{\Omega_t} \operatorname{div}_x \tilde{\mathbf{u}} (p(\tilde{\varrho}) - p(\varrho)) + (\tilde{\varrho} - \varrho) \partial_t H'(\tilde{\varrho}) + (\tilde{\varrho} \tilde{\mathbf{u}} - \varrho \mathbf{u}) \cdot \nabla_x H'(\tilde{\varrho}) \, dx dt.
\end{aligned}$$

Using the strong formulation of the momentum and continuity equations we find out that

$$\partial_t \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla_x \tilde{\mathbf{u}} = \frac{1}{\tilde{\varrho}} \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) - \nabla_x H'(\tilde{\varrho}) \quad (5.5)$$

in Q_T . Moreover, multiplying the strong formulation of the continuity equation by $H''(\tilde{\varrho})$ we obtain

$$\partial_t H'(\tilde{\varrho}) + \tilde{\mathbf{u}} \cdot \nabla_x H'(\tilde{\varrho}) = -\operatorname{div}_x \tilde{\mathbf{u}} \tilde{\varrho} H''(\tilde{\varrho}) = -\operatorname{div}_x \tilde{\mathbf{u}} p'(\tilde{\varrho}). \quad (5.6)$$

Finally, integrating by parts we have for a.a. $t \in (0, \tau)$

$$\int_{\Omega_t} \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) : (\nabla_x \tilde{\mathbf{u}} - \nabla_x \mathbf{u}) \, dx = - \int_{\Omega_t} \operatorname{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \, dx, \quad (5.7)$$

where the boundary integral vanishes due to boundary condition (1.7) and the fact, that $(\tilde{\mathbf{u}} - \mathbf{u}) \cdot \mathbf{n} = 0$ on Γ_t . Combining (5.4), (5.5), (5.6) and (5.7) we arrive to the following version of the relative energy inequality

$$\begin{aligned} & \mathcal{E}([\varrho, \mathbf{u}] | [\tilde{\varrho}, \tilde{\mathbf{u}}]) (\tau) + \int_0^\tau \int_{\Omega_t} (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \tilde{\mathbf{u}})) : (\nabla_x \mathbf{u} - \nabla_x \tilde{\mathbf{u}}) \, dx dt \\ & \leq \int_0^\tau \int_{\Omega_t} \varrho (\mathbf{u} - \tilde{\mathbf{u}}) \cdot \nabla_x \tilde{\mathbf{u}} \cdot (\tilde{\mathbf{u}} - \mathbf{u}) - \operatorname{div}_x \tilde{\mathbf{u}} (p(\varrho) - p'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - p(\tilde{\varrho})) \, dx dt \\ & + \int_0^\tau \int_{\Omega_t} \frac{1}{\tilde{\varrho}} (\varrho - \tilde{\varrho}) \operatorname{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \, dx dt. \end{aligned} \quad (5.8)$$

Now we would like to show that all the terms on the right-hand side of (5.8) can be absorbed by the left hand side and then use the Gronwall lemma. To do that we need the following estimate which can be easily checked

$$\begin{aligned} H(\varrho) - H'(r)(\varrho - r) - H(r) & \geq c(r)(\varrho - r)^2 \quad \text{for } \frac{r}{2} < \varrho < 2r \\ & \geq c(r)(1 + \varrho^\gamma) \quad \text{otherwise} \end{aligned} \quad (5.9)$$

and also the following Korn-type inequality

$$\|\mathbf{z}\|_{W^{1,2}(\Omega_t)} \leq C \|\mathbb{S}(\nabla_x \mathbf{z})\|_{L^2(\Omega_t)} \quad (5.10)$$

for all $\mathbf{z} \in W^{1,2}(\Omega_t)$.

Thus, it is not difficult to observe that

$$\left| \int_{\Omega_t} \varrho (\mathbf{u} - \tilde{\mathbf{u}}) \cdot \nabla_x \tilde{\mathbf{u}} \cdot (\tilde{\mathbf{u}} - \mathbf{u}) - \operatorname{div}_x \tilde{\mathbf{u}} (p(\varrho) - p'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - p(\tilde{\varrho})) \, dx \right| \leq C \|\nabla_x \tilde{\mathbf{u}}\|_{L^\infty(\Omega_t)} \mathcal{E}([\varrho, \mathbf{u}] | [\tilde{\varrho}, \tilde{\mathbf{u}}]) (t). \quad (5.11)$$

It remains to handle the last term on the right hand side of (5.8). We split the integral into three parts, considering first ϱ close to $\tilde{\varrho}$, then ϱ small and finally ϱ large. We have using the Hölder inequality, the Young inequality and (5.10)

$$\begin{aligned} & \left| \int_{\{\tilde{\varrho}/2 \leq \varrho \leq 2\tilde{\varrho}\}} \frac{1}{\tilde{\varrho}} (\varrho - \tilde{\varrho}) \operatorname{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \, dx \right| \\ & \leq C(\delta) \left\| \frac{1}{\tilde{\varrho}} \operatorname{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \right\|_{L^3(\Omega_t)}^2 \int_{\{\tilde{\varrho}/2 \leq \varrho \leq 2\tilde{\varrho}\}} (\varrho - \tilde{\varrho})^2 \, dx + \delta \|\tilde{\mathbf{u}} - \mathbf{u}\|_{L^6(\Omega_t)}^2 \\ & \leq C(\delta) \left\| \frac{1}{\tilde{\varrho}} \operatorname{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \right\|_{L^3(\Omega_t)}^2 \mathcal{E}([\varrho, \mathbf{u}] | [\tilde{\varrho}, \tilde{\mathbf{u}}]) (t) + \delta C \|\mathbb{S}(\nabla_x(\tilde{\mathbf{u}}) - \nabla_x(\mathbf{u}))\|_{L^2(\Omega_t)}^2. \end{aligned} \quad (5.12)$$

The last term can be absorbed into the left hand side for δ small enough, whereas the first term can be treated using the Gronwall lemma.

On the set where ϱ is small we can proceed in the following way

$$\begin{aligned}
& \left| \int_{\{0 \leq \varrho < \bar{\varrho}/2\}} \frac{1}{\bar{\varrho}} (\varrho - \bar{\varrho}) \operatorname{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \, dx \right| \\
& \leq \left| \int_{\{0 \leq \varrho < \bar{\varrho}/2\}} \operatorname{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \, dx \right| \\
& \leq C(\delta) \|\operatorname{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}})\|_{L^3(\Omega_t)}^2 \int_{\{0 \leq \varrho < \bar{\varrho}/2\}} 1 \, dx + \delta \|\tilde{\mathbf{u}} - \mathbf{u}\|_{L^6(\Omega_t)}^2 \\
& \leq C(\delta) \|\operatorname{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}})\|_{L^3(\Omega_t)}^2 \mathcal{E}([\varrho, \mathbf{u}][\bar{\varrho}, \tilde{\mathbf{u}}])(t) + \delta C \|\mathbb{S}(\nabla_x(\tilde{\mathbf{u}}) - \nabla_x(\mathbf{u}))\|_{L^2(\Omega_t)}^2.
\end{aligned} \tag{5.13}$$

Again, the last term is absorbed into the left hand side for δ small enough and the first term is treated using the Gronwall lemma.

Finally, consider the integral over the set where ϱ is large. Here we distinguish two cases. First for $\gamma \leq 2$ we have

$$\begin{aligned}
& \left| \int_{\{\varrho > 2\bar{\varrho}\}} \frac{1}{\bar{\varrho}} (\varrho - \bar{\varrho}) \operatorname{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \, dx \right| \\
& \leq \left| \int_{\{\varrho > 2\bar{\varrho}\}} \varrho \frac{\varrho - \bar{\varrho}}{\varrho \bar{\varrho}} \operatorname{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \, dx \right| \\
& \leq \left| \int_{\{\varrho > 2\bar{\varrho}\}} \varrho \frac{1}{\bar{\varrho}} \operatorname{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \, dx \right| \\
& \leq C(\delta) \left\| \frac{1}{\bar{\varrho}} \operatorname{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \right\|_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega_t)}^2 \left(\int_{\{\varrho > \bar{\varrho}/2\}} \varrho^\gamma \, dx \right)^{2/\gamma} + \delta \|\tilde{\mathbf{u}} - \mathbf{u}\|_{L^6(\Omega_t)}^2 \\
& \leq C(\delta) \left\| \frac{1}{\bar{\varrho}} \operatorname{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \right\|_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega_t)}^2 \mathcal{E}([\varrho, \mathbf{u}][\bar{\varrho}, \tilde{\mathbf{u}}])^{\frac{2}{\gamma}-1}(t) \mathcal{E}([\varrho, \mathbf{u}][\bar{\varrho}, \tilde{\mathbf{u}}])(t) + \delta C \|\mathbb{S}(\nabla_x(\tilde{\mathbf{u}}) - \nabla_x(\mathbf{u}))\|_{L^2(\Omega_t)}^2.
\end{aligned} \tag{5.14}$$

In this case the power $\frac{2}{\gamma} - 1$ is nonnegative and we use also the property $\mathcal{E}([\varrho, \mathbf{u}][\bar{\varrho}, \tilde{\mathbf{u}}]) \in L^\infty(0, T)$ to proceed further.

For $\gamma > 2$ we have

$$\begin{aligned}
& \left| \int_{\{\varrho > 2\bar{\varrho}\}} \frac{1}{\bar{\varrho}} (\varrho - \bar{\varrho}) \operatorname{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \, dx \right| \\
& \leq \left| \int_{\{\varrho > 2\bar{\varrho}\}} \varrho \frac{\varrho - \bar{\varrho}}{\varrho \bar{\varrho}} \operatorname{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \, dx \right| \\
& \leq \left| \int_{\{\varrho > 2\bar{\varrho}\}} \varrho^{\frac{\gamma}{2}} \frac{1}{\bar{\varrho}} \operatorname{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \, dx \right| \\
& \leq C(\delta) \left\| \frac{1}{\bar{\varrho}} \operatorname{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \right\|_{L^3(\Omega_t)}^2 \left(\int_{\{\varrho > \bar{\varrho}/2\}} \varrho^\gamma \, dx \right) + \delta \|\tilde{\mathbf{u}} - \mathbf{u}\|_{L^6(\Omega_t)}^2 \\
& \leq C(\delta) \left\| \frac{1}{\bar{\varrho}} \operatorname{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \right\|_{L^3(\Omega_t)}^2 \mathcal{E}([\varrho, \mathbf{u}] | [\bar{\varrho}, \tilde{\mathbf{u}}])(t) + \delta C \|\mathbb{S}(\nabla_x(\tilde{\mathbf{u}}) - \nabla_x(\mathbf{u}))\|_{L^2(\Omega_t)}^2.
\end{aligned} \tag{5.15}$$

Altogether we end up with the inequality

$$\mathcal{E}([\varrho, \mathbf{u}] | [\bar{\varrho}, \tilde{\mathbf{u}}])(\tau) \leq \int_0^\tau h(t) \mathcal{E}([\varrho, \mathbf{u}] | [\bar{\varrho}, \tilde{\mathbf{u}}])(t) \, dt \tag{5.16}$$

for some $h(t) \in L^1(0, T)$ and the Gronwall lemma finishes the proof.

Q.E.D.

Remark 5.1 *The case of nonzero bulk viscosity coefficient $\eta > 0$ in (1.3) as well as the case of nonzero boundary friction coefficient $\kappa > 0$ in (1.7) can be treated by obvious modifications just adding proper integrals to appropriate formulas.*

Remark 5.2 *The local-in-time existence of solutions belonging to the regularity class of the strong solution in Theorem 5.1 will be proved in our forthcoming paper [14]. This will show in particular, that the weak-strong uniqueness property stated in Theorem 5.1 is in fact a nonempty result.*

References

- [1] J. Carrillo, A. Jüngel, P.A. Markowich, G. Toscani and A. Unterreiter. Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities. *Monatshefte Math.*, **133**:1–82, 2001.
- [2] R.J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, **98**:511–547, 1989.
- [3] Doboszczak, S. Relative entropy and a weak-strong uniqueness principle for the compressible Navier-Stokes equations on moving domains *Appl. Math. Lett.*, **57**:60–68, 2016
- [4] E. Feireisl. Compressible Navier-Stokes equations with a non - monotone pressure law. *J. Differential Equations*, **184**:97–108, 2002.

- [5] E. Feireisl. *Dynamics of viscous compressible fluids*. Oxford University Press, Oxford, 2004.
- [6] E. Feireisl. On the motion of a viscous, compressible, and heat conducting fluid. *Indiana Univ. Math. J.*, **53**:1707–1740, 2004.
- [7] E. Feireisl, B.J. Jin, and A. Novotný. Relative entropies, suitable weak solutions, and weak-strong uniqueness for the compressible Navier-Stokes system. *J. Math. Fluid Mech.*, **14**(4):717-730, 2012.
- [8] E. Feireisl, J. Neustupa, and J. Stebel. Convergence of a Brinkman-type penalization for compressible fluid flows. *J. Differential Equations*, **250**(1):596–606, 2011.
- [9] E. Feireisl, O. Kreml, Š. Nečasová, J. Neustupa, J. Stebel. *Weak solutions to the barotropic Navier-Stokes system with slip boundary conditions in time dependent domains*. *J. Differential Equations* **254** (2013), no. 1, 125-140
- [10] E. Feireisl, A. Novotný, and H. Petzeltová. On the existence of globally defined weak solutions to the Navier-Stokes equations of compressible isentropic fluids. *J. Math. Fluid Mech.*, **3**:358–392, 2001.
- [11] E. Feireisl, A. Novotný and Y. Sun. Suitable weak solutions to the Navier-Stokes equations of compressible viscous fluids. *Indiana Univ. Math. J.*, **60**(2):611–631, 2011.
- [12] P. Germain. Weak-strong uniqueness for the isentropic compressible Navier-Stokes system. *J. Math. Fluid Mech.*, **13**(1):137–146, 2010.
- [13] D. Hoff. Local solutions of a compressible flow problem with Navier boundary conditions in general three-dimensional domains. *SIAM J. Math. Anal.*, **44**(2):633–650, 2012.
- [14] O. Kreml, Š. Nečasová and T. Piasecki. Local existence of strong solutions to the compressible Navier-Stokes system on moving domains. In preparation.
- [15] P.-L. Lions. *Mathematical topics in fluid dynamics, Vol.2, Compressible models*. Oxford Science Publication, Oxford, 1998.
- [16] N. Masmoudi. Incompressible inviscid limit of the compressible Navier-Stokes system. *Ann. Inst. H. Poincaré, Anal. Non Linéaire*, **18**:199–224, 2001.
- [17] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Kyoto Univ.*, **20**:67–104, 1980.
- [18] A. Matsumura and T. Nishida. Initial-boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids. *Comm. Math. Phys.*, **89**:445–464, 1983.
- [19] L. Saint-Raymond. Hydrodynamic limits: some improvements of the relative entropy method. *Ann. Inst. H. Poincaré, Anal. Non Linéaire*, **26**:705–744, 2009.
- [20] V. A. Vaigant, A. V. Kazhikov. On the existence of global solutions of two-dimensional Navier-Stokes equations of a compressible viscous fluid. *Siberian Math. J.*, **36**(6): 1108–1141, 1995.
- [21] A. Valli. An existence theorem for compressible viscous fluids. *Ann. Math. Pura Appl. (IV)*, **130**:197–213, 1982. *Ann. Math. Pura Appl. (IV)*, **132**:399–400, 1982.
- [22] A. Valli. Periodic and stationary solutions for compressible Navier-Stokes equations via a stability method. *Ann. Sc. Norm. Super. Pisa (IV)*, **10**:607–647, 1983.

- [23] S. Wang and S. Jiang. The convergence of the Navier-Stokes-Poisson system to the compressible Euler equation. *J. Math. Kyoto Univ.*, **26**(2):323–331, 1986.
- [24] W.M. Zajaczkowski. On nonstationary motion of a compressible barotropic viscous fluid with boundary slip condition. *J. Appl. Anal.*, **4**(2):167–204, 1998.