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Sur le problème des solutions dissipatives à valeurs mesure du système de Navier-Stokes-Fourier compressible

Note on the problem of dissipative measure-valued solutions to the compressible Navier-Stokes-Fourier system

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Abstract

We introduce a dissipative measure-valued solution to the full compressible Navier- Stokes- Fourier system. We derive a relative entropy inequality for measure-valued solution as a generalization of the "classical" entropy inequality introduced by Dafermos [2], Mellet-Vasseur [11], and Feireisl-Novotný [5].

Résumé

Nous considérons des solutions dissipatives à valeurs mesure du système de Navier-Stokes-Fourier compressible. Nous nous intéressons particulièrement à une inégalité d'entropie généralisant l'inégalité d'entropie "classique" introduite par Dafermos [2], Mellet, Vasseur [11] et Feireisl-Novotný [5].

1. Introduction

We consider measure-valued solutions of the compressible Navier-Stokes-Fourier system. The advantage of measure-valued solutions is the property that in many cases, the solutions can be obtained from weakly convergent sequences of approximate solutions.

Measure-valued solutions for systems of hyperbolic conservations laws were initially introduced by DiPerna [3]. He used Young measures to pass to limit in the artificial viscosity term. In the case of the incompressible Euler equations, DiPerna and Majda [4] also proved global existence of measure-valued solutions for any initial data with finite energy. They introduced generalized Young measures to take into account oscillation and concentration phenomena. Thereafter the existence of measure-valued solutions was finally shown for further models of fluids, e.g. compressible Euler and Navier-Stokes equations [13],[12].

Recently, weak-strong uniqueness for measure-valued solutions of isentropic Euler equations were proved in [9]. Inspired by previous results, the concept of dissipative measure-valued solution was finally applied to the barotropic compressible Navier-Stokes system [10].

In this note we introduce a dissipative measure-valued solution for the full Navier-Stokes-Fourier system and derive a relative entropy inequality in term of measure-valued solutions.

The motion of the fluid is governed by the standard field equations of classical continuum fluid mechanics describing the evolution of the mass density ρ, the velocity field $\mathbf{u}$, and the absolute temperature θ as functions of the time $t \in \mathbb{R}_+$ and the Eulerian spatial coordinate $x \in \Omega$, where Ω is a bounded region of $\mathbb{R}^3$. The evolution of the compressible viscous heat conductive flow equation reads

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \quad \text{in } (0,T) \times \Omega, \tag{1}$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \operatorname{div}\mathcal{S} \qquad \text{in } (0,T) \times \Omega, \tag{2}$$

$$\partial_t \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e(\rho,\theta) \right) + \operatorname{div} \left(\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e(\rho,\theta) + p \right) \mathbf{u} + q - \mathcal{S}\mathbf{u} \right) = 0 \tag{3}$$

The symbol $p = p(\rho,\theta)$ denotes the thermodynamic pressure and $e = e(\rho,\theta)$ is the specific internal energy, related through Maxwell's equation

$$\frac{\partial e}{\partial \rho} = \frac{1}{\rho^2}\left(p(\rho,\theta) - \theta \frac{\partial p}{\partial \theta} \right). \tag{4}$$

Furthermore, $\mathcal{S}$ is the viscous stress tensor determined by

$$\mathcal{S} = \mu \left(\nabla \mathbf{u} + \nabla^t \mathbf{u} - \frac{2}{3}\operatorname{div}\mathbf{u} \right) + \nu \operatorname{div}\mathbf{u}\mathcal{I}, \tag{5}$$

where μ is the shear viscosity coefficient and ν the bulk viscosity coefficient and both are effective functions of the temperature, q is the heat flux given by Fourier's law

$$q = -\kappa \nabla \theta, \tag{6}$$

with the heat conductivity coefficient $\kappa = \kappa(\theta) > 0$.

1.1. *Hypotheses*

We consider the pressure in the form

$$p(\rho,\theta) = \theta^{5/2} P\left(\frac{\rho}{\theta^{3/2}} \right) + \frac{a}{3}\theta^4, \ a > 0, \tag{7}$$

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where $P : [0,\infty) \to [0,\infty)$ is a given function with the following properties :

$$P \in C^1[0,\infty),\ P(0) = 0,\ P'(Z) > 0,\ \text{for all } Z \geq 0, \tag{8}$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z \geq 0, \tag{9}$$

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{10}$$

After Maxwell's equation (4), the specific internal energy e is

$$e(\rho,\theta) = \frac{3}{2}\left(\frac{\theta^{5/2}}{\rho}\right) P\left(\frac{\rho}{\theta^{3/2}}\right) + a\frac{\theta^4}{\rho}, \tag{11}$$

and the associated specific entropy reads

$$E(\rho,\theta) = M\left(\frac{\rho}{\theta^{3/2}}\right) + \frac{4a}{3}\frac{\theta^3}{\rho} \quad \text{with } M'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2} < 0. \tag{12}$$

The transport coefficients μ, η, and κ are continuously differentiable functions of the absolute temperature such that

$$0 < c_1(1+\theta) \leq \mu(\theta),\ \mu'(\theta) < c_2,\ 0 \leq \eta(\theta) \leq c(1+\theta), \tag{13}$$

$$0 < c_1(1+\theta^3) \leq \kappa(\theta) \leq c_2(1+\theta^3) \tag{14}$$

for any $\theta \geq 0$. As the term $\mathbb{S}\boldsymbol{u}$ in the total energy balance (3) is not controlled on the (hypothetical) vacuum zones of vanishing density, we will replace (3) by the internal energy equation

$$\partial_t(\rho e) + \operatorname{div}_x(\rho e\boldsymbol{u}) + \operatorname{div}_x\boldsymbol{q} = \mathcal{S} : \nabla_x\boldsymbol{u} - p\operatorname{div}_x\boldsymbol{u}, \tag{15}$$

moreover, dividing (15) on θ and using Maxwell's relation (4), we may rewrite (15) as the entropy equation

$$\partial_t(\rho E) + \operatorname{div}_x(\rho E\boldsymbol{u}) + \operatorname{div}_x\left(\frac{\boldsymbol{q}}{\theta}\right) = \frac{1}{\theta}\left(\mathcal{S} : \nabla_x\boldsymbol{u} - \frac{\boldsymbol{q}\cdot\nabla_x\theta}{\theta}\right) =: \varsigma, \tag{16}$$

where $\varsigma := \frac{1}{\theta}\left(\mathcal{S} : \nabla_x\boldsymbol{u} - \frac{\boldsymbol{q}\cdot\nabla_x\theta}{\theta}\right)$ is the (positive) matter entropy production.

1.2. *Dissipative measure-valued solutions to the compressible Navier-Stokes-Fourier system*

We introduce the concept of dissipative measure-valued solution to the full system of compressible Navier-Stokes-Fourier equations (in the spirit of [10,9])

Definition 1 *We say that a parameterized measure* $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$*,*

$$\nu \in L^\infty_{weak}\left((0,T)\times\Omega; \mathcal{P}\left([0,\infty)\times\mathbb{R}^N\right)\right), \quad \langle\nu_{t,x}; s\rangle \equiv \varrho, \quad \langle\nu_{t,x}; \mathbf{v}\rangle \equiv \mathbf{u}$$

$$\langle\nu_{t,x}; \eta\rangle \equiv \theta$$

is a dissipative measure-valued solution of the Navier-Stokes-Fourier system (1) - (3) in $(0,T)\times\Omega$*, with the initial conditions* ν_0 *and dissipation defect* $\mathcal{D}$*,*

$$\mathcal{D} \in L^\infty(0,T), \quad \mathcal{D} \geq 0,$$

if the following holds.

(i) Continuity equation : There exist a measure $r^C \in L^1\left([0,T], \mathcal{M}\left(\overline{\Omega}\right)\right)$ *and* $\chi \in L^1(0,T)$ *such that for a.a.* $\tau \in (0,T)$ *and every* $\psi \in C^1\left([0,T]\times\overline{\Omega}\right)$*,*

$$\left|\left\langle r^{C}\left(\tau\right);\nabla_{x}\psi\right\rangle\right|\leq\chi\left(\tau\right)\mathcal{D}\left(\tau\right)\left\|\psi\right\|_{C^{1}\left(\overline{\Omega}\right)} \tag{17}$$

and

$$\int_{\Omega}\left\langle \nu_{t,x};s\right\rangle \psi\left(\tau,\cdot\right)dx-\int_{\Omega}\left\langle \nu_{0};s\right\rangle \psi\left(0,\cdot\right)dx$$

$$=\int_{0}^{\tau}\int_{\Omega}\left[\left\langle \nu_{t,x};s\right\rangle \partial_{t}\psi+\left\langle \nu_{t,x};s\mathbf{v}\right\rangle \cdot\nabla_{x}\psi\right]dxdt+\int_{0}^{\tau}\left\langle r^{C};\nabla_{x}\psi\right\rangle dt. \tag{18}$$

(ii) *Momentum equation : Velocity* $\mathbf{u}=\left\langle \nu_{t,x};\mathbf{v}\right\rangle \in L^{2}\left(0,T;W_{0}^{1,2}\left(\Omega;\mathbb{R}^{N}\right)\right)$*, and there exists a measure* $r^{M}\in L^{1}\left(\left[0,T\right],\mathcal{M}\left(\overline{\Omega}\right)\right)$ *and* $\xi\in L^{1}\left(0,T\right)$ *such that for a.a.* $\tau\in\left(0,T\right)$ *and every* $\varphi\in C^{1}\left(\left[0,T\right]\times\overline{\Omega};\mathbb{R}^{N}\right)$, $\varphi|_{\partial\Omega}=0$,

$$\left|\left\langle r^{M}\left(\tau\right);\nabla_{x}\varphi\right\rangle\right|\leq\xi\left(\tau\right)\mathcal{D}\left(\tau\right)\left\|\varphi\right\|_{C^{1}\left(\overline{\Omega}\right)} \tag{19}$$

and

$$\int_{\Omega}\left\langle \nu_{t,x};s\mathbf{v}\right\rangle \varphi\left(\tau,\cdot\right)dx-\int_{\Omega}\left\langle \nu_{0};s\mathbf{v}\right\rangle \varphi\left(0,\cdot\right)dx$$

$$=\int_{0}^{\tau}\int_{\Omega}\left[\left\langle \nu_{t,x};s\mathbf{v}\right\rangle \partial_{t}\varphi+\left\langle \nu_{t,x};s\left(\mathbf{v}\otimes\mathbf{v}\right)\right\rangle :\nabla_{x}\varphi+\left\langle \nu_{t,x};p(s,\eta)\right\rangle div_{x}\varphi\right]dxdt$$

$$-\int_{0}^{\tau}\int_{\Omega}S\left(\eta,\nabla_{x}\mathbf{u}\right):\nabla_{x}\varphi dxdt+\int_{0}^{\tau}\left\langle r^{M};\nabla_{x}\varphi\right\rangle dt. \tag{20}$$

(iii) *Entropy inequality : Temperature* $\theta=\left\langle \nu_{t,x};\eta\right\rangle \in L^{2}\left(0,T;W^{1,2}\left(\Omega;\mathbb{R}^{N}\right)\right)$ *and there exists a measure* $r^{\xi}\in L^{1}\left(\left[0,T\right],\mathcal{M}\left(\overline{\Omega}\right)\right)$*,and* $\Psi\in L^{1}(0,T)$ *such that for a.a.* $\tau\in(0,T)$ *and any* $\sigma\in C^{1}((\left[0,T\right]\times\overline{\Omega}),\frac{\partial\sigma}{\partial n}=0$

$$\left|\left\langle r^{\xi}\left(\tau\right);\nabla_{x}\sigma\right\rangle\right|\leq\Psi\left(\tau\right)\mathcal{D}\left(\tau\right)\left\|\sigma\right\|_{C^{1}\left(\overline{\Omega}\right)} \tag{21}$$

and

$$-\int_{\Omega}\left\langle \nu_{t,x};sE(s,\eta)\right\rangle \sigma\left(\tau,\cdot\right)dx+\int_{\Omega}\left\langle \nu_{0};sE(s,\eta)\right\rangle \sigma\left(0,\cdot\right)\ dx \tag{22}$$

$$+\int_{0}^{\tau}\int_{\Omega}\left\langle \nu_{t,x},\frac{1}{\eta}\right\rangle \sigma\left[S(\eta,\nabla\mathbf{u}):\nabla\mathbf{u}-\frac{q(\eta,\nabla\eta)\nabla\eta}{\eta}\right]\ dx\ dt$$

$$\leq-\int_{0}^{\tau}\int_{\Omega}\left[\left\langle \nu_{t,x};sE(s,\eta)\right\rangle \partial_{t}\sigma+\left\langle \nu_{t,x};sE(s,\eta)\mathbf{v}\right\rangle \nabla\sigma+\left\langle \nu_{t,x},\frac{1}{\eta}\right\rangle q(\eta,\nabla\eta)\nabla\sigma\right]\ dx\ dt$$

$$+\int_{0}^{\tau}\left\langle r^{\xi};\nabla_{x}\sigma\right\rangle dt$$

(iv) *Balance of total energy :*

$$\int_{\Omega}\left\langle \nu_{t,x};\left(s|\mathbf{v}|^{2}+se(s,\eta)\right)\right\rangle\ dx=\ \int_{\Omega}\left\langle \nu_{0};\left(s|\mathbf{v}|^{2}+se(s,\eta)\right)\right\rangle dx,$$

for a.a. $\tau\in(0,T)$*. In addition, the following version of Poincaré's inequality holds for a.a.* $\tau\in(0,T)$ *:*

$$\int_{0}^{T}\int_{\Omega}\left\langle \nu_{t,x};|\mathbf{v}-\mathbf{u}|^{2}\right\rangle dxdt\leq c_{p}D(\tau). \tag{23}$$

2. Relative entropy inequality

We introduce the relative entropy functional

$$\mathcal{E}\left(\varrho, \mathbf{u}, \vartheta \mid r, \mathbf{U}, \Theta\right) = \int_\Omega \frac{1}{2}\varrho\left|\mathbf{u}-\mathbf{U}\right|^2 + H_\Theta(\varrho,\vartheta) - \partial_\varrho H_\Theta(r,\Theta)(\varrho - r) - H_\Theta(r,\Theta)dx,$$

$$H_\Theta(\varrho,\vartheta) = \varrho\left[e(\varrho,\vartheta) - \Theta s(\varrho,\vartheta)\right],$$

where $\varrho, \mathbf{u}, \vartheta$ is a weak solution and $r, \mathbf{U}, \Theta$ are arbitrary "test" functions satisfying the basic properties of $\varrho, \mathbf{u}, \vartheta$, specially r , Θ is positive, $\mathbf{U}$, Θ satisfy the relevant boundary conditions (see Feireisl et al. [5], Germain [8], Mellet and Vasseur [11], Dafermos [2]).

In fact it is shown in [6] that any finite energy weak solution $(\varrho, \mathbf{u})$ to the compressible barotropic Navier-Stokes system satisfies the relative entropy inequality for any pair $(r, \mathbf{U})$ of sufficiently smooth test functions such that $r > 0$ and $\mathbf{U}|_{\partial\Omega} = 0$ and this inequality is an essential tool in order to prove the convergence to a target system. For other details see [7].

In the framework of dissipative measure-valued solution (in the spirit of [10]-[9]) we define the functional

$$\mathcal{E}_{mv}\left(\varrho, \mathbf{u}, \vartheta \mid r, \mathbf{U}, \Theta\right) \equiv \int_\Omega \left\langle \nu_{t,x}; \frac{1}{2}s\left|\mathbf{v}-\mathbf{U}\right|^2 + H_\Theta(s,\eta) - \partial_\varrho H_\Theta(r,\Theta)(s-r) - H_\Theta(r,\Theta)\right\rangle dx.$$

Theorem 2.1 *Let the parameterized measure* $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$, *with*

$$\nu \in L^\infty_{weak}\left((0,T)\times\Omega; \mathcal{P}\left([0,\infty)\times\mathbb{R}^N\right)\right), \quad \langle \nu_{t,x}; s\rangle \equiv \varrho, \quad \langle \nu_{t,x}; \mathbf{v}\rangle \equiv \mathbf{u}, \ \ \langle \nu_{t,x}; \eta\rangle \equiv \theta,$$

be a dissipative measure-valued solution of the Navier-Stokes-Fourier system (1) - (3) in $(0,T)\times\Omega$, *with the initial conditions* ν_0 *and dissipation defect* $\mathcal{D}$.

Then $(s, \mathbf{v}, \theta)$ *satisfies the following relative entropy inequality*

$$\int_\Omega \left\langle \nu_{t,x}; \left(\frac{1}{2}s\left|\mathbf{v}-\mathbf{U}\right|^2 + H_\Theta(s,\eta) - \partial_\varrho H_\Theta(r,\Theta)(s-r) - H_\Theta(r,\Theta)\right)(\tau,\cdot)\right\rangle \ dx$$

$$+ \int_0^\tau \int_\Omega \left\langle \nu_{t,x}; \frac{1}{\eta}\right\rangle \Theta \left(S(\eta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\eta,\nabla_x\eta)\cdot\nabla_x\eta}{\eta}\right) \ dx \ dt$$

$$\leq \int_\Omega \left\langle \nu_{0,x}; \left(\frac{1}{2}s\left|\mathbf{v}-\mathbf{U}(0,\cdot)\right|^2 + \ H_{\Theta(0,\cdot)}(s,\eta) - \partial_\varrho H_{\Theta(0,\cdot)}(r(0,\cdot),\Theta(0,\cdot))(s - r(0,\cdot)) - H_{\Theta(0,\cdot)}(r(0,\cdot),\Theta(0,\cdot))\right)\right\rangle dx$$

$$+ \int_0^\tau \mathcal{R}(s,\mathbf{v},\theta,r,\mathbf{U},\Theta)(t)dt \qquad (24)$$

for a.a. $\tau \in (0,T)$ *and any pair of test functions* $(r, \mathbf{U}, \Theta)$ *such that* $\mathbf{U} \in C^1([0,T]\times\bar{\Omega}, R^n), \mathbf{U}|_{\partial\Omega} = 0$, $r \in C^\infty_c(\overline{Q_T})$, $r > 0$, $\Theta > 0$.

The remainder in the right hand side of (24) is given by

$$\int_0^\tau \mathcal{R}(s,\mathbf{v},\theta,r,\mathbf{U},\Theta)(t) \ dt = \int_0^\tau \int_\Omega \left(\langle \nu_{t,x}; s\rangle \partial_t \mathbf{U}\cdot(\mathbf{U}-\mathbf{u}) + \langle \nu_{t,x}; s\mathbf{v}\rangle \cdot \nabla_x \mathbf{U}\cdot(\mathbf{U}-\mathbf{u})\right) \ dx \ dt$$

$$\int_0^\tau \int_\Omega \left(\langle \nu_{t,x}; -p(s,\eta)\rangle \, div_x \mathbf{U} + S(\eta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U}\right) \ dx \ dt$$

$$- \int_0^\tau \int_\Omega \left(\langle \nu_{t,x}; s\rangle \left(E(s,\eta) - E(r,\Theta)\right)\partial_t \Theta\right) + \left(\langle \nu_{t,x}; s\mathbf{v}\rangle \left(E(s,\eta) - E(r,\Theta)\right)\mathbf{u}\cdot\nabla_x\Theta\right) dxdt$$

$$- \int_0^\tau \int_\Omega \left\langle \nu_{t,x}; \frac{1}{\eta}\right\rangle \mathbf{q}(\eta,\nabla_x\eta)\cdot\nabla_x\Theta dxdt + \int_0^\tau \int_\Omega \left(\left(1 - \frac{1}{r}\langle \nu_{t,x}; s\rangle\right)\partial_t p(r,\Theta)\right) - \frac{1}{r}\langle \nu_{t,x}; s\mathbf{v}\rangle \mathbf{u}\cdot\nabla_x p(r,\Theta)dxdt$$

$$+ \int_0^\tau \left\langle r^M; \nabla_x \mathbf{U}\right\rangle dt + \int_0^\tau \int_\Omega \left\langle r^C; \frac{1}{2}\nabla_x\left|\mathbf{U}\right|^2\right\rangle dxdt + \int_0^\tau \left\langle r^\xi; \nabla_x \Theta\right\rangle dt. \qquad (25)$$

The proof follows the method used in the analysis of relative entropy for the full system [5] together with the new concept of dissipative measure-valued solution [10]. More details will be found in [1].

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