On measure valued solutions to the compressible Euler equations

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We consider the compressible isentropic Euler system

$$\left\{\begin{array}{l} \partial_t \rho + \mathrm{div}_x(\rho v) \;=\; 0 \\ \partial_t(\rho v) + \mathrm{div}_x\left(\rho v \otimes v\right) + \nabla_x[p(\rho)] \;=\; 0 \\ \rho(\cdot, 0) \;=\; \rho^0 \\ v(\cdot, 0) \;=\; v^0 \,. \end{array}\right. \tag{1}$$

Unknowns:

- $\rho(x, t)$ ... density
- $v(x, t)$ ... velocity

The pressure $p(\rho)$ is given.

Properties

- Weak solutions nonunique $\Rightarrow$ Admissibility criteria $\Rightarrow$ Entropy conditions
- Existence of entropy weak solutions in general in multi-D not known
- What is correct notion of solution still unclear (ill-posedness results of entropy weak solutions by Chiodaroli, De Lellis and K.)
- Measure-valued solutions (MVS) introduced by DiPerna for general systems of conservation laws
- Existence of MVS for compressible Euler by Neustupa
- MVS criticized for being too weak, on the other hand may be useful in the weak-strong uniqueness results (Feireisl, Gwiazda, Świerczewska-Gwiazda, Wiedemann)
- Numerical schemes may not converge to entropy solutions, MVS are suggested instead (Fjordholm, Käppeli, Mishra, Tadmor)

Incompressible Euler system

Theorem 1 (Székelyhidi, Wiedemann, 2012)

Any measure-valued solution to the incompressible Euler system can be approximated by a sequence of weak solutions

This means that MVS are not substantially weaker than weak solutions, i.e. weak solutions are too weak.

What is the situation in the compressible case?

Young measures

For simplicity we work only with bounded measure-valued solutions - we ignore the effects of concentrations and avoid using generalized Young measures.
Young measure: map $\nu \in L^\infty_w(\Omega; \mathcal{M}^1(\mathbb{R}^d))$ ... assigns to almost every point $x \in \Omega$ a probability measure $\nu_x \in \mathcal{M}^1(\mathbb{R}^d)$ on the phase space $\mathbb{R}^d$.
Denote $\langle \nu_x, f \rangle := \int_{\mathbb{R}^d} f(z) \mathrm{d}\nu_x(z)$ ... the expectation of f with respect to the probability measure ν_x.

Measure valued solutions

In our context the domain takes the form $[0, T] \times \Omega$ and the phase space (we work in 3D) is $\mathbb{R}^+ \times \mathbb{R}^3$.
Denote the state variables by $\xi \in \mathbb{R}^+ \times \mathbb{R}^3$ and introduce

$$\xi = [\xi_0, \xi'] = [\xi_0, \xi_1, \xi_2, \xi_3] \in \mathbb{R}^+ \times \mathbb{R}^3$$

$$\langle \nu_{t,x}, \xi_0 \rangle = \overline{\rho}$$

$$\langle \nu_{t,x}, \sqrt{\xi_0}\xi' \rangle = \overline{\rho v}$$

$$\langle \nu_{t,x}, \xi' \otimes \xi' \rangle = \overline{\rho v \otimes v}$$

$$\langle \nu_{t,x}, p(\xi_0) \rangle = \overline{p(\rho)}.$$

In such a way ξ_0 is the state of the density ρ and ξ' is the state of $\sqrt{\rho}v$

Definition 2 (Measure-valued solution)

A *measure-valued solution to the compressible Euler equations* (1) is a Young measure $\nu_{t,x}$ on $\mathbb{R}^+ \times \mathbb{R}^3$ with parameters in $[0, T] \times \Omega$ which satisfies the Euler equations in an average sense, i.e.

$$\int_0^T \int_\Omega \partial_t \psi \overline{\rho} + \nabla_x \psi \cdot \overline{\rho v} \mathrm{d}x \mathrm{d}t + \int_\Omega \psi(0,x)\rho_0(x) \mathrm{d}x = 0$$

$$\int_0^T \int_\Omega \partial_t \varphi \cdot \overline{\rho v} + \nabla_x \varphi : \overline{\rho v \otimes v} + \mathrm{div}\,_x \varphi \overline{p(\rho)} \mathrm{d}x \mathrm{d}t$$
$$+ \int_\Omega \varphi(0,x) \cdot \rho_0(x) v_0(x) \mathrm{d}x = 0$$

for all $\psi \in C_c^\infty([0, T) \times \Omega)$ and all $\varphi \in C_c^\infty([0, T) \times \Omega; \mathbb{R}^3)$.

Some remarks

- Every weak solution defines naturally an atomic measure valued solution $\nu_{t,x} := \delta_{\rho(t,x),\sqrt{\rho}v(t,x)}$.
- We say that sequence $\{z_n\}$ *generates* the Young measure ν if for all bounded Carathéodory functions $f : \widetilde{\Omega} \times \mathbb{R}^d \to \mathbb{R}$

$$\lim_{n\to\infty} \int_{\widetilde{\Omega}} f(y, z_n(y))\varphi(y)\mathrm{d}y = \int_{\widetilde{\Omega}} \langle \nu_y, f(y,\cdot)\rangle \varphi(y)\mathrm{d}y$$

  for all $\varphi \in L^1(\widetilde{\Omega})$.
- Any sequence of functions bounded in $L^p(\Omega)$ (for any $p \geq 1$) generates, up to a subsequence, some Young measure [Fundamental theorem of Young measures].

Subsolutions

In order to formulate our first Theorem we need to define subsolutions. As usual we take the linearized system

$$\begin{aligned} \partial_t \rho + \operatorname{div}_x m &= 0 \\ \partial_t m + \operatorname{div}_x U + \nabla_x q &= 0, \end{aligned} \tag{2}$$

associated to the compressible Euler system. Here, as usual, $U \in S_0^3$ is a symmetric trace-free 3×3 matrix which replaces the traceless part of the matrix $\rho v \otimes v = \frac{m \otimes m}{\rho}$.
Weak solutions to (2) are functions (ρ, m, U, q) which satisfy (2) in the sense of distributions.

Measure valued subsolutions

We use the following notation:

$$
\begin{aligned}
[\zeta_0, \zeta', \mathbf{Z}, \tilde{\zeta}] &\in \mathbb{R}^+ \times \mathbb{R}^3 \times S_0^3 \times \mathbb{R}^+ \\
\langle \mu_{t,x}, \zeta_0 \rangle &= \overline{\rho} \\
\langle \mu_{t,x}, \zeta' \rangle &= \overline{m} \\
\langle \mu_{t,x}, \mathbf{Z} \rangle &= \overline{U} \\
\langle \mu_{t,x}, \tilde{\zeta} \rangle &= \overline{q}
\end{aligned}
$$

Measure values subsolutions II

Definition 3 (Measure valued subsolution)

A *measure-valued solution to the linear system* is a Young measure $\mu_{t,x}$ on $\mathbb{R}^+ \times \mathbb{R}^3 \times S_0^3 \times \mathbb{R}^+$ with parameters in $[0, T] \times \Omega$ which satisfies the linear system (2) in an average sense, i.e.

$$\int_0^T \int_\Omega \partial_t \psi \overline{\rho} + \nabla_x \psi \cdot \overline{m} \mathrm{d}x\mathrm{d}t + \int_\Omega \psi(0,x)\rho_0(x)\mathrm{d}x = 0$$

$$\int_0^T \int_\Omega \partial_t \varphi \cdot \overline{m} + \nabla_x \varphi : \overline{U} + \mathrm{div}_x \varphi \overline{q} \mathrm{d}x\mathrm{d}t + \int_\Omega \varphi(0,x) \cdot m_0(x)\mathrm{d}x = 0$$

for all $\psi \in C_c^\infty([0, T) \times \Omega)$ and all $\varphi \in C_c^\infty([0, T) \times \Omega; \mathbb{R}^3)$.

Lift

Similarly, as are subsolutions connected to solutions as functions, we need a procedure to connect measure valued solutions and subsolutions.

Definition 4 (Lift)

Let $\nu_{t,x}$ be a measure valued solution to the Euler equations. Denote $Q : \mathbb{R}^+ \times \mathbb{R}^3 \mapsto \mathbb{R}^+ \times \mathbb{R}^3 \times S_0^3 \times \mathbb{R}^+$

$$Q(\xi) := (\xi_0, \sqrt{\xi_0}\xi', \xi' \otimes \xi' - \frac{1}{3} |\xi'|^2 \mathbf{I}, p(\xi_0) + \frac{1}{3} |\xi'|^2).$$

We define the *lifted measure* $\tilde{\nu}_{t,x}$ as

$$\langle \tilde{\nu}_{t,x}, f \rangle := \langle \nu_{t,x}, f \circ Q \rangle$$

for $f \in C_0(\mathbb{R}^+ \times \mathbb{R}^3 \times S_0^3 \times \mathbb{R}^+)$ and a.e. (t, x).

Tartar framework

The linear system (2) fits into the so-called $\mathcal{A}$-free framework for linear partial differential constraints, introduced by Tartar. Consider a general linear system of l differential equations in $\mathbb{R}^N$ written as

$$\mathcal{A}z := \sum_{i=1}^{N} A^{(i)} \frac{\partial z}{\partial x_i} = 0, \tag{3}$$

where $A^{(i)}$ $(i = 1, ..., N)$ are $l \times d$ matrices and $z : \mathbb{R}^N \to \mathbb{R}^d$ is a vector-valued function.

Constant rank property

Next, we define the $l \times d$ matrix

$$\mathbb{A}(w) := \sum_{i=1}^{N} w_i A^{(i)}$$

for $w \in \mathbb{R}^N$.

Definition 5 (Constant rank)

We say that $\mathcal{A}$ has the *constant rank property* if there exists $r \in \mathbb{N}$ such that

$$\operatorname{rank} \mathbb{A}(w) = r$$

for all $w \in \mathcal{S}^{N-1}$.

Definition 6 ($\mathcal{A}$-Quasiconvexity)

A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be $\mathcal{A}$-*quasiconvex* if

$$f(z) \leq \int_{(0,1)^N} f(z + w(x))\mathrm{d}x \tag{4}$$

for all $z \in \mathbb{R}^d$ and all $w \in C^\infty_{per}((0,1)^N; \mathbb{R}^d)$ such that $\mathcal{A}w = 0$ and $\int_{(0,1)^N} w(x)\mathrm{d}x = 0$.

Finally recall that a sequence $\{z_n\}$ is called *p-equiintegrable* if the sequence $\{|z_n|^p\}$ is equiintegrable in the usual sense.

Theorem of Fonseca and Müller

Theorem 7

Let $1 \leq p < \infty$ *and let* $\{\nu_x\}_{x\in\Omega}$ *be a weakly measurable family of probability measures on* $\mathbb{R}^d$*. Let* $\mathcal{A}$ *have the constant rank property. There exists a p-equi-integrable sequence* $\{z_n\}$ *in* $L^p(\Omega;\mathbb{R}^d)$ *that generates the Young measure* ν *and satisfies* $\mathcal{A}z_n = 0$ *in* Ω *if and only if the following conditions hold:*

(i) *there exists* $z \in L^p(\Omega;\mathbb{R}^d)$ *such that* $\mathcal{A}z = 0$ *and* $z(x) = \langle \nu_x, \mathrm{id}\rangle$ *a.e.* $x \in \Omega$*;*

(ii) $\int_\Omega \int_{\mathbb{R}^d} |w|^p \, \mathrm{d}\nu_x(w)\mathrm{d}x < \infty$;

(iii) *for a.e.* $x \in \Omega$ *and all* $\mathcal{A}$*-quasiconvex functions* g *that satisfy* $|g(w)| \leq C(1+|w|^p)$ *for some* $C > 0$ *and all* $w \in \mathbb{R}^d$ *one has*

$$\langle \nu_x, g\rangle \geq g(\langle \nu_x, \mathrm{id}\rangle). \tag{5}$$

Our first main theorem is as follows

Theorem 8

Suppose the pressure function satisfies $c\rho^\gamma \leq p(\rho) \leq C\rho^\gamma$ *for some* $\gamma \geq 1$ *and* $\{(\rho_n, v_n)\}$ *is a sequence of weak solutions to the compressible Euler system* (1) *such that* $\{\rho_n\}$ *is* γ*-equiintegrable and* $\{\sqrt{\rho_n} v_n\}$ *is 2-equiintegrable. Suppose moreover* $\{(\rho_n, \sqrt{\rho_n} v_n)\}$ *generates a Young measure* ν *on* $\mathbb{R}^+ \times \mathbb{R}^3$*. Then* ν *is a measure-valued solution to the compressible Euler system* (1) *and the lifted measure* $\tilde{\nu}$ *on* $\mathbb{R}^+ \times \mathbb{R}^3 \times \mathcal{S}_0^3 \times \mathbb{R}^+$ *satisfies*

$$\langle \tilde{\nu}_{t,x}, g \rangle \geq g(\langle \tilde{\nu}_{t,x}, \mathrm{id} \rangle) \tag{6}$$

for all $\mathcal{A}_L$*-quasiconvex functions* g*.*

Our final aim is to give an example of a measure valued solution which cannot be generated by weak solutions. In fact we first prove a more general statement about $\mathcal{A}$-free rigidity which generalizes a well known result by Ball and James. Then we use this result to construct a desired example.
We start with some more definitions.

Wave cone

Definition 9

Consider a linear differential operator $\mathcal{A}$ as in (3). Its *wave cone* Λ is defined as the set of all $\bar{z} \in \mathbb{R}^d \setminus \{0\}$ for which there exists $\xi \in \mathbb{R}^N \setminus \{0\}$ such that

$$z(x) = h(x \cdot \xi)\bar{z}$$

satisfies $\mathcal{A}z = 0$ for any choice of profile function $h : \mathbb{R} \to \mathbb{R}$. Equivalently, $\bar{z} \in \Lambda$ if and only if $\bar{z} \neq 0$ and there exists $\xi \in \mathbb{R}^N \setminus \{0\}$ such that $\mathbb{A}(\xi)\bar{z} = 0$.

The wave cone Λ characterizes the directions of one dimensional oscillations compatible with (3).

Reformulation

Observe that

$$\sum_{i=1}^{N} A^{(i)} \frac{\partial z}{\partial x_i} = \left(\sum_{i=1}^{N} \sum_{k=1}^{d} A_{jk}^{(i)} \frac{\partial z_k}{\partial x_i} \right)_{j=1,\dots,l}$$
$$= \left(\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \sum_{k=1}^{d} A_{jk}^{(i)} z_k \right)_{j=1,\dots,l}.$$

Therefore, if we define the $l \times N$-matrix $Z_{\mathcal{A}}$ by

$$(Z_{\mathcal{A}})_{ji} = \sum_{k=1}^{d} A_{jk}^{(i)} z_k, \qquad j = 1, \dots, l, \qquad i = 1, \dots, N, \tag{7}$$

then (3) can be rewritten as

$$\operatorname{div} Z_{\mathcal{A}} = 0. \tag{8}$$

Reformulation II

Moreover, the condition $\mathbb{A}(\xi)\bar{z} = 0$ from the definition of the wave cone translates to $\bar{Z}_{\mathcal{A}}\xi = 0$ (where $\bar{Z}_{\mathcal{A}}$ is obtained from $\bar{z}$ via (7)), so that the following are equivalent:

1. $\bar{z} \in \Lambda$;
2. $\bar{z} \neq 0$ and $\mathrm{rank}\, \bar{Z}_{\mathcal{A}} < N$.

Rigidity theorem, part I

Theorem 10

Let $\Omega \subset \mathbb{R}^N$ *be a domain,* $\mathcal{A}$ *a linear operator of the form* (3)*, and* $1 < p < \infty$*. Let moreover* $z_n : \Omega \to \mathbb{R}^d$ *be a family of functions with*

$$\|z_n\|_{L^p(\Omega;\mathbb{R}^d)} \leq c,$$

$$\mathcal{A} z_n = 0 \text{ in } \mathcal{D}'(\Omega), \tag{9}$$

and suppose (z_n) *generates a compactly supported Young measure* $\nu_x \in \mathcal{M}^1(\mathbb{R}^d)$ *such that*

$$\mathrm{supp}[\nu_x] \subset \{\lambda \bar{z}_1 + (1-\lambda)\bar{z}_2,\ \lambda \in [0,1]\} \text{ for a.a. } x \in \Omega \tag{10}$$

and for some given constant states $\bar{z}_1, \bar{z}_2 \in \mathbb{R}^d$*,* $\bar{z}_1 \neq \bar{z}_2$*. Suppose that*

$$\bar{z}_2 - \bar{z}_1 \notin \Lambda.$$

Rigidity theorem, part II

Theorem 10 (cont.)

Then

$$z_n \to z_\infty \quad \text{in } L^p(\Omega),$$

which implies that

$$\nu_x = \delta_{z_\infty(x)}, \; z_\infty(x) \in \{\lambda \bar{z}_1 + (1-\lambda)\bar{z}_2, \; \lambda \in [0,1]\} \text{ for a.a. } x \in \Omega.$$

More specifically, z_∞ *is a constant function of the form*

$$z_\infty = \overline{\lambda}\bar{z}_1 + (1-\overline{\lambda})\bar{z}_2.$$

for some fixed $\overline{\lambda} \in [0,1]$.

Explicit example

Using the previous Theorem 10 we prove the following

Theorem 11

There exists a measure-valued solution of the compressible Euler system (1) *which is not generated by any sequence of L^p-bounded weak solutions to* (1) *(for any choice of $p > 1$).*

- Any reasonable sequence of approximate solutions of (1) will satisfy some uniform energy bound, so that the assumption of L^p-boundedness will always be met.
- As Theorem 10 did not require any equiintegrability, the statement of Theorem 11 is true even when the potential generating sequence is allowed to concentrate. I.e. there exists a generalized measure-valued solution which can not be generated by a sequence of weak solutions (take the measure from Theorem 11 as the oscillation part and choose the concentration part arbitrarily).

Thank you

Thank you for your attention.