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TOPOLOGICAL PROPERTIES OF FUNCTION SPACES OVER ORDINAL SPACES

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Abstract. Being motivated by the classical Ascoli theorem, a topological space X is said to be an *Ascoli space* if any compact subset $\mathcal{K}$ of $C_k(Y)$ is evenly continuous. We study the $k_{\mathbb{R}}$-property and the Ascoli property of $C_p(\kappa)$ and $C_k(\kappa)$ over ordinal spaces $\kappa = [0, \kappa)$. We prove that $C_p(\kappa)$ is always an Ascoli space, while $C_p(\kappa)$ is a $k_{\mathbb{R}}$-space iff the cofinality of κ is countable. In particular, this provides the first C_p-example of an Ascoli space which is not a $k_{\mathbb{R}}$-space, namely $C_p(\omega_1)$. We show that $C_k(\kappa)$ is Ascoli iff $\mathrm{cf}(\kappa)$ is countable iff $C_k(\kappa)$ is metrizable.

1. Introduction

The study of topological properties of function spaces is quite active and attract specialists both from topology and functional analysis, see for example [1, 3, 7, 13, 18, 19] and references therein. In the following diagram we select the most important compact type properties generalizing metrizability

$$\text{metric} \Longrightarrow \begin{matrix}\text{Fréchet–}\\ \text{Urysohn}\end{matrix} \Longrightarrow k\text{-space} \Longrightarrow k_{\mathbb{R}}\text{-space} \Longrightarrow \text{Ascoli},$$

and note that none of these implications is reversible, see [3, 5] (all relevant definitions are given in the next section).

For a Tychonoff topological space X, we denote by $C_k(X)$ and $C_p(X)$ the space $C(X)$ of all continuous real-valued functions on X endowed with the compact-open topology and the topology of pointwise convergence, respectively.

It is well-known that $C_p(X)$ is metrizable if and only if X is countable. Pytkeev, Gerlitz and Nagy (see §3 of [1]) characterized spaces X for which $C_p(X)$ is Fréchet–Urysohn or a k-space (these properties coincide for spaces of the form $C_p(X)$). The authors in [6] obtained some sufficient conditions on X for which the space $C_p(X)$ is an Ascoli space. Recall here that X is called an *Ascoli space* if any compact subset $\mathcal{K}$ of $C_k(X)$ is evenly continuous (or, equivalently, if the natural evaluation map $X \hookrightarrow C_k(C_k(X))$ is an embedding, see [3]).

A topological space X is called an *ordinal space* if there exists an ordinal number κ such that $X = [0, \kappa)$ (as usual we will write simply $X = \kappa$) and the topology on X is the usual ordered topology generated by the singleton $\{0\}$ and all intervals of the form $(\alpha, \beta]$ with $\alpha < \beta < \kappa$. Ordered spaces form an interesting class of linearly ordered topological spaces, and function spaces over them give a good source of (counter)examples in the corresponding theory. For instance, the space $C_p(\omega_1)$ is Lindelöf, see [19].

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Arhangel'skii showed in [2] that the space $C_p(\omega_1+1)$ is not normal. In [11] Gul'ko proved that there are no two distinct natural number n and m for which the powers $C_p(\omega_1)^n$ and $C_p(\omega_1)^m$ are homeomorphic. In [15] Morris and Wulbert observed that $C_k(\omega_1)$ is not barrelled.

In this short note we provide complete characterizations of those ordinals κ for which $C_p(\kappa)$ and $C_k(\kappa)$ are $k_{\mathbb{R}}$-spaces or Ascoli spaces. The following theorems are the main results of the paper.

Theorem 1.1. *For every ordinal κ the space $C_p(\kappa)$ is Ascoli.*

Denote by $\mathrm{cf}(\kappa)$ the cofinality of an ordinal κ.

Theorem 1.2. *For an ordinal κ, the space $C_p(\kappa)$ is a $k_{\mathbb{R}}$-space if and only if $\mathrm{cf}(\kappa)\le\omega$ if and only if $C_p(\kappa)$ is Fréchet–Urysohn.*

In light of the fact that for uncountable discrete X the space $C_p(X)=\mathbb{R}^X$ is a $k_{\mathbb{R}}$-space but not a k-space [5, 3.3.E], Theorem 1.2 also shows that the space $C_p(\omega_1)$ is Ascoli but not a $k_{\mathbb{R}}$-space. This answers Question 6.8 in [7] for spaces $C_p(X)$ in the affirmative.

Theorem 1.3. *For an ordinal κ, the space $C_k(\kappa)$ is an Ascoli space if and only if $\mathrm{cf}(\kappa)\le\omega$, so $C_k(\kappa)$ is complete and metrizable.*

Being motivated by the aforementioned results we conclude the paper with a number of facts about further topological properties of function spaces, as well as their properties related to the locally convex space theory, see Section 3.

2. Proofs

Below we recall some topological concepts used in Theorems 1.1 and 1.3, for other notions we refer the reader to the book [5].

Definition 2.1. A topological space X is

- a $k_{\mathbb{R}}$*-space* if a real-valued function f on X is continuous if and only if its restriction $f|_K$ to any compact subset K of X is continuous;
- *scattered* if every subspace A of X has an isolated point in A;
- μ*-space* if every functionally bounded subset of X has compact closure; $A\subset X$ is called functionally bounded if $f(A)$ is bounded in $\mathbb{R}$ for any $f\in C(X)$.

A *k-cover* $\mathcal{U}$ of a topological space X is a family of subsets of X such that every compact subset of X is contained in some member of $\mathcal{U}$.

Recall that an ordinal κ is *limit* if there is no α such that $\kappa=\alpha+1$, otherwise κ is called a *successor* ordinal. The *cofinality* $\mathrm{cf}(\kappa)$ of a limit ordinal number κ is the smallest ordinal α which is the order type of a cofinal subset of κ. If κ is a successor ordinal we write $\mathrm{cf}(\kappa)<\omega$.

The following simple facts should be well-known (for (i) see [9, § 5.11], other results are straightforward).

Lemma 2.2. (i) *κ is compact if and only if it is a successor;*
(ii) *κ is hemicompact non-countably compact if and only if $\mathrm{cf}(\kappa)=\omega$;*
(iii) *κ is countably compact non-compact if and only if $\mathrm{cf}(\kappa)>\omega$;*
(iv) *κ is a μ-space if and only if $\mathrm{cf}(\kappa)\le\omega$ if and only if κ is hemicompact;*
(v) *every open k-cover $\mathcal{U}$ of κ has a countable k-subcover if and only if $\mathrm{cf}(\kappa)\le\omega$;*
(vi) *κ is separable if and only if κ is countable;*
(vii) *κ has a dense σ-compact subspace if and only if $\mathrm{cf}(\kappa)\le\omega$.*

For the convenience of the reader we recall also the following two results.

Proposition 2.3 ([7]). *Assume X admits a family $\mathcal{U}=\{U_i : i\in I\}$ of open subsets of X, a subset $A=\{a_i : i\in I\}\subset X$ and a point $z\in X$ such that:* (i) *$a_i\in U_i$ for every $i\in I$,* (ii) *$\big|\{i\in I : C\cap U_i\ne\emptyset\}\big|<\infty$ for each compact subset C of X, and* (iii) *z is a cluster point of A. Then X is not an Ascoli space.*

A family $\{A_i\}_{i\in I}$ of subsets of a set X is said to be *point-finite* if the set $\{i \in I : x \in A_i\}$ is finite for every $x \in X$. A family $\{A_i\}_{i\in I}$ of subsets of a topological space X is called *strongly point-finite* if for every $i \in I$, there exists an open set U_i of X such that $A_i \subseteq U_i$ and $\{U_i\}_{i\in I}$ is point-finite. Following Sakai [17], a topological space X is said to have *property* (κ) if every pairwise disjoint sequence of finite subsets of X has a strongly point-finite subsequence. The following result is proved in [6].

Theorem 2.4. *If $C_p(X)$ is Ascoli, then X has property (κ).*

It is well-known that ordinal spaces are locally compact and scattered (for the last property we note that the smallest element of a subset A of X is isolated in A). The following proposition is of independent interest, it generalizes Corollary 1.5 of [6] and immediately implies Theorem 1.1.

Proposition 2.5. *Let X be a locally compact space. Then $C_p(X)$ is Asoli if and only if X is scattered.*

Proof. The "only if" part follows from Theorem 2.4 combined with the fact that property (κ) is preserved by subspaces, along with the fact that every compact space with property (κ) is scattered, see [17, Theorem 3.2]. For the "if" direction consider the one-point compactification $X^* = X \cup \{x_\infty\}$ of X and note that it is scattered. Therefore $C_p(X^*)$ is Frechet-Urysohn by [1, II.7.16], and hence so is its subspace Z consisting of those continuous $f : X^* \to \mathbb{R}$ such that $f(x_\infty) = 0$. Now it is easy to see that $Z \restriction X = \{f \restriction X : f \in Z\} \subset C_p(X)$ is homeomorphic to Z and is dense in $C_p(X)$. Therefore $C_p(X)$ has a dense Frechet–Urysohn subspace, and hence every function f belongs to a dense Ascoli subspace $f + Z$ of $C_p(X)$. Thus $C_p(X)$ is Ascoli by Proposition 5.10 of [3]. □

Below we prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Let $C_p(\kappa)$ be a $k_{\mathbb{R}}$-space and suppose towards a contradiction that $\mathrm{cf}(\kappa) > \omega$. Then κ is countably compact by Lemma 2.2. Hence the space $C(\kappa)$ endowed with the sup-norm is a Banach space. Therefore the space $C_p(\kappa)$ admits a stronger normed topology and is angelic by [13, Proposition 9.6]. Since every compact subset of $C_p(\kappa)$ is Fréchet–Urysohn, every sequentially continuous function on $C_p(\kappa)$ is continuous. In particular, every sequentially continuous functional on $C_p(\kappa)$ is continuous. So κ is a realcompact space by Theorem 1.1 of [20]. Being realcompact and pseudocompact the space κ is compact by [5, 3.11.1]. Hence κ is a successor ordinal, a contradiction. Thus $\mathrm{cf}(\kappa) \le \omega$.

Conversely, let $\mathrm{cf}(\kappa) \le \omega$. Then κ is hemicompact by Lemma 2.2. Thus $C_p(\kappa)$ is Fréchet–Urysohn by [1, II.7.16]. □

Proof of Theorem 1.3. Suppose for a contradiction that $\mathrm{cf}(\kappa) > \omega$. We shall use Proposition 2.3 and show that $C_k(\kappa)$ is not Ascoli. Let $\{\zeta_\alpha\}_{\alpha<\kappa}$ be an increasing enumeration of all even ordinals (including limit ones) and set $\eta_\alpha := \zeta_\alpha + 1$. So we obtain an enumeration of all ordinals $< \kappa$ by the family $\{\zeta_\alpha, \eta_\alpha\}_{\alpha<\kappa}$ such that

$$... < \zeta_\alpha < \eta_\alpha < \zeta_{\alpha+1} < \eta_{\alpha+1} < \dots .$$

For every $\alpha < \kappa$ we define $f_\alpha : [0, \kappa) \to [0, 1]$ by $f\big([0, \zeta_\alpha]\big) = \{0\}$ and $f\big([\eta_\alpha, \omega_1)\big) = \{1\}$ and set

$$U_\alpha := \{f : f(\zeta_\alpha) < 1/4,\ f(\eta_\alpha) > 3/4\}.$$

To prove that $C_k(\kappa)$ is not Ascoli it is enough to verify the assumptions of Proposition 2.3 for $\{f_\alpha\}_{\alpha<\kappa}$, $\{U_\alpha\}_{\alpha<\kappa}$ and 0. Clearly, (i) and (iii) hold true. Let us check (ii). Take any compact $C \subseteq C_k(\kappa)$ and assume, contrary to our claim, that there are infinitely many $\alpha < \kappa$ such that $C \cap U_\alpha \ne \emptyset$. Then there exists a strictly increasing sequence $\{\alpha_n\}_{n<\omega}$ such that $C \cap U_{\alpha_n} \ne \emptyset$. Let $\alpha = \lim \alpha_n$. As $\mathrm{cf}(\kappa) > \omega$ we obtain $\alpha + 1 < \kappa$. By the Ascoli theorem used for $[0, \alpha + 1] \subset [0, \kappa)$

and $1/2$ we can find a basic neighborhood O_α of α such that $|h(x)-h(y)|<1/4$ for all $x,y\in O_\alpha$ and $h\in C$. Take n such that $\zeta_{\alpha_n}\in O_\alpha$ and fix $h\in C\cap U_{\alpha_n}$. Then $\eta_{\alpha_n}=\zeta_{\alpha_n}+1\in O_\alpha$ and

$$\begin{aligned}\frac{1}{4} &> |h(\eta_{\alpha_n})-h(\zeta_{\alpha_n})|\\ &\geq |f_{\alpha_n}(\eta_{\alpha_n})-f_{\alpha_n}(\zeta_{\alpha_n})|-|f_{\alpha_n}(\eta_{\alpha_n})-h(\eta_{\alpha_n})|-|h(\zeta_{\alpha_n})-f_{\alpha_n}(\zeta_{\alpha_n})|\\ &> 1-\frac{1}{4}-\frac{1}{4}=\frac{1}{2},\end{aligned}$$

which is a contradiction. Thus $\mathrm{cf}(\kappa)\leq\omega$.

Conversely, if $\mathrm{cf}(\kappa)\leq\omega$, then κ is a hemicompact locally compact space by Lemma 2.2. Hence $C_k(\kappa)$ is complete metrizable by Corollary 5.2.2 of [14]. □

3. Final remarks

We conclude the paper with a number of facts concerning further topological properties of function spaces over ordinals. We assume that some of these results might be well-known to the specialists, but we were unable to locate them in literature. Recall from [10] that a topological space is a *σ-space* if X is regular and has a σ-locally finite network. In what follows κ is an infinite ordinal.

• *The space $C_p(\kappa)$ is paracompact if and only if it is normal if and only if $\kappa\leq\omega_1$.* Indeed, if $C_p(\kappa)$ is paracompact, then it is Lindelöf and hence normal, see [18, S.219]. If $\kappa>\omega_1$, then $[0,\kappa)$ contains the open compact subspace $K=[0,\omega_1]$. So the space $C_p(K)$ is also normal by [18, S.291]. However, $C_p(K)$ is not normal by Theorem 1.1 of [2]. So $\kappa\leq\omega_1$. Finally, if $\kappa\leq\omega_1$, then $C_p(\kappa)$ is Lindelöf by [18, S.316].

• *The space $C_p(\kappa)$ is a σ-space if and only if κ is countable if and only if $C_p(\kappa)$ is a metrizable space.* Indeed, if $C_p(\kappa)$ is a σ-space. Then $C_p(\kappa)$ has countable pseudocharacter, see [10]. So κ is countable by Corollary 4.3.3 of [14] and Lemma 2.2. The converse is clear.

• *The space $C_p(\kappa)$ has countable tightness if and only if* $\mathrm{cf}(\kappa)\leq\omega$. Indeed, if $C_p(\kappa)$ has countable tightness, then by the Arhangel'skii–Pytkeev theorem [1, II.1.1], the space $[0,\kappa)$ is Lindelöf, so $\mathrm{cf}(\kappa)\leq\omega$ by Lemma 2.2. The converse follows from Theorem 1.2.

• *The space $C_k(\kappa)$ has countable tightness if and only if it is a σ-space if and only if* $\mathrm{cf}(\kappa)\leq\omega$. Indeed, by [14, 4.7.2], $C_k(\kappa)$ has countable tightness if and only if every open k-cover of $[0,\kappa)$ has a countable k-subcover (applying Lemma 2.2) if and only if $\mathrm{cf}(\kappa)\leq\omega$. Now, if $C_k(\kappa)$ is a σ-space, then $C_k(\kappa)$ has countable pseudocharacter by [10]. Then by Corollary 4.3.2 of [14], $C_k(\kappa)$ has a dense σ-compact subspace. Applying Lemma 2.2 we obtain $\mathrm{cf}(\kappa)\leq\omega$. If $\mathrm{cf}(\kappa)\leq\omega$, then $C_k(\kappa)$ is complete metrizable, see Corollary 5.2.2 of [14].

We proceed with some properties of $C_p(\kappa)$ and $C_k(\kappa)$ related to their structures of locally convex spaces, for relevant definitions see [12].

• *The space $C_p(\kappa)$ is always quasibarrelled* because $C_p(X)$ is quasibarrelled for every Tychonoff space X, see [12, 11.7.3].

• *The space $C_p(\kappa)$ is barrelled if and only if $\kappa=\omega$, so $C_p(\kappa)=\mathbb{R}^\omega$,* by the Nachbin–Shirota theorem [12, 11.7.6].

• *The space $C_p(\kappa)$ is bornological if and only if* $\mathrm{cf}(\kappa)\leq\omega$, by the Buchwalter–Schmets theorem [4] (which states that $C_p(X)$ is bornological if and only if X is realcompact) and Lemma 2.2.

The next fact extends the remark of Morris and Wulbert stating that $C_k(\omega_1)$ is not barrelled, see [15].

• *The space $C_k(\kappa)$ is quasibarrelled if and only if* $\mathrm{cf}(\kappa)\leq\omega$ *if and only if $C_k(\kappa)$ is complete metrizable.* Indeed, if $C_k(\kappa)$ is quasibarrelled, then it is complete by [14, 5.1.2] (recall that κ is locally compact and hence is a k-space). So $C_k(\kappa)$ is barrelled by [12, 11.2.4]. Assuming that

$\mathrm{cf}(\kappa) > \omega$ we would obtain that $[0, \kappa)$ is pseudocompact non-compact by Lemma 2.2. So $C_k(\kappa)$ is not barrelled by the Nachbin–Shirota theorem [12, 11.7.5], a contradiction. Thus $\mathrm{cf}(\kappa) \leq \omega$.

Notice also the following fact:

• *The space $C_p(\kappa)$ is Čech-complete if and only if it is a Baire space if and only if $\kappa = \omega$, and hence $C_p(\kappa) = \mathbb{R}^\omega$.* This follows from Corollary I.3.3 and Theorem I.3.4 of [1].

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