# PhD Proposals: Geometry of Simplicial Finite Element Meshes

**Introduction.** Geometry of meshes plays a crucial role in the analysis and practice of the finite element method (FEM) which is one of the most powerful and widely used techniques in numerical solving of problems of mathematical physics. Thus, a suitably generated family of meshes guarantees the optimal interpolation order, convergence or even superconvergence of FEM, etc., see [KRC]. It also helps to derive various a posteriori error estimates increasing the controllability of real FEM calculations and to perform qualitatively correct FEM simulations. It seems that Milos Zlamal [Z] was the first who introduced in 1968 the so-called minimum angle condition used for proving convergence of FEM schemes on triangular meshes. From that time onward many other useful geometric (mostly of angle-type as the maximum angle condition, etc.) conditions on the shape of various finite elements were proposed (see e.g. the reference list for more details).

**Aim.** There are still several open research topics that have to be addressed in the context of this proposal:

1) Bisection-type algorithms for refining triangular partitions are sufficiently well studied. However, in higher dimensions, e.g. in the tetrahedral case, we do not know yet, whether the longest-edge bisection (or their natural generalizations, the so-called $n$-sections) algorithm produces simplices that do not degenerate when the mesh size tends to zero – see [HKK14].

2) Construction of nonobtuse simplicial meshes for bounded polyhedral or polytopic domains [BKKS]. For instance, an open problem is the following: Is there a face-to-face partition of an arbitrary tetrahedron (polyhedron) into nonobtuse simplices?

3) Path simplices are natural generalizations of a right triangle to higher dimensions. Can each simplex be decomposed into a finite number of path simplices?

4) Acute simplices (all of whose dihedral angles are acute) play an important role in the construction of qualitatively correct finite element schemes, since they produce monotone finite element matrices. There are many algorithms to decompose two- and three-dimensional space into acute simplices. It is known that the five-dimensional or higher dimensional Euclidean space cannot be decomposed into acute simplices [K]. However, the construction of a face-to-face

simplicial tiling of four-dimensional space is still an open problem [BKKS]. What are all the simplicial space-fillers in $d$-dimensions?

5) To find necessary and sufficient geometric conditions providing a convergence of the FEM applied to elliptic or parabolic-type problems, see [HKK12].

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