INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

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Preprint No. 63-2015

PRAHA 2015

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Abstract

We study a dynamic contact problem for a thermoelastic von Kármán plate vibrating against a rigid obstacle. The plate is subjected to a perpendicular force and to a heat source. The dynamics is described by a hyperbolic variational inequality for deflections. The parabolic equation for a thermal strain resultant contains the time derivative of the deflection. We formulate a weak solution of the system and verify its existence using the penalization method. A detailed analysis of the velocity, acceleration, and reaction force of the solution is given. The singular nature of the dynamic contact makes it necessary to treat the acceleration and contact force as time–dependent measures with nonzero singular parts in the zones of contact. Accordingly, the velocity field over the plate suffers (global) jumps at a countable number of times with natural physical interpretations of the signs of the jumps.

Keywords Thermoelastic plate, unilateral dynamic contact, rigid obstacle, penalization, measure valued accelerations and forces

Mathematics Subject Classification (2010) 35L87, 74M15, 35J87

1 Introduction

The dynamic contact problems are not frequently solved in the framework of variational inequalities if we disregard results obtained for rather rough approximate contact models (as e.g. the normal compliance one allowing an unrealistic unlimited interpenetration between the body and the obstacle or so called bilateral contact replacing the real contact condition by a homogeneous Dirichlet one). For the elastic problems only a very limited amount of results is available

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(cf. [8] and the literature cited there). We have solved these problems for geometrically nonlinear plates and shells in [2] and [3] respectively. We concentrate here not only on purely mechanical impact to the plate being under some load and possibly contacting a rigid obstacle, we also take into account its thermal deformation. However, we shall not consider the heat exchange between the plate and the obstacle. This can be guaranteed [14] if the lower and upper faces of the plate are thermally isolated. Our model is similar to that of [17], [12, Chapter 8], where the thermoelastic boundary contact for the radially symmetric body is considered and also no heat exchange between the body and the rigid foundation appeared.

We shall use the model derived in [15] under the assumption of a small variation of temperature compared with its reference temperature. The assumptions of thermally isolated faces of the plate enable us to consider the similar system as in [15] but with the unknown contact force in the equation for the deflection of its middle surface. In its variational form the originally hyperbolic equation for the deflections is substituted here by the variational inequality, involving also the geometrical nonlinearities in deflections.

After the formulation of the original problem in Section 2 we give the problem the variational formulation in Section 3 and analyze the properties of solutions. In Section 4 the existence theorem 3.5 is proved. First in Subsection 4.1 we formulate and solve the penalized initial-boundary value problem. Then with the help of a uniform estimate of the penalty term we achieve a (sub)sequence converging to a weak solution of the original problem in Subsection 4.2. Section 5 describes the spaces of vector valued measures $\mathscr{M}(I;V)$ and $\mathscr{M}(I;L_2(\Omega))$ and the associated space $\mathscr{M}_0(Q)$ occurring in our proofs. Section 6 proves Theorem 3.6, which analyses the measures representing the contact force and acceleration. Appendix A in Section 7 fixes our notation of the function spaces. Appendix B in Section 8 reviews basic notions of vector valued measures. Appendix C in Section 9 deals with some aspects of our evolution spaces.

2 The model

For convenience of the reader we describe the genesis of the applied model. We assume a thin isotropic elastic plate occupying the domain

$$G = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x = (x_1, x_2) \in \Omega,\ |x_3| < h/2\}.$$

Its middle plane $\Omega \subset \mathbf{R}^2$ is a bounded star–shaped domain with a piecewise C^2 boundary or the domain with a $C^{3,1}$ boundary $\partial\Omega$. Further, we work on a bounded time interval $I \equiv (0,T)$ and we let $Q = I \times \Omega$ and $S = I \times \partial\Omega$. The unit outer normal vector is denoted by $\mathbf{n} = (n_1, n_2)$.

The isotropic material of the plate is characterized by the following material constants:

$\rho > 0$	the density of the material,
$E > 0$	the Young modulus,
$\nu \in \left(-1, \frac{1}{2}\right)$	the Poisson ratio,
$c > 0$	the specific heat,
$\lambda > 0$	the thermal conductivity,
α	the thermal expansion coefficient.

The processes in the plate are described by the displacement $\mathbf{u} \equiv (u_1, u_2, u_3)$ and the deviation τ of temperature from a given constant reference temperature $\tau_\circ$. Changes in the displacement and temperature produce stresses $\boldsymbol{\sigma} = (\sigma_{i,j})_{i,j=1}^3$, fluxes of heat $\mathbf{q} = (q_1, q_2, q_3)$ and changes in energy, expressed below equivalently by the changes of entropy S. The strain tensor is defined as

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{3,i}u_{3,j}) - x_3\, u_{3,ij}, \;\; i,j = 1,2; \quad \epsilon_{i3} \equiv 0, \quad i = 1,2,3,$$

where the comma followed by one or more indices $i,\ j = 1,2,3$, denotes the partial derivatives with respect to the components of $x = (x_1, x_2, x_3)$. In the sequel we shall use the Einstein summation convention. The quantities $\boldsymbol{\sigma}$, $\mathbf{q}$, and S are given by

$$\begin{aligned}
&\sigma_{ij} = \frac{E}{1-\nu^2}\big((1-\nu)\epsilon_{ij} + \nu\epsilon_{kk}\delta_{ij} - (1+\nu)\alpha\tau\delta_{ij}\big), && i,\ j = 1,2,\\
&\sigma_{i3} = 0, && i = 1,2,3,\\
&q_i = -\lambda\tau_{,i}, \quad q_3 = 0, && i = 1,2,\\
&S = \frac{E\alpha}{1-2\nu}\epsilon_{kk} + \frac{\rho c}{\tau_\circ}\tau,
\end{aligned}$$

see, e.g., [16, Chapter 1]. The evolution of $\mathbf{u}$ and τ is governed by the equations of balance of linear momentum and energy

$$\left.\begin{aligned}
&\rho\ddot{\mathbf{u}} - \operatorname{div}\boldsymbol{\sigma} = \mathbf{b},\\
&\tau_\circ\dot{S} + \operatorname{div}\mathbf{q} = q
\end{aligned}\right\} \quad \text{on} \quad I \times G,$$

where $\mathbf{b}$ is the density of the external force (load), q is the heat source, the superimposed dot denotes the time derivative, and div is the three dimensional divergence. To eliminate the x_3 variable from these equations, we introduce the averaged displacement u and the thermal strain resultant function θ by

$$u(t,x) = \frac{1}{h}\int_{-h/2}^{h/2} u_3\, dx_3, \quad \theta(t,x) = \frac{12\alpha}{h^3}\int_{-h/2}^{h/2} x_3\, \tau\, dx_3.$$

Integrating the balance equations and making some approximations [15] we obtain the simplified form of balance equations

$$\left.\begin{aligned}
&\ddot{u} - a\,\Delta\,\ddot{u} + b\,\Delta^2\,u + \chi\,\Delta\,\theta - [u,v] = p,\\
&\Delta^2\,v + E[u,u] = 0,\\
&\dot{\theta} - \kappa\,\Delta\,\theta + d\theta - e\,\Delta\,\dot{u} = q
\end{aligned}\right\} \quad \text{on} \quad Q,$$

in which p is the perpendicular force on the plate, q is the appropriately modified heat source, v is the Airy stress function and Δ is now the two dimensional laplacean,

$$[u,v] \equiv u_{,11}v_{,22} + u_{,22}v_{,11} - 2u_{,12}v_{,12},$$

where the constants a, b, κ, d, and χ are as follows:

$$a = \frac{h^2}{12},\ b = \frac{Eh^2}{12\rho(1-\nu^2)},\ \kappa = \frac{\lambda}{\rho c},\ d = \frac{12\kappa}{h^2},\ e = \frac{\kappa\alpha^2\tau_\circ E}{\lambda(1-2\nu)},\ \chi = \frac{b(1+\nu)}{2}.$$

The presence of an obstacle imposes the condition $u \geq 0$; further, we write $p = f + g$ where f is the external force and g the reaction force arising from the contact with the obstacle, acting only if $u = 0$, thus imposing the conditions

$$u \geq 0, \quad g \geq 0, \quad ug = 0.$$

To summarize, we deal with a simply supported plate with zero lateral forces and zero thermal strain resultant on the boundary, acted upon by the perpendicular load f and the heat source q. The quadruple $\{u, v, g, \theta\}$ represents the unknown deflection of the middle plane, Airy stress function, contact force between the plate and the rigid obstacle and the rescaled thermal strain resultant. The evolution is governed by the following system of equations:

$$\left.\begin{array}{l} \ddot{u} - a\,\Delta\,\ddot{u} + b\,\Delta^2\,u + \chi\,\Delta\,\theta - [u,v] = f + g, \\ u \geq 0,\ g \geq 0,\ ug = 0, \\ \Delta^2\,v + E[u,u] = 0, \\ \dot{\theta} - \kappa\,\Delta\,\theta + d\theta - e\,\Delta\,\dot{u} = q \end{array}\right\} \text{ on } Q, \tag{1}$$

with the boundary conditions:

$$u = w, \quad M(u) = 0, \quad v = 0, \quad \partial_{\mathbf{n}} v = 0, \quad \theta = 0 \quad \text{on} \quad S, \tag{2}$$

$$M(u) \equiv \ b\,\Delta\,u + b(1-\nu)(2n_1 n_2 u_{,12} - n_1^2 u_{,22} - n_2^2 u_{,11}), \quad \partial_{\mathbf{n}} \equiv \partial/\partial\mathbf{n}$$

and the initial data:

$$u(0,\cdot) = u_0, \quad \dot{u}(0,\cdot) = v_0, \quad \theta(0,\cdot) = \theta_0 \quad \text{on} \quad \Omega. \tag{3}$$

2.1 Remarks

(i) The boundary conditions (2) for simply supported plate and zero boundary thermal stress resultant enable us to derive the *a priori* estimates in Sections 4.1 and 4.2, below. It is possible to consider also other types of boundary conditions with a slightly more complicated way of deriving *a priori* estimates unavoidable for the convergence process.

(ii) The assumption of the thermally isolated upper face of the plate may seem a bit artificial, but the symmetry of assumptions with respect to x_3 axis is inevitable in order to derive the mathematical model with an unknown thermal strain resultant.

3 Variational formulation of the problem

The singular nature of the set of instants of collision of the plate with the obstacle causes the velocity field $\dot{u}$ to exhibit jump discontinuities at these instances. Consequently the acceleration $\ddot{u}$ is a measure, not absolutely continuous with respect to the Lebesgue measure, at least as the dependence on time is concerned. For such measures, the values of integrands on sets of Lebesgue measure do matter. We therefore carefully distinguish between functions defined everywhere on their domains and the Lebesgue classes of equivalence in the treatment below. The reader is referred to Appendix 7 for the notation of function spaces.

3.1 Kinematics We assume that the function w occurring in the boundary condition $(2)_1$ belongs to $H^2(\Omega)$. The set of instantaneous values of deflections $u(t,\cdot):\Omega\to\mathbf{R}$ is the set of all nonnegative elements of the set $w+V$, where

$$V = H^2(\Omega)\cap \mathring{H}^1(\Omega);$$

the space of instantaneous values of velocity is $\mathring{H}^1(\Omega)$ and of the instantaneous values of acceleration is $L_2(\Omega)$. The character of the dependence of these quantities on time is embodied in the definitions of the evolution spaces (5), below. We introduce the following spaces to facilitate these definitions.

We denote by V^* the dual of V, and we let $\langle\cdot,\cdot\rangle$ be the duality pairing between V and V^*. Identifying the dual of $L_2(\Omega)$ with itself, we have the usual inclusions

$$V\subset L_2(\Omega)\subset V^*$$

with compact dense embeddings.

3.1.1 *Evolution of deflections*

Let $C_w(\bar I;V)$ denote the space of weakly continuous maps from the closure $\bar I$ of I to V, i.e., the set of all $u:\bar I\to V$ such that the real valued function $t\mapsto\langle\xi,u(t)\rangle$ is continuous on $\bar I$ for any $\xi\in V$. The evolutions of deflections will be the nonnegative elements of $w+C_w(\bar I;V)$.

3.1.2 *Evolution of velocities*

Extending the well known terminology, we say that a map $v:\bar I\to\mathring{H}^1(\Omega)$ is weakly regulated if the weak left and right weak limits $v(t-)$ and $v(t+)$ exist at every point of $\bar I$ (with one sided limits only at the endpoints of $\bar I$), i.e., for each $t\in\bar I$ there exist elements $v(t\pm)\in\mathring{H}(\Omega)$ such that $\langle F,v(t\pm)\rangle=\lim_{s\to t\,\pm}\langle F,v(s)\rangle$ for every $F\in H^{-1}(\Omega)$. By Remark 9.2, below, one has $v(t+)=v(t-)$ everywhere on $\bar I$ except at at most countable set of points of $\bar I$. (This will be proved as a general assertion about regulated maps; however, in the present case this follows from the fact that v is the velocity whose derivative [the acceleration] is a measure of a special form.) To remove the ambiguity in the value $v(t)$ if $v(t-)\neq v(t+)$, we put $v(t):=v(t+)$, obtaining thus a weakly right continuous weakly regulated map. We denote by $R_w(\bar I;\mathring{H}^1(\Omega))$ the set of all weakly right continuous weakly regulated maps from $\bar I$ to $\mathring{H}^1(\Omega)$.

3.1.3 *Evolution of accelerations*

Finally, time evolutions of accelerations will be modeled as signed measures on Q from the space $\mathscr{M}_0(Q)$ consisting of all measures $M\in\mathscr{M}(Q)$ such that

$$\Big|\int_Q\varphi\,dM\Big|\le c|\varphi|_{C_0(I;L_2(\Omega))}\tag{4}$$

for some c and all $\varphi\in C_0(Q)$. By Proposition 5.4 (below) this space can be identified with the space $\mathscr{M}_0(I;L_2(\Omega))$ of all countably additive $L_2(\Omega)$ valued measures of finite variation on I. Corollary 5.5 to Proposition 5.4 explains that the integral with respect to $\ddot u\in\mathscr{M}_0(Q)$ on the left hand side of $(10)_1$, below, is meaningful.

3.1.4 *Final forms of evolution spaces*

Accordingly, the evolution spaces occurring in the definition of solutions are defined as follows.

$$\left.\begin{aligned} \mathscr{W} &= \left\{p \in C_w(\bar{I}; V) \ : \ \dot{p} \in R_w(\bar{I}; \mathring{H}^1(\Omega)), \ddot{p} \in \mathscr{M}_0(Q)\right\}, \\ \mathscr{X} &= \left\{\theta \in C(\bar{I}; L_2(\Omega)) \cap L_2(I; \mathring{H}^1(\Omega)) : \dot{\theta} \in L_2(I; H^{-1}(\Omega))\right\}, \\ \mathscr{K} &= \left\{y \in w + C_w(\bar{I}; V : y \geq 0 \text{ on } Q\right\}, \\ \mathscr{Y} &= \left\{u \in w + \mathscr{W} : u \geq 0\right\}. \end{aligned}\right\} \tag{5}$$

The superimposed dots now denote the time derivatives in the sense of distributions. $\mathscr{Y}$ is the space of deflections, defined as a shifted cone in terms of $\mathscr{W}$. $\mathscr{K}$ is the space of test–deflections occurring in the variational inequality. With some license, we can define $\mathscr{W}$ by

$$\mathscr{W} = \left\{p \in L_\infty(I; V) : \dot{p} \in L_\infty(I; \mathring{H}^1(\Omega)), \ddot{p} \in \mathscr{M}_0(Q)\right\} \tag{6}$$

where now p and $\dot{p}$ are Lebesgue classes of equivalence. It will be shown in Remark 9.6, below, that it is possible to choose unique representatives p and $\dot{p}$ as in $(5)_1$. Similarly, the space of temperatures $\mathscr{X}$ consists of preferred representations of the space

$$\mathscr{X} = \left\{\theta \in L_2(I; \mathring{H}^1(\Omega)) : \dot{\theta} \in L_2(I; H^{-1}(\Omega))\right\} \tag{7}$$

as is well known, e.g., [10, Chapter IV, Theorem 1.17], [9, Theorem 3, Chapter 5].

To proceed to the variational formulation of the problem, we eliminate the Airy stress function v, introduce an abbreviation $\Lambda(u, \theta)$ for an integral which occurs frequently in the treatment below, and summarize useful properties of the elliptic operator $\mathbf{1} - a\,\Delta$.

3.2 Lemma

(i) ([13, Lemma 1]) *There exists a bilinear map $\Phi : H^2(\Omega)^2 \to \mathring{H}^2(\Omega)$ such that for each u, $v \in H^2(\Omega)$ the value $\Phi(u, v)$ is the unique solution of the variational equation*

$$\int_\Omega \Delta\,\Phi(u, v)\,\Delta\,\varphi\,dx = \int_\Omega [u, v]\varphi\,dx \quad \text{for all} \quad \varphi \in \mathring{H}^2(\Omega).$$

The map Φ is compact and symmetric. One has $\Phi : H^2(\Omega)^2 \to W_p^2(\Omega)$, $2 < p < \infty$, and

$$|\Phi(u, v)|_{W_p^2(\Omega)} \leq c|u|_{H^2(\Omega)}|v|_{W_p^1(\Omega)} \quad \text{for all} \quad u \in H^2(\Omega),\, v \in W_p^1(\Omega). \tag{8}$$

(ii) *If $f \in L_2(Q)$ then for each $(u, \theta) \in \mathscr{W} \times \mathscr{X}$ there exists an element*

$$\Lambda(u, \theta) \in L_2(I; V^*) \tag{9}$$

such that

$$\langle \Lambda(u, \theta)(t), \lambda \rangle = -\int_\Omega \big(A(u, \lambda) - \chi \nabla\theta \cdot \nabla\lambda + E\,[u, \Phi(u, u)]\,\lambda - f\lambda\big)\,dx$$

for a.e. $t \in I$ *and every* $\lambda \in V$, *where*

$$A(u,\lambda) = b\big(u_{,11}\lambda_{,22} \; + u_{,22}\lambda_{,22} \\ \qquad\qquad + \nu(u_{11}\lambda_{22} + u_{22}\lambda_{11}) + 2(1-\nu)u_{12}\lambda_{12}\big)$$

and we abbreviate $u = u(t)$, $\theta = \theta(t)$, $\quad f = f(t)$.

(iii) *Let* $\mathbf{1}$ *be the identity map on* V. *Then*

(a) *the operator* $\mathbf{1} - a\,\Delta$ *maps continuously and bijectively the space* V *onto* $L_2(\Omega)$;

(b) *its inverse* $(\mathbf{1} - a\,\Delta)^{-1} : L_2(\Omega) \to V$, *its adjoint* $(\mathbf{1} - a\,\Delta)^{*} : L_2(\Omega) \to V^*$ *and the inverse of the adjoint* $(\mathbf{1} - a\,\Delta)^{-*} : V^* \to L_2(\Omega)$ *are continuous and bijective also;*

(c) $(\mathbf{1} - a\,\Delta)^{-1}$ *maps nonnegative functions from* $L_2(\Omega)$ *into nonnegative functions from* V;

(d) *defining nonnegative elements* $\gamma \in V^*$ *as those satisfying* $\langle \gamma, \lambda \rangle \geq 0$ *for every nonnegative function* $\lambda \in V$, *we have that* $(\mathbf{1} - a\,\Delta)^{-*}$ *maps nonnegative elements of* V^* *into nonnegative functions from* $L_2(\Omega)$.

The symmetry of $\mathbf{1} - a\,\Delta$ and of its inverse reads

$$(\mathbf{1} - a\,\Delta)^{*}|V = \mathbf{1} - a\,\Delta, \quad (\mathbf{1} - a\,\Delta)^{-*}|L_2(\Omega) = (\mathbf{1} - a\,\Delta)^{-1};$$

therefore, it is customary to use the same expression for a symmetric differential operator and its adjoint; however, we do not follow this since the domains of the two operators are different.

Proof (ii): Using the definitions of $\mathscr{K}$, $\mathscr{W}$, and $\mathscr{X}$ and straightforward Hölder estimates, it is easily seen that for a.e. $t \in I$ the right hand side defines a continuous linear functional on V, i.e., an element of V^*, and the same estimates show that we have the relation (9).

(iii) (a): This is just the existence and uniqueness of the solution of the Dirichlet problem for the elliptic equation $(\mathbf{1} - a\,\Delta)u = f$ with $f \in L_2(\Omega)$, see, e.g., [11, Theorem 9.15], under the assumption that $\partial\Omega$ is of class $C^{1,1}$, which is covered by our assumptions.

(b): The assertions are consequences of (a). For example, the continuity of $(\mathbf{1} - a\,\Delta)^{-1}$ follows from Banach's inverse mapping theorem and the rest of (ii) is just the continuity of adjoints.

(c): & (d) follow from the weak maximum principle for elliptic operators [11, Theorem 8.1], which gives (c), and by a dualization of (c), which is (d). $\square$

Lemma 3.2 enables us to reformulate the variational formulation of the system (1)–(3) in the following form:

3.3 Problem $\mathscr{P}$ *Find* $(u,\theta) \in \mathscr{Y} \times \mathscr{X}$ *such that*

$$\left.\begin{aligned} \int_Q (\mathbf{1} - a\,\Delta)(y-u)\,d\ddot{u} - \int_I \langle \Lambda(u,\theta), y-u \rangle \; dt \geq 0, \\ \int_I \langle \dot{\theta}, z \rangle \; dt + \int_Q (\, d\theta z + \kappa \nabla\theta \cdot \nabla z + e \nabla \dot{u} \cdot \nabla z) dt\,dx = \int_Q qz\,dt\,dx \end{aligned}\right\} \tag{10}$$

for every $(y,z) \in \mathscr{K} \times L_2(I; \mathring{H}^1(\Omega))$ *and*

$$u(0) = u_0, \quad \dot{u}(0) = v_0, \quad \theta(0) = \theta_0.$$

In view of Definitions (5), the initial values of the deflection and velocity $u(0)$ and $\dot{u}(0)$ are well defined. The data of the problem are assumed to satisfy the following

3.4 Hypothesis Let

$$\left.\begin{array}{l} w \in H^2(\Omega),\ w(x) \ge w_{\min} > 0 \ \text{ for all } \ x \in \Omega; \quad w|\partial\Omega = u_0|\partial\Omega, \\ u_0 \in H^2(\Omega),\ u_0 \ge 0 \ \text{ on } \ \Omega;\ v_0 \in \mathring{H}^1(\Omega),\ \theta_0 \in L_2(\Omega), \\ f,\, q \in L_2(Q). \end{array}\right\} \tag{11}$$

In the subsequent treatment we shall prove the following two theorems, the main results of the paper.

3.5 Theorem *Under Hypotheses* (11) *there exists a solution of Problem* $\mathscr{P}$.

The following proposition introduces the reaction force G between the plate and the rigid obstacle for any solution of the contact problem Let $\mathscr{M}_1^+(Q)$ be the set of all nonnegative Radon measures M on Q (not necessarily finite) such that

$$\int_Q |q|\, dM \le c|q|_{C_0(I;V)} \quad \text{for some } c \text{ and each } q \in C_0(I;V) \subset C_0(I;C_0(\Omega)) \tag{12}$$

where the last inclusion, which makes the integral in (12) unambiguous, follows from the imbedding $V \subset C_0(\Omega)$, well known from the theory of Sobolev spaces. Condition (12) is automatically satisfied if M is finite.

3.6 Theorem *For every solution* $(u, \theta) \in \mathscr{Y} \times \mathscr{X}$ *of* $\mathscr{P}$ *we have the following statements.*

(i) *There exists a unique nonnegative measure* $G \in \mathscr{M}_1^+(Q)$ *such that*

$$\int_Q (\mathbf{1} - a\,\Delta) p\, d\ddot{u} - \int_I \langle \Lambda(u,\theta), p\rangle\ dt = \int_Q p\, dG \tag{13}$$

for every $p \in C_0(I;V)$.

(ii) *The Lebesgue decomposition*

$$\ddot{u} = \ddot{u}_{\mathrm{r}} + \ddot{u}_{\mathrm{s}} \tag{14}$$

into the absolutely continuous part $\ddot{u}_{\mathrm{r}}$ (*with respect to* $\mathscr{L}^3$) *and the singular part* $\ddot{u}_{\mathrm{s}}$ (*with respect to* $\mathscr{L}^3$) *has the form*

$$\ddot{u}_{\mathrm{r}} = h\,\mathscr{L}^3 \llcorner Q, \quad \ddot{u}_{\mathrm{s}} = \pi\,\phi \otimes \mathscr{L}^2 \llcorner \Omega \tag{15}$$

where $h \in L_1(I; L_2(\Omega))$ *and* ϕ *is a finite nonnegative measure on* I *supported by a* $\mathscr{L}^1$ *null set* $J \subset I$ *and* $\pi \in L_1(\phi; L_2(\Omega))$, $\pi \ge 0$. *Here* $\llcorner$ *denotes the restriction of a measure, see* (55), *below. Thus* $\ddot{u}_{\mathrm{s}}$ *is supported by a* $\mathscr{L}^3$ *null set whose projection onto the time axis* I *has a null* $\mathscr{L}^1$ *measure.*

(iii) *If* G_1 *is the restriction of* G *to* $J \times \Omega$ *then*

$$\ddot{u}_{\mathrm{s}} = (\mathbf{1} - a\,\Delta)^{-*} G_1 \tag{16}$$

in the sense of distributions, i.e.,

$$\int_Q \varphi \, d\ddot{u}_{\rm s} = \int_Q (\mathbf{1} - a\,\Delta)\varphi \, dG_1$$

for each $\varphi \in \mathscr{D}(Q)$*.*

(iv) *The supports of* $\ddot{u}_{\rm s}$ *and* G *are contained in the contact zone*

$$C = \big\{(t,x) \in Q : u(t,x) = 0\big\}\,.$$

The countable set of times t of jump discontinuities of $\dot{u}$ coincides with the set of atoms of ϕ (where $\phi(\{t\}) > 0$). Thus

$$\dot{u}(t,\cdot) - \dot{u}(t-,\cdot) = \pi(t,\cdot)\phi(\{t\}) \geq 0.$$

The sign corresponds to the intuitive idea that the plate is bounced off the obstacle. The compatibility of the picture requires $\dot{u}(t,\cdot) - \dot{u}(t-,\cdot) \in \mathring{H}^1(\Omega)$ despite the fact that the measure $\ddot{u}_{\rm s}$ generally ranges only in $L_2(\Omega)$.

4 Proof of Theorem 3.5

4.1 Penalized problem

For any $\eta > 0$ we formulate the *penalized problem*

$$\left.\begin{aligned} &\ddot{u} - a\,\Delta\,\ddot{u} + b\,\Delta^2\,u + \chi\,\Delta\,\theta - [u,v] = f + \eta^{-1}u^-,\\ &\Delta^2\,v + E[u,u] = 0,\\ &\dot{\theta} - \kappa\,\Delta\,\theta + d\theta - e\,\Delta\,\dot{u} = q \end{aligned}\right\} \quad \text{on} \;\; Q,$$

and the initial conditions (3) *hold.* The problem has the following variational formulation after applying the bilinear operator Φ in the same way as above:

4.1 Problem $\mathscr{P}_\eta$ *Find* $(u,\theta) \in (w + L_\infty(I;V)) \times \mathscr{X}$ *with* $\dot{u} \in L_\infty(I;\mathring{H}^1(\Omega))$ *and* $\ddot{u} \in L_2(Q)$ *such that*

$$\left.\begin{aligned} &\int_Q \ddot{u}(\mathbf{1} - a\,\Delta)y\,dt\,dx - \int_I \langle \Lambda(u,\theta), y\rangle\;dt - \eta^{-1}\int_Q u^- y\,dt\,dx = 0,\\ &\int_{\bar{I}} \big\langle \dot{\theta}, z\big\rangle\;dt + \int_Q (\,d\theta z + \kappa\nabla\theta\cdot\nabla z + e\nabla\dot{u}\cdot\nabla z)dt\,dx = \int_Q qz\,dt\,dx \end{aligned}\right\} \tag{17}$$

for every $(y,z) \in L^2(I;V) \times L_2(I;\mathring{H}^1(\Omega))$ *and the initial conditions* (3) *hold.*

We shall verify the existence of a solution to the penalized problem.

4.2 Theorem *For every* $\eta > 0$ *there exists a solution* $\{u,\theta\}$ *of Problem* $\mathscr{P}_\eta$*.*

Proof Let us denote by $\{v_i \in V;\, i \in \mathbf{N}\}$ a basis of V orthonormal with respect to the inner product

$$(u,y)_a = \int_\Omega (uy + a\nabla u\cdot\nabla y)\,dx,\;\; u,\,y \in \mathring{H}^1(\Omega)$$

and by $\{w_i \in \mathring{H}^1(\Omega);\, i \in \mathbf{N}\}$ a basis of $\mathring{H}^1(\Omega)$, orthonormal with respect to the standard inner product in $L_2(\Omega)$. We construct the Galerkin approximation $\{u_m, \theta_m\}$ of a solution in the form

$$u_m(t) = w + \sum_{j=1}^{m} \alpha_j(t) v_j; \quad \theta_m(t) = \sum_{j=1}^{m} \beta_j(t) w_j; \quad \{\alpha_j(t), \beta_j(t)\} \in \mathbf{R}^2,\ t \in I,$$

$j = 1, ..., m$, to satisfy the following system of equations

$$\int_\Omega (\ddot{u}_m v_i + a\,\nabla \ddot{u}_m \cdot \nabla v_i)\,dx + \langle \Lambda(u_m, \theta_m), v_i\rangle = \int_\Omega (\eta^{-1} u_m^- + f)\, v_i\,dx, \tag{18}$$

$$\int_\Omega \big(\dot{\theta}_m w_i + \kappa \nabla\theta_m \cdot \nabla w_i + d\theta_m w_i + e\nabla \dot{u}_m \cdot \nabla w_i\big)\,dx = \int_\Omega q w_i\,dx, \tag{19}$$

$i = 1, ..., m$, and the initial conditions

$$\begin{aligned} u_m(0) &= u_{0m}, \quad u_{0m} \to u_0 \quad \text{in } \ H^2(\Omega);\\ \dot{u}_m(0) &= v_{0m}, \quad v_{0m} \to v_0 \quad \text{in } \ \mathring{H}^1(\Omega);\\ \theta_m(0) &= \theta_{0m}, \quad \theta_{0m} \to \theta_0 \quad \text{in } \ L_2(\Omega). \end{aligned} \tag{20}$$

The initial value problem (18)–(20) fulfils the conditions for the local existence of solution $\{u_m, \theta_m\}$ on some interval $I_m \equiv [0, t_m]$, $0 < t_m < T$.

Let us set $\gamma = \chi/e$. To derive the *a priori* estimates for solutions of (18)–(20) we multiply the equations (18) by $\dot{\alpha}_i(t)$ and (19) by $\gamma\beta_i(t)$, respectively, sum with respect to i, and integrate on $[0, t_m]$. Taking in mind

$$\int_\Omega [u, v] y\,dx = \int_\Omega [u, y] v\,dx$$

if at least one element of $\{u, v, y\}$ belongs to $\mathring{H}^2(\Omega)$ (cf. [4, Lemma 2.2.2, Chapter 2]), we obtain after integrating for $Q_m := I_m \times \Omega$ the relation

$$\begin{aligned} \int_{Q_m} \Big[\frac{1}{2}\partial_t\big(\dot{u}_m^2 + a|\nabla \dot{u}_m|^2 \ &+ A(u_m, u_m) + E(\Delta\,\Phi(u_m, u_m))^2/2\\ &+\gamma\theta_m^2 + \eta^{-1}(u_m^-)^2\big) + \gamma(\kappa|\nabla\theta_m|^2 + d\theta_m^2)\Big]\,dt\,dx\\ &= \int_{Q_m} (f\dot{u}_m + \gamma\, q\,\theta_m)\,dt\,dx, \end{aligned}$$

which leads to the estimate

$$\begin{aligned} |\dot{u}_m|^2_{L_\infty(I;\mathring{H}^1(\Omega))} &+ |u_m|^2_{L_\infty(I;H^2(\Omega))} + |\Phi(u_m, u_m)|^2_{L_\infty(I;H^2(\Omega))}\\ &+\eta^{-1}|u_m^-|^2_{L_\infty(I;L_2(\Omega))} + |\theta_m|^2_{L_\infty(I;L_2(\Omega))} + |\theta_m|^2_{L_2(I;\mathring{H}^1(\Omega))}\\ &\le C_1 \equiv C_1(f, q, u_0, v_0, \theta_0). \end{aligned} \tag{21}$$

The prolongation to the whole interval I is due to the fact that the original estimate for I_m does not depend on m. Moreover the estimate (8) implies

$$|\Phi(u_m, u_m)|_{L_\infty(I;W_p^2(\Omega))} \le c_p \equiv c_p(f, u_0, u_1) \ \text{ for all } \ p > 2.$$

The estimate (21) further implies that with $r = 2p/(p+2)$ we have

$$\begin{gathered} [u_m, \Phi(u_m, u_m)] \in L_2(I; L_r(\Omega)),\\ \big|[u_m, \Phi(u_m, u_m)]\big|_{L_2(I;L_r(\Omega))} \le c_r \equiv c_r(f, u_0, u_1). \end{gathered}$$

From the equation (19) we obtain straightforwardly the estimate

$$|\dot{\theta}_m|_{L_2(I;W_m{}^*)} \leq C_2(f,q,u_0,v_0,\theta_0), \quad m \in \mathbf{N}, \tag{22}$$

where $W_m \subset \mathring{H}^1(\Omega)$ is the linear hull of $\{w_i\}_{i=1}^m$. From (18) we obtain

$$|(\mathbf{1}-a\,\Delta)^*\ddot{u}_m|^2_{L_2(I;V_m{}^*)} \leq C_3(\eta), \quad m \in \mathbf{N}, \tag{23}$$

where $V_m \subset H^2(\Omega)$ is the linear hull of $\{v_i\}_{i=1}^m$.

We proceed with the convergence of the Galerkin approximation. Applying the estimate (21), the compact embedding theorem and interpolation in Sobolev spaces we obtain on the base of the well known Alaoglu principle used for duals of separable Banach spaces the existence of subsequences of $\{u_m\}$, $\{\theta_m\}$ (denoted again by $\{u_m\}$, $\{\theta_m\}$), and functions u, θ with the convergences

$$\left.\begin{array}{lll} u_m \rightharpoonup^* u & \text{in} & L_\infty(I;H^2(\Omega)), \\ \dot{u}_m \rightharpoonup^* \dot{u} & \text{in} & L_\infty(I;\mathring{H}^1(\Omega)), \\ \theta_m \rightharpoonup^* \theta & \text{in} & L_\infty(I;L_2(\Omega)), \\ \theta_m \rightharpoonup \theta & \text{in} & L_2(I;\mathring{H}^1(\Omega)). \end{array}\right\} \tag{24}$$

The estimates (22), (23) imply the convergences

$$\dot{\theta}_m \rightharpoonup \dot{\theta} \ \text{ in } \ L_2(I;W^*), \tag{25}$$

$$(\mathbf{1}-a\,\Delta)^*\ddot{u}_m \rightharpoonup (\mathbf{1}-a\,\Delta)^*\ddot{u} \ \text{ in } \ L_2(I;Y^*), \tag{26}$$

where $W = \bigcup_{m\in\mathbf{N}} W_m$, $\overline{W} = \mathring{H}^1(\Omega)$ and $Y = \bigcup_{m\in\mathbf{N}} V_m$, $\overline{Y} = V$. The convergences (25), (26) imply

$$|\dot{\theta}_m|_{L_2(I;H^{-1}(\Omega))} \leq C_2(f,q,u_0,v_0,\theta_0), \quad m \in \mathbf{N}, \tag{27}$$

$$\dot{\theta}_m \rightharpoonup \dot{\theta} \text{ in } \ L_2(I;H^{-1}(\Omega)), \tag{28}$$

$$|(\mathbf{1}-a\,\Delta)^*\ddot{u}_m|^2_{L_2(I;V^*)} \leq C_3(\eta), \quad m \in \mathbf{N}. \tag{29}$$

Moreover we obtain from (29) a better acceleration estimate

$$|\ddot{u}_m|_{L_2(Q)} \leq C_4(\eta) \tag{30}$$

and the convergence

$$\ddot{u}_m \ \rightharpoonup \ \ddot{u} \ \text{ in } \ L_2(Q) \tag{31}$$

for a chosen subsequence denoted again by $\{\ddot{u}_m\}$. We note that (29) implies (30) since (cf. [1])

$$|\ddot{u}_m|_{L_2(Q)} = |(\mathbf{1}-a\,\Delta)^{-*}(\mathbf{1}-a\,\Delta)^*\ddot{u}_m|_{L_2(Q)} \leq c|(\mathbf{1}-a\,\Delta)^*\ddot{u}_m|_{L_2(I;V^*)}$$

as $(\mathbf{1}-a\,\Delta)^{-*} : V^* \to L_2(\Omega)$ is bounded, see Lemma 3.2(iii).

The estimate (30) implies after considering (24) the uniform convergences

$$\begin{array}{lll} u_m \to u & \text{in} & C(\bar{I};H^{2-\epsilon}(\Omega)), \\ \dot{u}_m \to \dot{u} & \text{in} & C(\bar{I};H^{1-\epsilon}(\Omega)) \end{array} \tag{32}$$

and the convergences

$$\begin{aligned}
&\Phi(u_m,u_m) \to \Phi(u,u) \quad \text{in} \quad L_2(I;H^2(\Omega)),\\
&\Phi(u_m,u_m) \rightharpoonup^* \Phi(u,u) \quad \text{in} \quad L_\infty(I;W_p^2(\Omega)).
\end{aligned} \tag{33}$$

In fact to get the first two uniform convergences we use the following pattern: we start from the proved weak convergences of the time and space derivatives. The standard extension technique (cf. e.g. [8]) allows to extend in an appropriate way all the employed functions from their domains to the whole spaces. The Fourier transform and the suitable use of the Hölder inequality allows to prove the week convergences in the spaces $H^{1/2+\epsilon}(I;H^r(\Omega))$ for a small $\epsilon > 0$ with $r \nearrow 1$ for u and $r \nearrow 1/2$ for $\dot{u}$ as $\epsilon \searrow 0$. The compact embedding of such spaces to $C(\bar{I};H^{r'}(\Omega))$ valid for any $r' < r$ gives the starting strong convergence. Then we use the interpolation of this with the results in (24) and we are done. The last two convergences are based on the previous ones and estimates from [13].

Let $\mu \in \mathbf{N}$, $y_\mu = \sum_{i=1}^{\mu} \phi_i(t)v_i$, $z_\mu = \sum_{i=1}^{\mu} \phi_i(t)w_i$, $\phi_i \in \mathscr{D}(0,T)$, $i = 1, ..., \mu$. We have for arbitrary $t \in I$ the relations

$$\int_\Omega \ddot{u}_m(\mathbf{1} - a\,\Delta)y_\mu dx + \langle \Lambda(u_m,\theta_m)(t), y_\mu\rangle - \int_\Omega \eta^{-1}u_m y_\mu\,dx = \int_\Omega f y_\mu\,dx,$$

$$\int_\Omega \left(\dot{\theta}_m z_\mu + \kappa\nabla\theta_m \cdot \nabla z_\mu + d\theta_m z_\mu + e\nabla\dot{u}_m \cdot \nabla z_\mu\right)dx = \int_\Omega q z_\mu\,dx, \quad \text{for all} \quad m \geq \mu.$$

The convergences (24), (28), (31) imply that functions $\{u,\,\theta\} \in (w+L_\infty(I;V))\times L_2(I;\mathring{H}^1(\Omega))$ fulfil

$$\int_Q \ddot{u}(\mathbf{1} - a\,\Delta)y_\mu dt\,dx + \langle \Lambda(u,\theta)(t), y_\mu\rangle - \int_\Omega \eta^{-1}u y_\mu\,dt\,dx = \int_Q f y_\mu\,dt\,dx, \tag{34}$$

$$\int_Q \left(\dot{\theta} z_\mu + \kappa\nabla\theta \cdot \nabla z_\mu + d\theta z_\mu + e\nabla\dot{u} \cdot \nabla z_\mu\right)dt\,dx = \int_Q q z_\mu\,dt\,dx. \tag{35}$$

The functions $\{y_\mu\}$, $\{z_\mu\}$ form dense subsets of the spaces $L_2(I;V)$ and $L_2(I;\mathring{H}^1(\Omega))$ respectively. Then we obtain from (34), (35) the relations (17). Moreover the relation $\theta \in L_2(I;\mathring{H}^1(\Omega)) \cap H^1(I;H^{-1}(\Omega))$ implies $\theta \in C(\bar{I};L_2(\Omega))$ ([9, Theorem 3, Chapter 5]). The convergence (28), the uniform convergence (32), a continuity of $t \mapsto \theta(t) \in H^{-1}(\Omega)$ and the properties (20) imply the initial conditions (3) and the proof of the existence of a solution is complete.

4.2 Solvability of the original problem

The estimates (21), (27) imply the following η-independent estimates:

$$\begin{aligned}
&|\dot{u}_\eta|^2_{L_\infty(I;\mathring{H}^1(\Omega))} + |u_\eta|^2_{L_\infty(I;H^2(\Omega))} + |\Phi(u_\eta,u_\eta)|^2_{L_\infty(I;W_p^2(\Omega))} + \eta^{-1}|u_\eta^-|^2_{L_\infty(I;L_2(\Omega))}\\
&\quad + |\theta_\eta|^2_{L_\infty(I;L_2(\Omega))} + |\theta_\eta|^2_{L_2(I;\mathring{H}^1(\Omega))} + |\dot{\theta}_\eta|^2_{L_2(I;H^{-1}(\Omega))} \leq C_5,
\end{aligned} \tag{36}$$

$C_5 \equiv C_5(f,q,u_0,v_0,\theta_0)$ for a solution $\{u_\eta,\theta_\eta\}$, $\eta > 0$, of the penalized problem. The acceleration term $\ddot{u}_\eta$ does not appear in (36). It is then suitable to transform the penalized relation $(17)_1$ using the integration by parts with respect to t. We obtain the system

$$\left.\begin{aligned}\int_I \langle \Lambda(u_\eta,\theta_\eta), y\rangle\,dt\ -\int_Q(\mathbf{1}-a\,\Delta)\dot u_\eta\,\dot y\,dt\,dx+\int_\Omega(\mathbf{1}-a\,\Delta)\dot u_\eta\,y(T,\cdot)\,dx&\\ =\int_\Omega(\mathbf{1}-a\,\Delta)v_0\,y(0,\cdot)\,dx+\int_Q(f+\eta^{-1}u_\eta^-)y\,dt\,dx,&\\ \int_Q\big(\dot\theta_\eta z+\kappa\nabla\theta_\eta\cdot\nabla z+d\theta_\eta z+e\nabla\dot u_\eta\cdot\nabla z\big)\,dt\,dx=\int_Q qy\,dt\,dx&\end{aligned}\right\}\quad(37)$$

holding for any $\{y,z\}\in L_2(I;V)\times L_2(I;\mathring{H}^1(\Omega))$ with $\dot y\in L_2(I;\mathring{H}^1(\Omega))$. We derive the following crucial $\eta-$independent estimate of the penalty term $\eta^{-1}u_\eta^-$. Applying Assumptions (11) and the definition of u_η^- we obtain

$$0\le w_{\min}\int_Q\eta^{-1}u_\eta^-\,dt\,dx\le\int_Q\eta^{-1}u_\eta^- w\,dt\,dx\le\int_Q\eta^{-1}u_\eta^-(w-u_\eta)\,dt\,dx$$

based on its sign. After inserting $y=w-u_\eta$ in $(37)_1$ and invoking estimates (36) we achieve the crucial estimate

$$|\eta^{-1}u_\eta^-|_{L_1(Q)}\le C_6\equiv C_6(f,q,u_0,v_0,\theta_0).\tag{38}$$

By this estimate we have $\ddot u_\eta\in L_1(I;V^*)$ for each $\eta>0$. Abbreviating $\Lambda_\eta=\Lambda(u_\eta,\theta_\eta)$, we rewrite (34) in the form

$$\int_Q(\mathbf{1}-a\,\Delta)^*\ddot u_\eta p\,dt\,dx=\int_I\langle\Lambda_\eta,p\rangle\,dt+\int_Q\eta^{-1}u_\eta^-p\,dt\,dx+\int_Q fp\,dt\,dx$$

for every $p\in C_0(I;V)$. There exists a unique measure $g_\eta\in\mathscr{M}(I;V^*)$ such that $\int_I\langle p,dg_\eta\rangle=\int_Q\eta^{-1}u_\eta^-p\,dt\,dx$ for every $p\in C_0(I;V)$. Then

$$(\mathbf{1}-a\,\Delta)^*\ddot u_\eta\mathscr{L}^1\llcorner I=\Lambda_\eta\mathscr{L}^1\llcorner I+g_\eta\tag{39}$$

and estimates (36) and (38) give

$$|\Lambda_\eta|_{L_2(I;V^*)}\le C_7,\quad |g_\eta|_{\mathscr{M}(I;V^*)}\le C_8\tag{40}$$

for some constants $C_7=C_7(f,q,u_0,v_0,\theta_0)$, $C_8=C_8(f,q,u_0,v_0,\theta_0)$ and all $\eta>0$. The application of the bounded operator $(\mathbf{1}-a\,\Delta)^{-*}$ to (39) yields by Lemma 3.2(iii) also

$$|\ddot u_\eta\mathscr{L}^1\llcorner I|_{\mathscr{M}(I;L_2(\Omega))}\le c\tag{41}$$

for some $c<\infty$ and all $\eta>0$.

We now choose arbitrary sequence $\eta_k\to0$ and write $(\cdot)_k:=(\cdot)_{\eta_k}$. Using the estimates (36), (38), and (40), on the basis of the Alaoglu principle there exist functions $\{u,\theta\}\in(w+L_\infty(I;V))\times[L_\infty(I;L_2(\Omega))\cap L_2(I;\mathring{H}^1(\Omega))]$ and measures $g\in\mathscr{M}(I;V^*)$ and $G\in\mathscr{M}(Q)$ such the following convergences hold up to a subsequence:

$$\left.\begin{array}{ll}
u_k \rightharpoonup^* u & \text{in } \ w + L_\infty(I;V), \\
u_k \to u & \text{uniformly in } \ C(\operatorname{cl} Q), \\
\dot u_k \rightharpoonup^* \dot u & \text{in } \ L_\infty(I;\mathring{H}^1(\Omega)), \\
\ddot u_k \mathscr{L}^1 \llcorner I \rightharpoonup^* \ddot u & \text{in } \ \mathscr{M}(I;L_2(\Omega)), \\
u_k \to u & \text{in } \ C(I;H^{2-\epsilon}(\Omega)) \ \text{ for any } \ \epsilon > 0, \\
g_k \rightharpoonup^* g & \text{in } \ \mathscr{M}(I;V^*) \\
\eta^{-1} u_k^- \mathscr{L}^3 \rightharpoonup^* G & \text{in } \ \mathscr{M}(Q), \\
\Phi(u_k,u_k) \to \Phi(u,u) & \text{in } \ L_2(I;H^2(\Omega))), \\
\Phi(u_k,u_k) \rightharpoonup^* \Phi(u,u) & \text{in } \ L_\infty(I;W^2_p(\Omega)), \\
\theta_k \rightharpoonup^* \theta & \text{in } \ L_\infty(I;L_2(\Omega)) \cap L_2(I;\mathring{H}^1(\Omega)), \\
\dot\theta_k \rightharpoonup \dot\theta & \text{in } \ L_2(I;H^{-1}(\Omega)), \\
\Lambda_k \rightharpoonup \Lambda & \text{in } \ L_2(I;V^*)
\end{array}\right\} \qquad (42)$$

where $\Lambda = \Lambda(u,\theta)$. As mentioned above, the weak* convergences follow from the estimates, where we use the identifications (6) and (7) and where we note that *a priori* the sequences $\dot u_k$, $\ddot u_k$ and $\dot\theta_k$ weak* converge to some limits which only subsequently turn out to be $\dot u$, $\ddot u$ and $\dot\theta$ by the linearity of the definitions of weak derivatives. The convergence $(42)_2$ is a consequence of Aubin's lemma (see (64), below). The strong convergences are derived in the same way as in (32) and (33) (observe that the strong convergence for $\dot u$ is missing here). The above convergences prove the initial conditions $u(0,\cdot) = u_0$, $\theta(0,\cdot) = \theta_0$ and $\dot u(0,\cdot) = v_0$.

We now complete the proof of Theorem 3.5. The uniform convergence $(42)_2$ implies $u_k^\pm \to u^\pm$ uniformly. On the other hand, (38) implies that $u_k^- \to 0$ for a.e. point of Q and hence $u_k^- \to 0$ uniformly on $\operatorname{cl} Q$. Thus the limit u is nonnegative and

$$u_k^+ \to u \quad \text{uniformly in } \ \operatorname{cl} Q. \qquad (43)$$

We have $\int_Q u_k^+ \eta^{-1} u_k^- \, dt\, dx = 0$ and thus using the limits (43) and $(42)_7$ we obtain

$$\int_Q u \, dG = 0.$$

The limit with (39) provides

$$(\mathbf{1} - a\,\Delta)^* \ddot u = \Lambda \mathscr{L}^1 \llcorner I + g$$

where $\int_I \langle p, dg\rangle = \int_Q p\, dG$ for every $p \in C_0(I;V)$. Then for $y \in \mathscr{K}$

$$\int_Q (\mathbf{1} - a\,\Delta)(y-u)\, d\ddot u - \int_I \langle \Lambda(u,\theta), y - u\rangle \, dt = \int_Q (y-u)\, dG = \int_Q y\, dG \ge 0,$$

i.e., we have $(10)_1$. A similar but easier proof applies to the proof of $(10)_2$. $\square$

5 Measures in $\mathscr{M}(I;V)$, $\mathscr{M}(I;L_2(\Omega))$ and $\mathscr{M}_0(Q)$

The contact force and the acceleration as treated above are measures generally not absolutely continuous with respect to the Lebesgue measure. The subsequent treatment will show that these quantities are most naturally modeled as elements of the spaces $\mathscr{M}(I;V^*)$ and $\mathscr{M}(I;L_2(\Omega))$ of vector valued measures on I ranging in V^* and $L_2(\Omega)$, respectively. The reader is referred to Appendix B (Section 8) for details on vector valued measures.

5.1 Definition A measure $\nu \in \mathscr{M}(I;V^*)$ is said to be nonnegative if $\int_I \langle p, d\nu\rangle \geq 0$ for every $p \in C_0(I;V)$ such that $p \geq 0$.

We determine all nonnegative measures in $\mathscr{M}(I;V^*)$, which will be helpful in analyzing the contact force between the plate and the obstacle. The result is not entirely straightforward, see (46), below.

To formulate the next result, we recall that Ω is assumed to be a bounded two-dimensional manifold which is either star–shaped with a piecewise class C^2 boundary or possesses a class $C^{3,1}$ boundary (cf. requirements in [13]). We denote by $C^2_{\mathrm{pw}}(\partial\Omega)$ the set of all piecewise class C^2 functions on $\partial\Omega$. Then in accordance with the system of notation summarized in Appendix A (Section 7, below), $C_0(I;C^2_{\mathrm{pw}}(\partial\Omega))$ is the set of all continuous maps $p : \mathbf{R} \to C^2_{\mathrm{pw}}(\partial\Omega)$ (continuous in time with respect to the norm of $C^2_{\mathrm{pw}}(\partial\Omega)$) such that $p = 0$ outside I.

5.2 Lemma *For every $\kappa \in C_0(I;C^2_{\mathrm{pw}}(\partial\Omega))$ there exists a function $p \in C_0(I;V)$ with the restriction of p to $I \times \partial\Omega$ in $C_0(I;C^2_{\mathrm{pw}}(\partial\Omega))$ such that*

$$-\nabla p \cdot \mathbf{n} = \kappa \tag{44}$$

everywhere on $I \times \partial\Omega$ with the exception of the corner points of $\partial\Omega$. This function can be chosen nonnegative if κ is nonnegative. Actually, there exists a uniformly bounded sequence of nonnegative functions $p_k \in C_0(I;C^2_{\mathrm{pw}}(\partial\Omega))$ such that

$$\left.\begin{aligned} -\nabla p_k \cdot \mathbf{n} = \kappa \quad &\text{on } I \times \partial\Omega \text{ and} \\ p_k \to 0 \quad &\text{pointwise on } Q \end{aligned}\right\} \tag{45}$$

with the exception of the corner points of $\partial\Omega$. Recall that $\mathbf{n}$ is the outer normal to $\partial\Omega$.

Proof To simplify the notation, assume that $\partial\Omega$ is of class C^2 and $\kappa \in C_0(I;C^2(\partial\Omega))$. (The case of piecewise class C^2 boundary and $\kappa \in C_0(I;C^2_{\mathrm{pw}}(\partial\Omega))$ involves minor modifications.) The restriction ω of $\mathrm{dist}(\cdot,\partial\Omega)$ to $\mathrm{cl}\,\Omega$ is class C^2 in a neighborhood N of $\partial\Omega$ with $\omega = 0$ and $\nabla\omega = -\mathbf{n}$ on $\partial\Omega$. For the given $\kappa \in C_0(I;C^2_{\mathrm{pw}}(\partial\Omega))$ let $\tilde\kappa \in C_0(I;C^2(\mathrm{cl}\,\Omega))$ be any extension of κ such that $\tilde\kappa = 0$ on $Q \setminus I \times N$, chosen nonnegative if κ is nonnegative. Thus we require $\tilde\kappa(t,x) = \kappa(t,x)$ for every $(t,x) \in I \times \partial\Omega \subset I \times \mathrm{cl}\,\Omega$ and $\tilde\kappa(t,x) \neq 0$ only if $(t,x) \in I \times N$. The function $\tilde\kappa$ is easy to construct by locally flattening $\partial\Omega$ by a class C^2 diffeomorphism Φ, performing an (easy) construction of $\tilde\kappa$ in the flattened picture, returning back to the curved $\partial\Omega$ via Φ^{-1}, and then to globalize the local construction using the compactness of $\partial\Omega$ and partition of unity.

Then $p := \tilde{\kappa}\omega$ satisfies (44). Further, let $\psi_k : \operatorname{cl}\Omega \to [0,1]$ be any sequence of functions on $\operatorname{cl} Q$ such that each ψ_k is equal to 1 in a neighborhood $I \times N_k$ of $I \times \partial\Omega$ and $\psi_k \to 0$ pointwise on Q. Then $p_k := \psi_k p$ satisfies (45). $\square$

5.3 Proposition *A measure $\nu \in \mathscr{M}(I; V^*)$ is nonnegative if and only if there exists a unique measure $G \in \mathscr{M}_1^+(Q)$, a unique finite nonnegative measure ζ on I and a unique function $F \in L_\infty(I; L_2(\partial\Omega), \zeta)$ with*

$$|F(t)|_{L_2(\partial\Omega)} = 1 \quad \text{for} \quad \zeta \quad \text{almost} \quad \text{every} \quad t \in I$$

such that

$$\int_I \langle p, d\nu \rangle = \int_Q p\, dG - \int_I \int_{\partial\Omega} F(t)\, \nabla p(t) \cdot \mathbf{n}\; d\mathscr{H}^1 d\zeta(t) \tag{46}$$

for every $p \in C_0(I; V)$. Here $\mathscr{H}^1$ is the length element on $\partial\Omega$.

Proof Uniqueness: Let G_i, F_i and ζ_i, $i = 1, 2$, be two pairs corresponding to the same ν. Then in particular $\int_Q p\, dG_1 = \int_Q p\, dG_2$ for each $p \in C_0(I; V)$ such that the support of p is in $I \times \Omega$. Then $G_1 = G_2$. Consequently,

$$\int_I \int_{\partial\Omega} F_1(t)\, \nabla p(t) \cdot \mathbf{n}\; d\mathscr{H}^1 d\zeta_1(t) = \int_I \int_{\partial\Omega} F_2(t)\, \nabla p(t) \cdot \mathbf{n}\; d\mathscr{H}^1 d\zeta_2(t)$$

for each $p \in C_0(I; V)$. By Lemma 5.2 then

$$\int_I \int_{\partial\Omega} F_1(t)\, \kappa\; d\mathscr{H}^1 d\zeta_1(t) = \int_I \int_{\partial\Omega} F_2(t)\, \kappa\; d\mathscr{H}^1 d\zeta_2(t)$$

for every κ as in the statement of that lemma. This implies that the signed measures $F_1\zeta_1$ and $F_2\zeta_2$ coincide; the condition $|F_1(t)|_{L_2(\partial\Omega)} = |F_2(t)|_{L_2(\partial\Omega)} = 1$ then gives $\zeta_1 = \zeta_2$ and $F_1(t) = F_2(t)$ at $\zeta_1 = \zeta_2$ almost every point t of I.

Sufficiency: If ν is of the form (46) then ν is nonnegative. Indeed, if $p \in C_0(I; V)$ is a nonnegative function then the first term on the right hand side of (46) is clearly nonnegative while the second term is nonnegative since $\nabla p(t) \cdot \mathbf{n} \le 0$ on $I \times \partial\Omega$ since p vanishes on $I \times \partial\Omega$ and is nonnegative on $I \times \Omega$.

Necessity: We prove the existence of the measure G and the function F as follows. The restriction of the functional $p \mapsto \int_I \langle p, d\nu \rangle$ to p from $\mathscr{D}(Q)$ gives a nonnegative Schwartz's distribution and hence by the well known theorem on nonnegative distributions [19, Theorem III] there exists a nonnegative measure G on Q such that

$$\int_I \langle p, d\nu \rangle = \int_Q p\, dG \tag{47}$$

for every $p \in \mathscr{D}(Q)$. We now extend (47) to all $p \in C_0(I; \mathring{H}^2(\Omega))$ as follows. If $p \in C_0(I; \mathring{H}^2(\Omega))$, there exists a sequence $p_k \in \mathscr{D}(Q)$ such that $p_k \to p$ in $C_0(I; V)$ and in particular, $p_k \to p$ uniformly on Q is view of the embedding $\mathring{H}^2(\Omega)$ into $C(\operatorname{cl}\Omega)$. Then

$$\int_Q p_k\, dG = \int_I \langle p_k, d\nu \rangle \to \int_Q p\, dG \quad \text{and} \quad \int_I \langle p_k, d\nu \rangle \to \int_I \langle p, d\nu \rangle$$

and thus (47) holds. The continuity gives (12).

To complete the proof of (46), note that

$$p \mapsto \int_I \langle p,\, d\nu \rangle - \int_Q p\, dG$$

a continuous linear functional on $C_0(I;V)$. Theorem 8.5 then gives a measure $\nu_{\rm b}\in\mathscr{M}(I;V^*)$ such that

$$\int_I \langle p, d\nu_{\rm b}\rangle = \int_I \langle p,\, d\nu\rangle - \int_Q p\, dG$$

for every $p\in C_0(I;V)$. Since (47) holds for all $p\in C_0(I;\mathring{H}^2)$, we have

$$\int_I \langle p, d\nu_{\rm b}\rangle = 0 \ \text{ for every } p\in C_0(I;\mathring{H}^2(\Omega)).$$

We shall prove that as a consequence,

$$\int_I \langle p, d\nu_{\rm b}\rangle = -\int_I\int_{\partial\Omega} F(t)\,\nabla p(t)\cdot\mathbf{n}\; d\mathscr{H}^1 d\zeta(t) \tag{48}$$

for every $p\in C_0(I;V)$ where F and ζ are as in the statement of the proposition except the nonnegativity. By the general form of a general measure in $\mathscr{M}(I;V^*)$ we have $\nu_{\rm b}=\gamma_{\rm b}|\nu_{\rm b}|_{V^*}$ where $\gamma_{\rm b}: I\to V^*$ is $|\nu_{\rm b}|_{V^*}$ integrable map. Property (47) gives that

$$\langle\gamma_{\rm b}(t),\lambda\rangle = 0$$

for each $\lambda\in\mathring{H}^2(\Omega)$ and $|\nu_{\rm b}|_{V^*}$ almost every $t\in I$. Fix such a t and put $\gamma=\gamma_{\rm b}(t)$ for brevity. Prove that there exists a function $f\in L_2(\partial\Omega)$ such that

$$\langle\gamma,\lambda\rangle = -\int_{\partial\Omega} f\nabla\lambda\cdot\mathbf{n}\, d\mathscr{H}^1 \tag{49}$$

for each $\lambda\in V$. The function f is determined uniquely. Since the scalar product

$$(\lambda_1,\lambda_2)\equiv\int_\Omega \Delta\,\lambda_1\cdot\Delta\,\lambda_2\, dx$$

gives rise to the norm equivalent to the standard norm on V, by the Riesz representation theorem for Hilbert spaces there exists a $\sigma\in V$ such that

$$\langle\gamma,\lambda\rangle = (\sigma,\lambda) = \int_\Omega \Delta\,\sigma\cdot\Delta\,\lambda\, dx$$

for each $\lambda\in V$. Denoting $f := -\Delta\,\sigma\in L_2(\Omega)$, the condition $\langle\gamma,\lambda\rangle=0$ for each $\lambda\in\mathring{H}^2(\Omega)$ is seen to imply that

$$\Delta\, f = 0 \tag{50}$$

on Ω in the sense of distributions. The conditions $f\in L_2(\Omega)$, $\Delta\, f = 0\in L_2(\Omega)$ are known to imply $f\in H^2(\Omega)$. Since

$$\int_\Omega \nabla f\cdot\nabla\lambda\, dx = -\int_\Omega \Delta\, f\cdot\lambda\, dx = 0$$

by (50), Green's theorem then gives

$$\langle\gamma,\lambda\rangle = -\int_\Omega f\,\Delta\,\lambda\, dx = -\int_{\partial\Omega} f\nabla\lambda\cdot\mathbf{n}\, d\mathscr{H}^1.$$

Thus we have (49) with f the trace of f on $\partial\Omega$. Consequently we have (48) by putting $F(t) = f/|f|_{L^2(\partial\Omega)}$, $\zeta = |f|_{L^2(\partial\Omega)}|\nu_{\rm b}|$.

We complete the proof by showing that F is nonnegative. We invoke Lemma 5.2 to this end. Let κ be a nonnegative function as in that lemma and let p_k be the sequence assured by the statement of lemma. Then

$$\begin{aligned}\int_I \langle p_k, d\nu\rangle &= \int_Q p_k\,dG - \int_I\int_{\partial\Omega} F(t)\,\nabla p_k(t)\cdot\mathbf{n}\;d\mathscr{H}^1 d\zeta(t)\\ &= \int_Q p_k\,dG + \int_I\int_{\partial\Omega} F(t)\,\kappa(t)\;d\mathscr{H}^1 d\zeta(t) \geq 0.\end{aligned}$$

Since $\int_Q p_k\,dG \to 0$, we must have $\int_I\int_{\partial\Omega} F(t)\,\kappa(t)\;d\mathscr{H}^1 d\zeta(t)\geq 0$ and the arbitrariness of $\kappa \geq 0$ gives $F(t)\geq 0$. □

For each $M\in\mathscr{M}_0(Q)$ denote by $|M|_{\mathscr{M}_0(Q)}$ the smallest constant such that (4) holds. Then $|\cdot|_{\mathscr{M}_0(Q)}$ is a norm which converts $\mathscr{M}_0(Q)$ into a Banach space. We have the following statements in which we identify maps $p : I\to L_2(\Omega)$ with real valued functions $p : Q\to\mathbf{R}$:

5.4 Proposition

(i) *the space $\mathscr{M}_0(Q)$ is isometrically isomorphic to the space $\mathscr{M}(I;L_2(\Omega))$ under the identification of $M\in\mathscr{M}(Q)$ with $\mu\in\mathscr{M}(I;L_2(\Omega))$ by the requirement that*

$$\int_I (p, d\mu)_{L_2(\Omega)} = \int_Q p\,dM \tag{51}$$

for each $p\in C_0(Q)$;

(ii) *under the above identification of M and μ, a general map $p : I\to L_2(\Omega)$ is $|\mu|_{L_2(\Omega)}$ integrable if and only if p is M integrable and then* (51) *holds.*

Proof (i): Since $C_0(Q)$ is dense in $C_0(I;L_2(\Omega))$, by Inequality (4) for any $M\in\mathscr{M}_0(Q)$ the functional $p\mapsto\int_Q p\,dM$ extends by continuity to a functional F_0 on $C_0(I;L_2(\Omega))$. By Theorem 8.5 that functional has a representation in terms of a measure $\mu\in\mathscr{M}(I;L_2(\Omega))$ as in (51).

Conversely, given $\mu\in\mathscr{M}(I;L_2(\Omega))$, then by restricting the functional $p\mapsto\int_I (p,d\mu)_{L_2(\Omega)}$ from $C_0(I;L_2(\Omega))$ to $C_0(Q)$ one obtains a functional F that satisfies

$$|F(p)| \leq |\mu|_{\mathscr{M}(I;L_2(\Omega))}|p|_{C_0(I;L_2(\Omega))} \leq |\mu|_{\mathscr{M}(I;L_2(\Omega))}\mathscr{L}^2(\Omega)|p|_{C_0(Q)}.$$

Thus the scalar Riesz representation theorem gives a measure M as in (51). One then has

$$\Big|\int_Q p\,dM\Big| = \Big|\int_I (p,d\mu)_{L_2(\Omega)}\Big| \leq |\mu|_{\mathscr{M}(I;L_2(\Omega))}|p|_{C_0(I;L_2(\Omega))}$$

and hence $M\in\mathscr{M}_0(Q)$ as we have (4).

This establishes a one to one correspondence between the measures M and μ. An easy examination of the involved constants shows that the correspondence is isometric.

(ii): By Subsection 8.4(c), the measure μ has the form $\mu = \pi|\mu|_{L_2(\Omega)}$ where $\pi : I\to L_2(\Omega)$ is $|\mu|_{L_2(\Omega)}$ integrable and $|\pi(t)|_{L_2(\Omega)} = 1$ for $|\mu|_{L_2(\Omega)}$ almost every $t\in I$. Therefore, if we define the measure $M_1 := \pi|\mu|_{L_2(\Omega)}\otimes(\mathscr{L}^2 \llcorner \Omega)\in\mathscr{M}(Q)$, then

$$\int_I (p,d\mu)_{L_2(\Omega)} \equiv \int_I (p,\pi)_{L_2(\Omega)} d|\mu|_{L_2(\Omega)} = \int_Q p\,dM_1$$

for each $p \in C_0(Q)$. Equation (51) then gives

$$\int_Q p\, dM_1 = \int_Q p\, dM$$

for each $p \in C_0(Q)$. The uniqueness assertion in the scalar Riesz representation theorem in the class of regular signed measures (see, e.g., [18, Theorem 6.19]) and the fact that finite measures on $\mathbf{R}^n$ are automatically regular ([18, Theorem 2.18]) implies that the measures M and M_1 are the same. Thus $M_1(B) = M(B)$ for each Borel subset of Q. This extends the validity of (51) to characteristic functions of Borel subsets of Q; hence to simple functions, and as any integrable function is the limit of simple functions, we have (51) generally. We leave the details to the reader. □

5.5 Corollary

(i) *The class $\mathscr{M}_0(Q)$ permits only the following ambiguity of integrands: if p_1, p_2 are two Borel measurable scalar functions on Q then*

$$\int_Q p_1\, dM = \int_Q p_2\, dM \tag{52}$$

for all $M \in \mathscr{M}_0(Q)$ if and only if

$$\text{for every } t \in I \text{ we have } \ p_1(t,x) = p_2(t,x) \ \text{ for } \mathscr{L}^2 \text{ almost every } x \in \Omega. \tag{53}$$

(ii) *If $p : I \to L_2(\Omega)$ is a weakly continuous norm bounded map then p is M integrable for any $M \in \mathscr{M}_0(Q)$; in particular, if $p \in C_w(\bar{I}; V)$ then $t \mapsto (\mathbf{1} - a\,\Delta)p(t)$ is a weakly continuous map from $\bar{I}$ to $L_2(\Omega)$ and hence M integrable for any $M \in \mathscr{M}_0(Q)$.*

Proof (i): If $f \in L_2(\Omega)$, $t \in I$, and δ_t is the Dirac measure at t, then $M := f\delta_t \otimes \mathscr{L}^2 \llcorner \Omega \in \mathscr{M}_0(Q)$; hence testing (52) on this M and varying f and t yields (53). Conversely, invoking the integrability of p_1 an p_2 with respect to any $M \in \mathscr{M}_0(Q)$, the particular case of M as above yields $p_1(t,\cdot)$, $p_2(t,\cdot) \in L_2(\Omega)$ for any $t \in I$, Equation (53) then guarantees that p_1 and p_2, are the same maps from $I \to L_2(\Omega)$ and thus (52) holds for any $M \in \mathscr{M}_0(Q)$ by the identification in Item (i).

(ii): Let $\mu \in \mathscr{M}(I; L_2(\Omega))$ be the measure related to M as in Proposition 5.4(i). Then p is $|\mu|_{L_2(\Omega)}$ measurable by the Pettis measurability theorem mentioned after Definitions 8.3 since $L_2(\Omega)$ is separable. The norm boundedness of p then implies that p is $|\mu|_{L_2(\Omega)}$ integrable and hence M integrable by Proposition 5.4(ii). In particular, if $v \in C_w(\bar{I}; V)$, the weak continuity of v in V and the continuity of $\mathbf{1} - a\,\Delta : V \to L_2(\Omega)$ implies the weak continuity of $t \mapsto (\mathbf{1} - a\,\Delta)v(t)$ on $\bar{I}$ in $L_2(\Omega)$. Since $\bar{I}$ is compact, the continuous numerical function $t \mapsto ((\mathbf{1} - a\,\Delta)v(t), g)_{L_2(\Omega)}$ is bounded on $\bar{I}$ for each $g \in L_2(\Omega)$. Hence it is norm bounded in the sense that $|(\mathbf{1} - a\,\Delta)v(t)|_{L_2(\Omega)} \le c < \infty$ for some c and all $t \in \bar{I}$ by the uniform boundedness principle. The conclusion then follows from the first part of the assertion. □

6 Proof of Theorem 3.6

Let (u, θ) be a solution of Problem $\mathscr{P}$. Write $\Lambda = \Lambda(u, \theta)$.

(i): Since by (9) and (4) the map

$$p \mapsto \int_Q (\mathbf{1} - a\,\Delta) p \, d\ddot{u} - \int_I \langle \Lambda, p \rangle \; dt$$

is a continuous linear functional on $C_0(I;V)$, by Theorem 8.5 there exists a unique measure $g \in \mathscr{M}(I;V^*)$ such that

$$\int_Q (\mathbf{1} - a\,\Delta) p \, d\ddot{u} - \int_I \langle \Lambda(u,\theta), p \rangle \; dt = \int_I \langle p, dg \rangle$$

for every $p \in C_0(I;V)$. Let us prove that g is a nonnegative measure. Indeed, let $p \in C_0(I;V)$ be nonnegative. For every $\tau > 0$ put $y = u + \tau p$. Inserting this y into $(10)_1$, dividing by τ and letting $\tau \to \infty$ we obtain

$$\int_Q (\mathbf{1} - a\,\Delta) p \, d\ddot{u} - \int_I \langle \Lambda, p \rangle \; dt \ge 0$$

which proves that g is nonnegative. Thus g has the form (46) where G, F and ζ are as in Proposition 5.3. If $y \in \mathscr{K}$ is such that $y(0) = u(0)$ and $y(T) = u(T)$ then $y - u \in C_0(I;V)$ and hence

$$\int_I \langle y - u, dg \rangle = \int_Q (\mathbf{1} - a\,\Delta)(y - u) \, d\ddot{u} - \int_I \langle \Lambda(u,\theta), y - u \rangle \; dt \ge 0$$

by $(10)_1$. Thus

$$\int_Q (y - u) \, dG - \int_I \int_{\partial\Omega} F \, \nabla(y - u) \cdot \mathbf{n} \; d\mathscr{H}^1 d\zeta \ge 0 \tag{54}$$

for every $y \in \mathscr{K}$ such that $y(0) = u(0)$ and $y(T) = u(T)$. If $\kappa \in C_0(I \times \partial\Omega, \mathbf{R})$ is a class C^2 function, we can choose a uniformly bounded sequence $y_k \in \mathscr{K}$ such that $y_k(0) = u(0)$ and $y_k(T) = u(T)$, $-\nabla(y_k - u) \cdot \mathbf{n} = \kappa$ on $I \times \partial\Omega$ and $y_k - u \to 0$ pointwise on Q by Lemma 5.2. The limit in (54) provides

$$-\int_I \int_{\partial\Omega} F \, \kappa \; d\mathscr{H}^1 d\zeta \ge 0$$

and thus $F = 0$ by the arbitrariness of κ. Equation (46) then reduces to $\int_I \langle p, dg \rangle = \int_I p \, dG$, which completes the proof of (i).

(ii): Interpreting $\ddot{u}$ as an element of $\mathscr{M}(I; L_2(\Omega))$ by Proposition 5.4, we denote the regular and singular parts relative to $\mathscr{L}^1$ in the sense of the Lebesgue decomposition theorem [5, Theorem 9, p. 31] by $\ddot{u}_{\mathrm{r}}$ and $\ddot{u}_{\mathrm{s}}$. Thus we have (14) and $\ddot{u}_{\mathrm{r}}$ is absolutely continuous with respect to $\mathscr{L}^1$ in the sense that if B is a Borel subset of I such that $\mathscr{L}^1(B) = 0$ then $\ddot{u}_{\mathrm{s}}(B) = 0$ while $\ddot{u}_{\mathrm{s}}$ is singular to $\mathscr{L}^1$ in the sense that there exists a Borel subset J of I with $\mathscr{L}^1(J) = 0$ such that $\ddot{u}_{\mathrm{s}}(B) = 0$ for each Borel subset B of $I \setminus J$. Since by the results of Section 8, all measures in $\mathscr{M}(I; L_2(\Omega))$ are representable by densities via the Radon–Nikodým theorem, we have $\ddot{u}_{\mathrm{r}} = h\mathscr{L}^1 \llcorner I$ for some $h \in L_1(I; L_2(\Omega))$ and $\ddot{u}_{\mathrm{s}} = \pi\phi$ for a $\mathscr{L}^1$ singular nonnegative measure ϕ and some $\pi \in L_1(I; L_2(\Omega), \phi)$. It follows that the Lebesgue decomposition of $\ddot{u}$, now interpreted as a scalar measure from $\mathscr{M}_0(Q)$, takes the form (14)–(15) with h, π and ϕ having the properties required by (ii), except for the nonnegativity of π, which will be proved below.

(iii): Equation (13) shows that

$$(\mathbf{1} - a\,\Delta)^* \ddot{u} = \Lambda \mathscr{L}^1 \llcorner I + g$$

where $(\mathbf{1} - a\,\Delta)^* \ddot{u} \in \mathscr{M}(I; V^*)$ is a measure defined by $[(\mathbf{1} - a\,\Delta)^* \ddot{u}](B) = (\mathbf{1} - a\,\Delta)^*[\ddot{u}(B)]$ for any Borel subset B of I. Since $\Lambda\mathscr{L}^1 \llcorner I$ is absolutely continuous with respect to $\mathscr{L}^1$, we see from the last relation that the singular parts g_{s} and $\ddot{u}_{\mathrm{s}}$ of g and $\ddot{u}$, respectively, in the sense of the Lebesgue decomposition theorem on I relative to $\mathscr{L}^1$, are related by

$$(\mathbf{1} - a\,\Delta)^* \ddot{u}_{\mathrm{s}} = g_{\mathrm{s}}, \quad \ddot{u}_{\mathrm{s}} = (\mathbf{1} - a\,\Delta)^{-*} g_{\mathrm{s}}.$$

Since g is nonnegative, also g_{s} is nonnegative. Lemma 3.2(iii) then provides that $\ddot{u}_{\mathrm{s}}$ is nonnegative. Consequently also π is nonnegative and we have (16).

(iv): Prove that the support of G is contained in C. Inserting $p = y - u$ into (13) one obtains

$$\int_I ((\mathbf{1} - a\,\Delta)(y - u), d\ddot{u})_{L_2(\Omega)} - \int_I \langle \Lambda, u - y\rangle \, dt = \int_Q (y - u)\, dG$$

for every $y \in \mathscr{K}$; inequality $(10)_1$ then gives

$$\int_Q (y - u)\, dG \geq 0.$$

Taking any uniformly bounded sequence $y_k \in \mathscr{K}$ such that $y_k \to 0$ pointwise on Q we obtain $-\int_Q u\, dG \geq 0$ and hence $\int_Q u\, dG = 0$ as both u and G are nonnegative. Then $\operatorname{supp} G \subset C$ follows.

Finally, prove also that the support of $\ddot{u}_{\mathrm{s}}$ is contained in C. Since $\ddot{u}_{\mathrm{s}}$ and u are nonnegative, we have

$$0 \leq \int_I (u, d\ddot{u}_{\mathrm{s}})_{L_2(\Omega)} = \int_I \langle (\mathbf{1} - a\,\Delta)u, dg_{\mathrm{s}}\rangle = -a \int_I \langle \Delta\, u, dg_{\mathrm{s}}\rangle \leq 0$$

since $\int_I \langle \Delta\, u, dg_{\mathrm{s}}\rangle \geq 0$ as we now show. Indeed, the integration in the last integral is effective only on the contact zone C as $\Delta\, u = 0$ outside C. However, on C we have $\Delta\, u \geq 0$. To explain it, note that we have $\Delta\, u = 0$ in the interior of C. Further, every boundary point of C is a point of minimum of u and thus the second differential of u is positive semidefinite; consequently the laplacian is nonnegative. □

7 Appendix A: Remarks on notation

Let $\mathscr{L}^n$ denote the Lebesgue measure in $\mathbf{R}^n$ for any n and let $\phi \llcorner M$ denote the restriction of a measure ϕ to a Borel set M, given by

$$(\phi \llcorner M)(A) = \phi(M \cap A) \tag{55}$$

for each ϕ measurable set A. Further, $\mathscr{M}(Q)$ denotes the set of all (finite) signed scalar Borel measures on Q.

If K is a compact space and X a Banach space then $C(K; X)$ denotes the set of all continuous maps from K into X with the maximum norm $|\cdot|_{C(K;X)}$. If $P \subset \mathbf{R}^n$ then $C_0(P; X)$ is the set of all continuous maps from $\mathbf{R}^n$ into X which vanish outside P. This defines in particular $C_0(I; X)$ where $I \subset \mathbf{R}$ is the time interval. $\mathscr{D}(Q)$ denotes the Schwartz's space of class C^∞ functions $p : \mathbf{R}^3 \to \mathbf{R}$ with compact support contained in Q.

If M stands for any of Ω, Q or $\partial\Omega$ and $1 \le p \le \infty$, we define the Lebesgue space $L_p(M)$ of (Lebesgue classes of) real valued Lebesgue measurable functions, integrable with power p if $p < \infty$, and essentially bounded if $p = \infty$. If X is a Banach space, then $L_p(I;X)$ denotes the Lebesgue space of X valued maps on I [5, Chapter IV, Section 1], [21, Sections 23.2 and 23.3].

By $H^k(M) \subset L_2(M)$ with $k \ge 0$ we denote the Hilbert–type Sobolev (for a noninteger k the Sobolev-Slobodetskii) spaces of functions defined on M. By $\mathring{H}^1(\Omega)$ we denote the subspace of functions from $H^1(\Omega)$ with zero traces on $\partial\Omega$. The dual of $\mathring{H}^1(\Omega)$ is denoted by $H^{-1}(\Omega)$ with the duality pairing $\langle\cdot,\cdot\rangle$ introduced in Section 3.

8 Appendix B: Vector valued measures

For convenience of the reader, we present a summary of Banach space valued measures on I. In practically every assertion, the interval I can be replaced by a locally compact space. We refer to Dinculeanu [6], Diestel & Uhl [5], and Dunford & Schwartz [7, Chapter IV]. Section 5 applies the material presented here to vector valued measures pertinent to Problem $\mathscr{P}$.

Throughout the section, let X be a Banach space with the norm $|\cdot|_X$; we denote by $\langle\gamma,\xi\rangle \equiv \langle\xi,\gamma\rangle$ the value of $\gamma \in X^*$ on an element $\xi \in X$.

8.1 Definitions (Cf. [6, Definition 3, p. 16 and Definition 1, p. 302])

(i) Let $\mathscr{A}$ be the collection of Borel subsets of I. A map $\mu : \mathscr{A} \to X$ is called an X valued measure on I if

(a) it is countably additive in the norm sense;

(b) its variation $|\mu|_X$ is finite, i.e., if for any Borel set $B \subset I$ one has

$$|\mu|_X(B) := \sup\Bigg\{ \sum_{j=1}^{k} |\mu(B_j)|_X : B_j \in \mathscr{A},\ j = 1,\ldots,k, \text{ is}$$
$$\text{a pairwise disjoint partition of } B \Bigg\} < \infty.$$

(ii) The function $|\mu|_X$ is a finite nonnegative Radon measure on I, called the total variation measure of μ.

(iii) Denote by $\mathscr{M}(I;X)$ the set of all X valued measures on I with the norm $|\mu|_{\mathscr{M}(I;X)} := |\mu|_X(I)$, under which $\mathscr{M}(I;X)$ becomes a Banach space.

8.2 Remarks

(i) In the scalar case, i.e., if $X = \mathbf{R}$, Property (b) in Definition 8.1 is a consequence of (a). See, e.g., [18, Theorem 6.4]. Not so for a general X.

(ii) Since every open subset of I is the union of a countable family of compact sets, $|\mu|_X$ is automatically regular [18, Theorem 2.18], i.e., for every $B \in \mathscr{A}$ and every $\epsilon > 0$ there exist a compact set C and an open set U such that $C \subset B \subset U$ and
$$|\mu|_X(U) - |\mu|_X(C) < \epsilon;$$
consequently $|\mu(C) - \mu(B)|_X < \epsilon$, $|\mu(B) - \mu(U)|_X < \epsilon$.

8.3 Definitions Let $\alpha : I \to X$ and let ϕ be a nonnegative measure on I.

(i) α is said to be simple if its range is finite and for each $\xi \in X$ the set $\alpha^{-1}(\{\xi\}) \subset I$ is borelian.

(ii) α is said to be ϕ measurable if there exists a sequence of simple functions that converges to α at ϕ almost every point of I.

(iii) α is said to be ϕ integrable if it is ϕ measurable and there exists a sequence of simple maps α_k such that

$$\int_I |\alpha_k - \alpha|_X \, d\phi \to 0$$

where it is noted that the ϕ measurability of α implies that the numerical sequence $|\alpha_k - \alpha|$ is ϕ measurable. We denote by $\mathscr{L}(\phi, X)$ the set of all X valued ϕ integrable maps. Clearly, $\int_I |\alpha|_X \, d\phi < \infty$ for any $\alpha \in \mathscr{L}(\phi, X)$.

By the Pettis measurability theorem [5, Theorem 2, p. 42], α is ϕ measurable if and only if α is ϕ essentially separably valued, i.e., there exists a Borel set $E \subset I$ of null ϕ measure such that $\alpha(I \setminus E)$ is norm separable and for each $\gamma \in X^*$, the numerical function $\langle \alpha, \gamma \rangle$ is ϕ measurable. In particular if α is norm continuous or weakly continuous and essentially separably valued then α is ϕ integrable.

8.4 Integrals Let $\mu \in \mathscr{M}(I; X)$, $\phi \in \mathscr{M}(I; \mathbf{R})$, $\phi \geq 0$. The theory of vector valued measures provides definitions of three types of integrals in (a)–(c) below. We shall reduce the definitions of these vector valued integrals to the scalar case.

(a) The Bochner integral of a ϕ integrable map $\alpha : I \to X$; the result is a unique element $\int_I \alpha \, d\phi$ of X such that

$$\left\langle \gamma, \int_I \alpha \, d\phi \right\rangle = \int_I \langle \gamma, \alpha \rangle \, d\phi$$

for any $\gamma \in X^*$. Note that such an element exists in X and not only in the second dual X^{**}; this has to be proved via the limits of integrals of simple maps. One has $|\int_I \alpha \, d\phi|_X \leq \int_I |\alpha|_X \, d\phi$.

(b) The integral of a $|\mu|_X$ integrable scalar valued function $\varphi : I \to \mathbf{R}$ with respect to μ; the result is the unique vector $\int_I \varphi \, d\mu$ in X such that

$$\left\langle \gamma, \int_I \varphi \, d\mu \right\rangle = \int_I \varphi \, d \langle \gamma, \mu \rangle$$

for any $\gamma \in X^*$. One has $|\int_I \varphi \, d\mu|_X \leq \int_I |\varphi| \, d|\mu|_X$.

(c) The integral $\int_I \langle \alpha, d\mu \rangle$ of a $|\mu|_X$ integrable map $\alpha : I \to X^*$; the result is a real number $\int_I \langle \alpha, \mu \rangle$. We give the definition only in the particular case when X is either reflexive or a separable dual of some Banach space. Namely, then the Radon Nikodým theorem holds [5, Corollary 13, p. 76 and Theorem 1, p. 79]. Therefore, since μ is absolutely continuous with respect to $|\mu|_X$ (i.e., $\mu(B) = 0$ whenever $|\mu|_X(B) = 0$), there exists a $|\mu|_X$ integrable map $\pi : I \to X$ such that $\mu = \pi|\mu|_X$, where the last is a measure defined by

$$(\pi|\mu|_X)(B) = \int_B \pi \, d|\mu|_X$$

for any Borel set $B \subset I$. One has $|\pi|_X = 1$ for $|\mu|_X$ almost every point of I. We then define

$$\int_I \langle \alpha, d\mu \rangle := \int_I \langle \alpha, \pi \rangle \; d|\mu|_X. \tag{56}$$

One has $|\int_I \langle \alpha, d\mu\rangle| \le \int_I |\alpha|_{X^*} d|\mu|_X$. In the subsequent treatment, we apply the above notions only to the choices $X = L_2(\Omega)$ and $X = V^*$; both these are separable duals, so the above hypotheses hold. If $X = L_2(\Omega)$, we write $\int_I (\alpha, d\mu)_{L_2(\Omega)}$ for (56). We note in passing that the Radon Nikodým theorem, which does not hold generally, is one of the central topics of the theory of vector valued measures. See [5, Chapters III & IV]; the above result is a simple particular case.

The following direct generalization of the Riesz representation theorem is the main result of this section.

8.5 Theorem (Singer [20]; see also [6, Corollary 2, p. 387] and [5, Theorem, p. 182]) *If X is a Banach space then $C_0(I;X)^*$ is isometrically isomorphic with $\mathscr{M}(I;X^*)$ under the identification of $F \in C_0(I;X)^*$ with $\mu \in \mathscr{M}(I;X^*)$ via*

$$\langle \alpha, F \rangle = \int_I \langle \alpha, d\mu \rangle$$

for any $\alpha \in C_0(I;X)$.

9 Appendix C: Weakly differentiable maps

Throughout the section, X and Y are Banach spaces with norms $|\cdot|_X$ and $|\cdot|_Y$ with X reflexive and compactly and densely embedded in Y.

9.1 Remark *We have*

$$\left.\begin{aligned} C_w(\bar I;X) &= L_\infty(I;X) \cap C(\bar I;Y),\\ R_w(\bar I;X) &= L_\infty(I;X) \cap R(\bar I;Y); \end{aligned}\right\} \tag{57}$$

so

$$\left.\begin{aligned} C_w(\bar I;V) &= L_\infty(I;V) \cap C(\bar I;\mathring{H}^1(\Omega)),\\ R_w(\bar I;\mathring{H}^1(\Omega)) &= L_\infty(I;\mathring{H}^1(\Omega)) \cap R(\bar I;L_2(\Omega)). \end{aligned}\right\} \tag{58}$$

Here $C_w(\bar I;X)$ is the space of weakly continuous maps, i.e., such that $t \mapsto \langle \gamma, p(t)\rangle$ is a continuous function for each $\gamma \in X^*$, $C(\bar I;Y)$ the set of norm continuous maps, $R_w(\bar I;X)$ the set of weakly regulated maps, defined in the same way as in Section 3.1 with $\mathring{H}^1(\Omega)$ replaced by X, $R(\bar I;Y)$ the set of norm regulated maps, defined by replacing the weak topology by the norm topology.

Proof Let $p \in L_\infty(I;X) \cap C(\bar I;Y)$. Let us first prove that $p(t) \in X$ for every $t \in \bar I$ (and not just for a.e. $t \in \bar I$). Assume, on the contrary, that $t \in \bar I$ be such that $p(t) \notin X$. Let $t_k \to t$ be any sequence in $\bar I$ such that $p(t_k) \in X$ for each k. The sequence $|p(t_k)|_X$ is bounded and therefore, by the reflexivity of X, there exists a subsequence, again denoted by $p(t_k)$, such that $p(t_k) \rightharpoonup w$ for some $w \in X$ and $\liminf_{k\to\infty} |p(t_k)|_X \ge |w|_X$. We have $Y^* \subset X^*$ and thus if

$\gamma \in Y^*$, we have $\langle \gamma, p(t_k) \rangle \to \langle \gamma, w \rangle$ However, the Y continuity of p implies that $\langle \gamma, p(t_k) \rangle \to \langle \gamma, p(t) \rangle$. Thus

$$\langle \gamma, w \rangle = \langle \gamma, p(t) \rangle \quad \text{for all} \quad \gamma \in Y^*. \tag{59}$$

Since Y^* dense in X^* by the density of X in Y, (59) implies $p(t) = w \in X$. Hence $p(t) \in X$ for every $t \in \bar{I}$ and $|p(t)|_X \le |u|_{L_\infty(I;X)}$. The same argument shows that if $t \in \bar{I}$ then

$$\langle \gamma, p(s) \rangle \to \langle \gamma, p(t) \rangle \tag{60}$$

for $s \to t$ for every $\gamma \in Y^*$. The boundedness of the X norm of p then extends (60) to all $\gamma \in X^*$ by density. Thus p is weakly continuous as a map from $\bar{I}$ to X. This proves the inclusion $\supset$ in $(57)_1$.

Conversely, let $p \in C_w(\bar{I}; X)$. By the Banach Steinhaus theorem, p is norm bounded. Since the embedding of X into Y is compact, $p \in C_w(\bar{I}; X)$ implies $p \in C(\bar{I}; Y)$, which proves the opposite inclusion.

The assertion about the regulated maps is proved similarly. □

9.2 Remark *If X^* is separable then for every $p \in R_w(\bar{I}; X)$ we have $p(t+) = p(t-)$ for all $t \in \bar{I}$ except possibly an at most countable set $L \subset \bar{I}$.*

Proof Let S_0 be a countable dense subset of X^*. Let L be the set of all t such that $p(t+) \ne p(t-)$ and for each $\gamma \in S_0$ the symbol L_γ be the set of all t such that $\langle \gamma, p(t+) \rangle \ne \langle \gamma, p(t-) \rangle$. The assumption implies that

$$L = \bigcup_{\gamma \in S_0} L_\gamma.$$

By a well–known property of scalar valued regulated functions, each L_γ is at most countable. □

9.3 Theorem

(i) *For every $p \in L_\infty(I; X)$ with $\dot{p} \in \mathscr{M}(I; Y)$ we can choose a unique representative p that belongs to $R_w(\bar{I}; X)$. There exists a $\beta \in Y$ such that we have*

$$p(t) = \beta + \dot{p}((0, t)) \tag{61}$$

for every $t \in \bar{I}$,

$$p(t-) = \dot{p}((0, t)), \quad p(t) - p(t-) = \dot{p}\left(\{t\}\right)$$

with the limits in the sense of the norm $|\cdot|_Y$.

(ii) *If $p \in L_\infty(I; X)$ and $\dot{p} \in L_1(I; Y)$ then the unique representative p from* (i) *belongs to $C_w(\bar{I}; X)$ and* (61) *reduces to*

$$p(t) = \beta + \int_0^t \dot{p}\, dt$$

for some $\beta \in Y$ and all $t \in \bar{I}$.

Here $\dot{p} \in \mathscr{M}(I; Y)$ and $\dot{p} \in L_1(I; Y)$ are generalized derivatives, obtained by interpreting p and $\dot{p}$ as X or Y valued distributions on $(0, T)$ in the usual way [10, Chapter IV], [21, Chapter 21], and defining the derivatives in the same way as in the scalar case.

Proof (i): By [5, Theorem 9, p. 49] the integrability of p implies that for a.e. $s \in \bar{I}$ we have

$$\lim_{h\to 0} h^{-1} \int_s^{s+h} |p(t) - p(s)|_X \, dt = 0,$$

$$\lim_{h\to 0} h^{-1} \int_s^{s+h} p(t)\, dt = p(s) \quad \text{in the } |\cdot|_X \text{ norm sense in } X. \tag{62}$$

Let $s \in \bar{I}$ be such that (62) holds. Let $J = [s,t] \subset \operatorname{int} \bar{I}$ and for every $h > 0$ let $\varphi_h : \mathbf{R} \to \mathbf{R}$ be defined by $\varphi_h(\tau) = \max\{1 - h^{-1}\operatorname{dist}(\tau, [s, t-h]), 0\}$. Applying the definition $\int_I \dot\varphi p\, dt = -\int_I \varphi d\dot p$ to φ_h, evaluating the derivative of φ_h and using (62) one finds that

$$\int_I \dot\varphi_h p\, dt \to p(t) - p(s)$$

in the norm $|\cdot|_X$ sense. Since φ_h converges pointwise to the characteristic function $1_{[s,t]}$ of the interval $[s,t]$, the vectorial Lebesgue dominated convergence theorem ([6, Theorem 3, p. 136]) gives $\int_I \varphi_h \, d\dot p \to \dot p([s,t])$ and thus the limit with

$$\int_I \dot\varphi_h p\, dt = -\int_I \varphi_h \, d\dot p \tag{63}$$

gives

$$p(t) - p(s) = \dot p([s,t])$$

which holds for a.e. $s, t \in \bar{I}$. As there is at most countable set of s such that $\dot p(\{s\}) \neq 0$, we also have

$$p(t) - p(s) = \dot p((s,t])$$

for a.e. $s, t \in \bar{I}$. It follows that there is a $\beta \in Y$ such that the p given by (61) differs from any element of the Lebesgue class on a set of Lebesgue measure 0. The representative from (61) then satisfies

$$|p(t) - p(s)|_Y \le |\dot p|((s,t])$$

and the scalar Lebesgue dominated convergence theorem gives the continuity of p from the right. The rest of (i) is immediate in view of Remark 9.1.

(ii) is a direct consequence of (i). □

9.4 Definition Let $v \in L_1(I;Y)$ and w be a representative of v.

(i) Then the variation $\operatorname{Var}(w)$ is defined as

$$\operatorname{Var}(w) \equiv \sup\left\{ \sum_{j=1}^{k} |w(t_{j+1}) - w(t_j)|_Y : 0 = t_1 < \cdots < t_{k+1} = T, k = 2, 3, \ldots \right\}$$

and the essential variation $\operatorname{ess}\operatorname{Var}(v)$ as

$$\operatorname{ess}\operatorname{Var}(v) = \inf\left\{ \operatorname{Var}(w) : w \in v \right\}.$$

(ii) We say that v has finite $|\cdot|_Y$ variation if $\operatorname{ess}\operatorname{Var}(v) < \infty$. We denote by $BV(I;Y)$ the set of all maps $v \in L_1(I;Y)$ of finite essential variation.

9.5 Theorem *Let $v \in L_1(I;V)$. Then $v \in BV(I;Y)$ if and only if v has a weak derivative represented by a measure $\dot{v}$ from $\mathscr{M}(I;V)$. If these two equivalent conditions are satisfied then*

$$\operatorname{ess}\operatorname{Var}(v) = |\dot{v}|_Y(I).$$

Proof From [6, Theorem 1, p. 358] one deduces the existence of a measure $\dot{v} \in \mathscr{M}(I;Y)$ such that (61) holds for every $t \in \bar{I}$ and some $\beta \in Y$. If $\gamma \in Y^*$ then the measure $\langle \gamma, \dot{v} \rangle$ is a numerical signed measure on $\bar{I}$ and the standard scalar integration by parts gives

$$\Big\langle \gamma, \int_I \dot{\varphi} v \, dt \Big\rangle = -\Big\langle \gamma, \int_I \varphi \, d\dot{v} \Big\rangle$$

which in turn gives (63) for every $\varphi \in C_0^\infty(\bar{I})$, which proves the direct implication. The converse implication is immediate. □

9.6 Remark

(i) *The definitions of $\mathscr{W}$ in $(5)_1$ and (6) are equivalent.*

(ii) *If $u_k \in \mathscr{W}$ is a sequence such that $u_k \rightharpoonup^* u$ in $w + L_\infty(I;V)$, then $u \in w + \mathscr{W}$ and*

$$u_k \to u \quad \text{uniformly in} \quad C(\operatorname{cl} Q). \tag{64}$$

Proof (i): If u belongs to the right hand side of $(5)_1$ then (58) shows that u belongs to the right hand side of (6). Conversely, assume that u belongs to the right hand side of (6) and prove that we can choose the representatives u and $\dot{u}$ as in the right hand side of $(5)_1$. Indeed, if $\ddot{p} \in \mathscr{M}_0(\bar{I}; L_2(\Omega))$ and $\dot{p} \in L_\infty(I; \mathring{H}^1(\Omega))$ then $\dot{p} \in R_w(\bar{I}; \mathring{H}^1(\Omega))$ by Theorem 9.3(i). Similarly, if $\dot{p} \in L_\infty(I; \mathring{H}^1(\Omega))$ and $p \in L_\infty(I;V)$ then $p \in C_w(\bar{I};V)$ by Theorem 9.3(ii).

(ii): Equation (64) follows from Aubin's lemma where $u_k \in w + L_\infty(I;V)$ where V is compactly embedded into $C(\operatorname{cl}\Omega))$ and the derivative is taken as a bounded sequence in $L_\infty(I;V^*)$; we thus have $C(\operatorname{cl}\Omega) \subset V^*$. □

Acknowledgement The work presented here was partially supported by the Grant No. P201/1 2/0671 of the Grant Agency of the Czech Republic and under the Institutional research plan RVO 67985840 and by the grants APVV-0246-1 2, VEGA 1/0426/1 2 of the Grant Agencies of the Slovak Republic.

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