Relative entropies in fluid mechanics and their applications

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Compressible (isentropic) Navier-Stokes (Euler) system:

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \tag{1}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) = \operatorname{div} \mathbb{S}(\nabla \mathbf{u}), \tag{2}$$

where

$$\mathbb{S}(\nabla \mathbf{u}) = \mu \left(\nabla \mathbf{u} + \nabla^t \mathbf{u} - \frac{2}{3} \operatorname{div} \mathbf{u} \mathbb{I} \right) + \eta \operatorname{div} \mathbf{u} \mathbb{I} \tag{3}$$

with $\mu \geq 0$, $\eta \geq 0$ and $p(\varrho)$ is a given function, typically

$$p(\varrho) \sim \varrho^\gamma$$

Compressible Navier-Stokes-Fourier (NSF) system:

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \tag{4}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho, \vartheta) = \operatorname{div} \mathbb{S}(\vartheta, \nabla \mathbf{u}), \tag{5}$$

$$\partial_t(\varrho E) + \operatorname{div}((\varrho E + p)\mathbf{u}) + \operatorname{div} \mathbf{q} = \operatorname{div}(\mathbb{S}(\vartheta, \nabla \mathbf{u})\mathbf{u}). \tag{6}$$

where E denotes the total energy density, $E = \frac{1}{2}|\mathbf{u}|^2 + e(\varrho, \vartheta)$ with $e(\varrho, \vartheta)$ being an internal energy density.
In this case

$$\mathbb{S}(\vartheta, \nabla \mathbf{u}) = \mu(\vartheta)\left(\nabla \mathbf{u} + \nabla^t \mathbf{u} - \frac{2}{3} \operatorname{div} \mathbf{u} \mathbb{I}\right) + \eta(\vartheta) \operatorname{div} \mathbf{u} \mathbb{I} \tag{7}$$

with $\mu, \eta \geq 0$ given functions and

$$\mathbf{q} = -\kappa(\vartheta) \nabla \vartheta \tag{8}$$

Equations of fluid mechanics III

Instead of equation (6) it is more suitable to work with the entropy equation (more precisely inequality)

$$\partial_t(\rho s(\varrho,\vartheta)) + \mathrm{div}(\rho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma, \qquad (9)$$

with σ being the entropy dissipation measure satisfying

$$\sigma \geq \frac{1}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla\mathbf{u}) : \nabla\mathbf{u} - \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta}\right).$$

The system is then completed with the total energy conservation

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)dx = 0$$

Initial and boundary conditions

For simplicity throughout this talk:

- Problems studied on $[0, T] \times \Omega$
- Ω is a bounded and sufficiently regular domain in $\mathbb{R}^3$
- $\mathbf{u} = 0$ on $\partial\Omega$ (most of the results hold also for complete slip BC $\mathbf{u} \cdot \mathbf{n} = 0$ and $[\mathbb{S} \cdot \mathbf{n}] \times \mathbf{n} = 0$ on $\partial\Omega$ and for Navier slip BC)
- If working with NSF system: $\mathbf{q} \cdot \mathbf{n} = 0$ on $\partial\Omega$
- $\varrho(0, \cdot) = \varrho_0(\cdot) \geq 0$
- $\varrho\mathbf{u}(0, \cdot) = (\varrho\mathbf{u})_0(\cdot)$ and $(\varrho\mathbf{u})_0 = 0$ whenever $\varrho_0 = 0$
- If working with NSF system: $\vartheta(0, \cdot) = \vartheta_0 > 0$

Weak formulation

Continuity equation:

$$\int_\Omega \varrho(\tau,\cdot)\varphi(\tau,\cdot)dx - \int_\Omega \varrho_0\varphi(0,\cdot)dx = \int_0^\tau \int_\Omega (\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla\varphi)dxdt,$$

for any $\varphi \in C^\infty([0,T]\times\overline{\Omega})$ and any $\tau \in [0,T]$

Continuity equation renormalized:

$$\int_\Omega b(\varrho)(\tau,\cdot)\varphi(\tau,\cdot)dx - \int_\Omega b(\varrho_0)\varphi(0,\cdot)dx$$

$$= \int_0^\tau \int_\Omega \left(b(\varrho)\partial_t\varphi + b(\varrho)\mathbf{u}\cdot\nabla\varphi + \left(b(\varrho) - b'(\varrho)\varrho\right)\operatorname{div}\mathbf{u}\varphi\right)dxdt$$

for any $\tau \in [0,T]$, any $\varphi \in C^\infty([0,T]\times\overline{\Omega})$, and any $b \in C^1([0,\infty))$, $b(0) = 0$, $b'(r) = 0$ for large r.

Weak solutions II

Momentum equation:

$$\int_\Omega \varrho \mathbf{u} \cdot \boldsymbol{\varphi}(\tau, \cdot) dx - \int_\Omega (\varrho \mathbf{u})_0 \cdot \boldsymbol{\varphi}(0, \cdot) dx$$

$$= \int_0^\tau \int_\Omega \left(\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla \boldsymbol{\varphi} + p(\varrho) \operatorname{div} \boldsymbol{\varphi} - \mathbb{S}(\nabla \mathbf{u}) : \nabla \boldsymbol{\varphi} \right) dxdt$$

for any $\tau \in [0, T]$ and any test function $\boldsymbol{\varphi} \in C_c^\infty([0, T] \times \Omega)$.

Weak solutions III

In the case of full NSF - **entropy inequality:**

$$\int_\Omega \varrho_0 s(\varrho_0, \vartheta_0)\varphi(0,\cdot)dx - \int_\Omega \varrho s(\varrho,\vartheta)(\tau,\cdot)\varphi(\tau,\cdot)dx$$

$$+\int_0^\tau \int_\Omega \frac{1}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla\mathbf{u}) : \nabla\mathbf{u} - \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta}\right)\varphi dxdt$$

$$\leq -\int_0^\tau \int_\Omega \left(\varrho s(\varrho,\vartheta)\partial_t\varphi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla\varphi + \frac{\mathbf{q}\cdot\nabla\varphi}{\vartheta}\right) dxdt$$

for any $\varphi \in C^\infty([0,T]\times\overline{\Omega})$, $\varphi \geq 0$ and almost all $\tau \in [0,T]$.
And finally **the total energy balance:**

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(\tau,\cdot)dx = \int_\Omega \left(\frac{1}{2\varrho_0}|(\varrho\mathbf{u})_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0)\right)dx$$

for almost all $\tau \in [0,T]$.

Compressible NS:

- Lions ($\gamma > \frac{9}{5}$)
- Feireisl, Novotný and Petzeltová ($\gamma > \frac{3}{2}$)

These solutions satisfy moreover the energy inequality

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right)(\tau, \cdot)dx + \int_0^\tau \int_\Omega \mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{u} dxdt$$

$$\leq \int_\Omega \left(\frac{1}{2\varrho_0}|(\varrho \mathbf{u})_0|^2 + P(\varrho_0) \right) dx$$

for almost all $\tau \in [0, T]$.

Here $P(\varrho) = \varrho \int_1^\varrho \frac{p(r)}{r^2} dr$ is the pressure potential and in particular for $p(\varrho) = \varrho^\gamma$ it holds $P(\varrho) = c\varrho^\gamma$ (for $\gamma > 1$)

Existence of weak solutions II

Compressible NSF:
In the presented framework due to Feireisl and Novotný (2009) with certain assumptions on the form of

- $p(\varrho, \vartheta)$, $e(\varrho, \vartheta)$, $s(\varrho, \vartheta)$
- $\mu(\vartheta)$, $\eta(\vartheta)$ and $\kappa(\vartheta)$

Relative entropy: Nonnegative quantity providing a kind of distance between two solutions of the same problem, one of which typically is regular

- Dafermos (1979) - application of a kind of relative entropy in thermoelasticity
- Carillo, Jüngel, Markowich, Toscani and Unterreiter (2001) - rel. entropy w.r.t. a stationary solution - long-time behavior of quasilinear parabolic equations
- Saint-Raymond (2009) - incompressible Euler limit of the Boltzmann equation
- other applications: Grenier (1997), Masmoudi (2001), Ukai (1986), Wang and Jiang (2006), ...

Relative entropy for compressible fluids

Germain (2010)

- introduced a class of weak solutions to the CNS satisfying relative entropy inequality w.r.t. a (hypothetical) strong solution
- establishes weak-strong uniqueness in this class
- existence of solutions in this class is an open problem, he needs $\nabla \varrho \in L^{2\gamma}(0, T, L^{\beta}(\Omega))$ for certain β

Relative entropies II

Relative entropy for compressible NS:

$$\mathcal{E}(\varrho, \mathbf{u} | r, \mathbf{U}) := \frac{1}{2} \varrho \left| \mathbf{u} - \mathbf{U} \right|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r)$$

Relative entropy inequality:

$$\int_\Omega \mathcal{E}(\varrho, \mathbf{u} | r, \mathbf{U})(\tau, \cdot) dx + \int_0^T \int_\Omega (\mathbb{S}(\nabla \mathbf{u}) - \mathbb{S}(\nabla \mathbf{U})) : (\nabla \mathbf{u} - \nabla \mathbf{U}) dxdt$$

$$\leq \int_\Omega \mathcal{E}(\varrho_0, \mathbf{u}_0 | r(0), \mathbf{U}(0))(\tau, \cdot) dx + \int_0^T \int_\Omega \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) dxdt$$

with

$$\mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) = \varrho(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla \mathbf{U}) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\nabla \mathbf{U}) : \nabla(\mathbf{U} - \mathbf{u})$$

$$+ (r - \varrho) \partial_t P'(r) + (r\mathbf{U} - \varrho \mathbf{u}) \cdot \nabla P'(r) - (p(\varrho) - p(r)) \operatorname{div} \mathbf{U}$$

Most important results

Feireisl, Novotný, Sun (2011)

- existence of suitable weak solutions (i.e. solutions satisfying moreover REI)
- weak-strong uniqueness in this class
- are all weak solutions also suitable? (left open)

Feireisl, Jin, Novotný (2012) - all finite energy weak solutions are in fact suitable

Theorem 1 (Feireisl, Jin, Novotný)

Let $\varrho, \mathbf{u}$ be a (finite energy) weak solution to the compressible NS. Then $(\varrho, \mathbf{u})$ satisfies the relative entropy inequality with respect to any couple of smooth functions $(r, \mathbf{U})$ with $r > 0$ and $\mathbf{U} = 0$ on $\partial\Omega$.

Sketch of the proof

- test momentum equation by $\mathbf{U}$
- test continuity equation by $\frac{1}{2}|\mathbf{U}|^2$
- test continuity equation by $P'(r)$
- use conservation of mass ($\int_\Omega \varrho dx$ is constant in time)
- sum all of this with the energy inequality and calculate a little bit

Weak-strong uniqueness

Natural application of the relative entropy inequality is the weak-strong uniqueness principle.

Theorem 2 (Feireisl, Jin, Novotný)

Let $(\varrho, \mathbf{u})$ be a finite energy weak solution to the compressible NS. Let $(r, \mathbf{U})$ be a strong solution belonging to the class

$$0 < \inf r(t,x) \leq r(t,x) \leq \sup r(t,x) < \infty$$

$$\nabla r \in L^2(0, T, L^q), \nabla^2 \mathbf{U} \in L^2(0, T, L^q)$$

with $q > max\{3, \frac{3}{\gamma-1}\}$, emanating from the same initial data. Then

$$\varrho = r, \mathbf{u} = \mathbf{U}$$

in $(0, T) \times \Omega$.

Sketch of the proof

With some computing we get

$$\int_\Omega \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) dx = \int_\Omega \varrho(\mathbf{u} - \mathbf{U}) \cdot \nabla \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) dx$$

$$- \int_\Omega \operatorname{div} \mathbf{U} (p(\varrho) - p'(r)(\varrho - r) - p(r)) dx$$

$$+ \int_\Omega \frac{1}{r} (\varrho - r) \operatorname{div} \mathbb{S}(\nabla \mathbf{U}) \cdot (\mathbf{U} - \mathbf{u}) dx$$

and because we have

$$P(\varrho) - P'(r)(\varrho - r) - P(r) \geq c(r) \begin{cases} (\varrho - r)^2 \text{ for } \frac{r}{2} < \varrho < 2r \\ 1 + \varrho^\gamma \text{ otherwise} \end{cases}$$

we can bound the first two terms above by $c \left\| \nabla \mathbf{U} \right\|_{L^\infty} \mathcal{E}(\varrho, \mathbf{u} | r, \mathbf{U})$.

Full Navier-Stokes Fourier system

- much more complicated, one needs to find a function measuring distance between densities as well as temperatures
- Ballistic free energy: $H^{\Theta}(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \Theta \varrho s(\varrho, \vartheta)$ for some given Θ
- $\varrho \mapsto H^{\Theta}(\varrho, \vartheta)$ is strictly convex
- $\vartheta \mapsto H^{\Theta}(\varrho, \vartheta)$ attains global minimum at $\vartheta = \Theta$
- Relative entropy: $\mathcal{E}(\varrho, \mathbf{u}, \vartheta | r, \mathbf{U}, \Theta) :=$

$$\frac{1}{2}\varrho\,|\mathbf{u} - \mathbf{U}|^2 + H^{\Theta}(\varrho, \vartheta) - \partial_{\varrho} H^{\Theta}(r, \Theta)(\varrho - r) - H^{\Theta}(r, \Theta)$$

Relative entropy inequality

$$\int_\Omega \mathcal{E}(\varrho, \mathbf{u}, \vartheta | r, \mathbf{U}, \Theta)(\tau, \cdot) dx$$

$$+ \int_0^T \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} - \frac{\mathbf{q} \cdot \nabla \vartheta}{\theta} \right) dxdt$$

$$\leq \int_\Omega \mathcal{E}(\varrho_0, \mathbf{u}_0, \vartheta_0 | r(0), \mathbf{U}(0), \Theta(0)) dx$$

$$+ \int_0^T \int_\Omega \mathcal{R}(\varrho, \mathbf{u}, \vartheta, r, \mathbf{U}, \Theta) dxdt$$

Remainder term for NSF

$$\mathcal{R} = \varrho(\mathbf{u} - \mathbf{U}) \cdot \nabla \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) + \varrho(s(\varrho, \vartheta) - s(r, \Theta))(\mathbf{U} - \mathbf{u}) \cdot \nabla \Theta$$

$$+\varrho(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U}) \cdot (\mathbf{U} - \mathbf{u}) - p(\varrho, \vartheta) \mathrm{div}\, \mathbf{U} + \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{U}$$

$$-\varrho(s(\varrho, \vartheta) - s(r, \Theta))(\partial_t \Theta + \mathbf{U} \cdot \nabla \Theta) - \frac{\mathbf{q}}{\vartheta} \cdot \nabla \Theta$$

$$+ \left(1 - \frac{\varrho}{r}\right) \partial_t p(r, \Theta) - \frac{\varrho}{r} \mathbf{u} \cdot \nabla p(r, \Theta)$$

Theorem 3 (Feireisl, Novotný)

Any weak solution to the compressible NSF system satisfies the above stated relative entropy inequality with respect to smooth functions $r, \mathbf{U}, \Theta$ such that r, Θ are bounded and bounded away from zero and $\mathbf{U} = 0$ on $\partial\Omega$.

Weak-strong uniqueness for NSF

Theorem 4 (Feireisl, Novotný)

Let $(\varrho, \mathbf{u}, \vartheta)$ *be a weak solution to the compressible NSF and let* $(r, \mathbf{U}, \Theta)$ *be a smooth solution to the same system emanating from the same initial data. Then*

$$\varrho = r, \mathbf{u} = \mathbf{U}, \vartheta = \Theta.$$

The proof consists of

- very long series of estimates of the terms in $\mathcal{R}$
- adjusting the viscous terms on the left hand side of the relative entropy inequality
- final Gronwall argument

Other weak-strong uniqueness results

- **Gwiazda, Świerczewska-Gwiazda, Wiedemann (2015)** - WSU for admissible measure valued solutions for compressible Euler and Savage-Hutter model in 1D and 2D
- **Feireisl, Gwiazda, Świerczewska-Gwiazda, Wiedemann (2016)** - notion of dissipative measure valued solutions for compressible NS, in a sense weakest possible solutions for which WSU holds
- **Dobosczak (2016)** - WSU on moving domains with prescribed motion of the boundary

Riemann problem for compressible Euler

Riemann problem:

$$\varrho_0 = \begin{cases} \varrho_L & \text{for } x_1 \leq 0, \\ \varrho_R & \text{for } x_1 > 0, \end{cases}$$

$$u_0^1 = \begin{cases} u_L^1 & \text{for } x_1 \leq 0, \\ u_R^1 & \text{for } x_1 > 0, \end{cases} \quad u_0^j = 0 \text{ for } j > 1.$$

1D data $\Rightarrow$ 1D selfsimilar BV solution consisting in this case in general of shocks and/or rarefaction waves

- **Chiodaroli, De Lellis, K. (2015)** - There exist Riemann initial data generating a selfsimilar solution consisting of 1 shock and 1 rarefaction wave for which there is infinitely many admissible weak solutions
- **Chiodaroli, K. (2014)** - For every Riemann initial data generating a selfsimilar solution consisting of 2 shocks there is infinitely many admissible weak solutions

Riemann problem for compressible Euler II

Theorem 5 (Feireisl, K. (2015))

The selfsimilar solution to the above mentioned Riemann problem consisting only of rarefaction waves is unique within the class of all multiD admissible bounded weak solutions.

Similar result holds also for full compressible Euler system (also with equation for total energy)

Theorem 6 (Feireisl, K., Vasseur (2015))

The selfsimilar solution to the Riemann problem for the full compressible Euler system consisting only of rarefaction waves is unique within the class of all multiD bounded weak solutions.

Singular limits

- **Sueur**: Compressible NS to compressible Euler in 3D (no-slip BC with some boundary layer condition, Navier slip ok)
- **Feireisl, Novotný**: rotating 3D fluids to 2D incompressible Euler
- **Feireisl, Lu, Novotný**: rotating 3D fluids to 2D incompressible damped Euler
- **Donatelli, Feireisl, Novotný**: plasma to incompressible Euler
- **Feireisl, Klein, Novotný, Zatorska**: stratified flows towards anelastic NS
- and many others...

Dimension reduction

- **Maltese, Novotný** - compressible NS: 3D to 2D
- **Ducomet, Caggio, Nečasová, Pokorný** - compressible NSF coupled with Poisson equation 3D to 2D
- **Bella, Feireisl, Novotný** - compressible NS: 3D to 1D
- **Březina, K., Mácha** - compressible NSF: 3D to 1D

Thank you

Thank you for your attention.