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A VARIATIONAL APPROACH TO BIFURCATION POINTS OF A REACTION-DIFFUSION SYSTEM WITH OBSTACLES AND NEUMANN BOUNDARY CONDITIONS

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Abstract. Given a reaction-diffusion system which exhibits Turing's diffusion-driven instability, we study the influence of unilateral obstacles of opposite sign on bifurcation and critical points. The approach is based on a variational approach to a non-variational problem which even after transformation to a variational problem has an unusual structure for which usual variational methods do not apply.

1. Introduction

Let us consider a system

$$\begin{aligned} u_t &= d_1\Delta u + f_1(u,v) \\ v_t &= d_2\Delta v + f_2(u,v) \end{aligned} \quad \text{in } (0,\infty)\times\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with a Lipschitzian boundary $\partial\Omega$ and f_i are differentiable functions, $f_i(0,0) = 0$. We are interested in existence and displacement of bifurcation points of nontrivial stationary solutions of the system (1.1) with Neumann boundary conditions and some unilateral obstacles for v. An example are boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,\ \dfrac{\partial v}{\partial n} = 0 \text{ on } \partial\Gamma_N, \\ \pm v \geq 0,\ \pm\dfrac{\partial v}{\partial n} \geq 0,\ v\cdot\dfrac{\partial v}{\partial n} = 0 \text{ on } \Gamma_\pm, \end{cases} \tag{1.2}$$

where Γ_+, Γ_-, Γ_N be pairwise disjoint subsets of $\partial\Omega$,

$$\operatorname{mes}\Gamma_+, \operatorname{mes}\Gamma_- > 0, \quad \operatorname{mes}(\partial\Omega\setminus(\Gamma_-\cup\Gamma_+\cup\Gamma_N)) = 0 \tag{1.3}$$

(the $(d-1)$-dimensional Lebesgue measure). Clearly, $(0,0)$ is a solution of (1.1) with pure Neumann boundary conditions

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{1.4}$$

as well as with (1.2), and also with the other unilateral obstacles we will consider. We will use a certain non-direct variational approach, which will force us to deal in fact with a particular case

$$\begin{aligned} d_1\Delta u + b_{11}u + b_{12}v &= 0 \\ d_2\Delta v + b_{21}u + b_{22}v + n(v) &= 0 \end{aligned} \quad \text{in } \Omega \tag{1.5}$$

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where

$$n(0) = n'(0) = 0, \tag{1.6}$$

or even with the linearized stationary system

$$\begin{aligned} d_1\Delta u + b_{11}u + b_{12}v &= 0 \\ d_2\Delta v + b_{21}u + b_{22}v &= 0. \end{aligned} \quad \text{in } \Omega \tag{1.7}$$

However, in order to explain the meaning of our results, let us recall some facts concerning the general reaction-diffusion system (1.1) and its relations to (1.7) and (1.5). We will denote $b_{ij} = \frac{\partial f_i}{\partial u_j}(0,0)$ and assume that

$$\begin{gathered} b_{11} + b_{22} < 0, \quad \det B := b_{11}b_{22} - b_{12}b_{21} > 0, \\ b_{11} > 0 > b_{22}, \quad b_{12}b_{21} < 0. \end{gathered} \tag{1.8}$$

The first line in (1.8) guarantees that the equilibrium $(0,0)$ is asymptotically stable as a solution of the corresponding system of ODE's without any diffusion ($d_1 = d_2 = 0$). If also the second line is fulfilled then $(0,0)$ as a solution of the whole system (1.1) with Neumann conditions (1.4) is linearly stable only for values (d_1, d_2) from a certain open domain $D_S \subseteq \mathbb{R}^2_+$, but linearly unstable for (d_1, d_2) from the interior of the complement $D_U := \mathbb{R}^2_+ \setminus \overline{D}_S$. For (d_1^0, d_2^0) from the boundary C_E between D_S and D_U it usually happens that there is a bifurcation of spatially non-constant stationary solutions, that is, each neighborhood of $(d_1^0, d_2^0, 0, 0)$ in $\mathbb{R}^2 \times W^{1,2}(\Omega)$ contains stationary solutions (d_1, d_2, u, v) of (1.1), (1.4) with spatially non-constant (u, v), see e.g. [17], [19]. Such solutions can describe Turing's spatial patterns having interpretation in biology, see e.g. [4], [12], [18].

Let us note that standard linearization and compactness arguments imply that such a bifurcation point $(d_1, d_2) = (d_1^0, d_2^0)$ must necessarily be a critical point of (1.1), (1.4), that is, the system (1.7), (1.4) has a nontrivial solution (u, v) which in view of $\det B \neq 0$ is necessarily spatially non-constant.

If the system under consideration describes a chemical reaction, then the second line in (1.8) means that our system is of an activator-inhibitor type (the case $b_{12} < 0 < b_{21}$) or of a positive feedback (substrate-depletion) type, respectively. See e.g. [4], [12], [18]. In the first case, u and v are related to the concentration of the activator and inhibitor, respectively. In fact, in applications u and v typically describe the *difference* of the concentration of some chemicals to some spatially constant equilibrium $(\bar{u}, \bar{v})$ so that, after a variable substitution in an original model, it is no loss of generality to assume $(\bar{u}, \bar{v}) = (0,0)$, and also negative values of u and v have a natural physical interpretation (they correspond to concentrations under the equilibrium threshold).

The domain of stability D_S contains, in particular, all points $(d_1, d_2) \in \mathbb{R}^2_+$ with $d_1 > b_{11}/\kappa_1$ where κ_1 is the first positive eigenvalue of $-\Delta$ with Neumann boundary conditions (1.4) so that bifurcations of stationary solutions to (1.1), (1.4) do not occur with $d_1 > b_{11}/\kappa_1$.

An influence of unilateral obstacles to the bifurcation of spatially non-constant stationary solutions of systems (1.1) was studied already in the past, but usually for the case that also a Dirichlet condition is imposed in some part of the boundary (e.g. [2], [5], [8], [9], [14], [20], [22], [23]). It was shown that if a unilateral condition is prescribed for v, then there are bifurcation points also in D_S. However, also in all these results a bifurcation in fact cannot occur if $d_1 > b_{11}/\kappa_1$.

A surprisingly different situation occurs if no Dirichlet boundary data are prescribed and if unilateral conditions of only one sign are imposeded for v, e.g. unilateral boundary conditions (1.2) are considered and one of the two sets Γ_+ or Γ_- is empty. It has been shown in [16] that in this case for every sufficiently large d_1, in particular for some $d_1 > b_{11}/\kappa_1$ (in dimension $d = 1$ even for *every* $d_1 > 0$, see [10]), there is some $d_2 > 0$ such that there is a bifurcation of stationary spatially non-constant solutions of (1.1) with unilateral obstacles at (d_1, d_2). In fact, there are bifurcation points with d_1/d_2 arbitrarily small. By standard arguments (see e.g. [16]) one obtains again that each bifurcation point $(d_1, d_2) \in \mathbb{R}^2_+$ of (1.1) with unilateral conditions (e.g. with (1.2)) is necessarily a critical point, that is, (1.7) with unilateral conditions has a nontrivial solution.

However, the methods used in the cited papers [10], [16] break down if unilateral conditions of opposite sign are given on different parts of the boundary or of the interior, that is, if simultaneously there are unilateral sources and sinks for v, e.g. if both of the sets Γ_+ and Γ_- are non-empty in (1.2). In the current paper, we will show that in this case there might be bifurcation (hence critical) points $(d_1, d_2) \in \mathbb{R}^2_+$ with $d_1 > b_{11}/\kappa_1$ of (1.5) with unilateral obstacles, but it might also happen that there are no such critical points, that is, that (1.7) with unilateral obstacles has only the trivial solution $(u, v) = (0, 0)$ in $W^{1,2}$ for all $d_2 > 0$, $d_1 > b_{11}/\kappa_1$. In fact, using a variational approach, we will be able to give a *necessary and sufficient* criterion for the existence of such critical points. This criterion will relate in a rather implicit manner the geometry and location of the unilateral obstacles with the values of the Jacobi matrix $B := (b_{ij}) = (D_j f_i(0, 0))$.

We emphasize that, although (1.7) is linear, unilateral obstacles are of an inherently nonlinear nature so that one cannot expect to use any tools from linear theory or from linearization methods. We use variational methods in spite of that the matrix B is non-symmetric because of (1.8), and thus the original problem has no potential. We apply a modification of a trick which was used in a primitive form already in [13], [14], and then for more detailed study of systems with unilateral conditions in [1]. We will work with a weak formulation written as a system of an operator equation and a variational inequality in $W^{1,2}(\Omega)$, we fix an arbitrary $d_1 = d_1^0$ and consider only d_2 as a parameter. Expressing u from the equation and substituting it into the inequality, we get a single variational inequality for v with a potential operator and a parameter d_2. By a variational approach we obtain the maximal bifurcation point d_2^0 of this variational inequality, which is simultaneously maximal eigenvalue of the inequality with the linearized operator, and consequently $[d_1^0, d_2^0]$ is a critical and simultaneously bifurcation point of the system (1.5) with unilateral conditions. However, in the lack of a Dirichlet condition considered in [13], [14] and [1], this inequality has a structure for which "standard" variational methods for inequalities do not apply, and therefore the situation is more complicated.

Unfortunately, analogously as in [1], the approach mentioned cannot be used for the proof of bifurcation in the case when a nonlinearity appears also in the first equation or if n in the second equation of (1.5) depends also on u. In these cases, even if it were possible to express u from the first equation, the potentiality of the operator obtained would not be clear. So, in general the question if the critical point obtained by our procedure is simultaneously a bifurcation point of the full system (1.1) with both nonlinear f_1 and f_2 remains open. However, in some particular situations it is known that an eigenvalue of a variational inequality is also a bifurcation point (see [7], [21]) and that a critical point of a unilateral problem for (1.7) is also a bifurcation point of the unilateral problem for (1.1), see [15]. (Sometimes it is also possible to determine the direction of the bifurcation branch, see [6]). In concrete examples discussed in all these papers, a Dirichlet boundary condition on a part of the boundary is considered, which simplifies the situation. However, it seems that also in our case of purely Neumann conditions, the results of the current paper give in fact an information about bifurcations for the general system (1.1), at least for non-local (integral) unilateral conditions as in [15] or for the one-dimensional case as in Example 3.1 below.

2. Abstract Formulation

Let us assume that n is a continuous function satisfying (1.6) and that there exists $c \in \mathbb{R}$ such that

$$|n(u)| \leq c(1+|u|)^{q-1} \tag{2.1}$$

with some $q > 2$ or $2 < q < \dfrac{2d}{d-2}$ in the case $d \leq 2$ or $d > 2$, respectively (in the case $d = 1$, we do not need the hypothesis (2.1) and put $q = \infty$ in the following). We equip the (real) Hilbert space $\mathbb{H} = W^{1,2}(\Omega)$ with the usual scalar product

$$\langle u, \varphi \rangle = \int_\Omega \big(\nabla u(x) \cdot \nabla \varphi(x) + u(x)\varphi(x)\big)\, dx \qquad \text{for all } u, \varphi \in \mathbb{H}, \tag{2.2}$$

and the corresponding norm $\|\varphi\|^2 = \langle \varphi, \varphi \rangle$ and define operators $A, N \colon \mathbb{H} \to \mathbb{H}$ by

$$\langle Au, \varphi \rangle = \int_\Omega u(x)\varphi(x)\, dx \quad \text{for all } u, \varphi \in \mathbb{H}, \tag{2.3}$$

$$\langle N(u), \varphi \rangle = \int_\Omega n(u(x))\varphi(x)\, dx \quad \text{for all } u, \varphi \in \mathbb{H}. \tag{2.4}$$

It follows from the compactness of the embedding $\mathbb{H} \hookrightarrow\hookrightarrow L^q(\Omega)$ and the continuity of the Nemyckij operator of $L^q(\Omega)$ into $L^{q^*}(\Omega)$, $\frac{1}{q} + \frac{1}{q^*} = 1$ (see e.g. [11]) that under the assumption (2.1)

$$A \text{ is linear, symmetric, positive and compact with the largest simple eigenvalue } 1, \tag{2.5}$$

$$N \text{ is nonlinear, continuous and compact.} \tag{2.6}$$

Furthermore, under the conditions (1.6), (2.1)

$$N \text{ is Fréchet differentiable at } 0,\ N(0) = 0,\ N'(0) = 0, \tag{2.7}$$

see e.g. [3]. Moreover, let us introduce the functional $G_N \colon \mathbb{H} \to \mathbb{R}$ by

$$G_N(u) = \int_\Omega \int_0^{u(x)} n(s)\, ds\, dx.$$

Under the assumptions (2.1), this functional is well defined, Fréchet differentiable, and we have

$$G_N'(u) = N(u), \tag{2.8}$$

i.e. G_N is a potential of the operator N.

It is natural to define (weak) solutions of (1.7), (1.4) or (1.5), (1.4) as pairs (u, v) satisfying

$$\begin{cases} u, v \in \mathbb{H}, \\ d_1 u - d_1 Au - b_{11} Au - b_{12} Av = 0, \\ d_2 v - d_2 Av - b_{21} Au - b_{22} Av = 0 \end{cases} \tag{2.9}$$

or

$$\begin{cases} u, v \in \mathbb{H}, \\ d_1 u - d_1 Au - b_{11} Au - b_{12} Av = 0, \\ d_2 v - d_2 Av - b_{21} Au - b_{22} Av - N(v) = 0, \end{cases} \tag{2.10}$$

respectively. In order to treat the unilateral conditions (1.2), we define the cone

$$K := \{ v \in \mathbb{H} : v|_{\Gamma_+} \geq 0 \text{ and } v|_{\Gamma_-} \leq 0 \}, \tag{2.11}$$

where the inequalities are understood in the sense of traces. We define correspondingly solutions of the problems (1.7), (1.2) or (1.5), (1.2), as couples (u, v) satisfying the variational inequalities

$$\begin{cases} u \in \mathbb{H},\ v \in K, \\ d_1 u - d_1 Au - b_{11}Au - b_{12}Av = 0, \\ \langle d_2 v - d_2 Av - b_{21}Au - b_{22}Av, \varphi - v\rangle \geq 0 \text{ for all } \varphi \in K \end{cases} \tag{2.12}$$

or

$$\begin{cases} u \in \mathbb{H},\ v \in K, \\ d_1 u - d_1 Au - b_{11}Au - b_{12}Av = 0, \\ \langle d_2 v - d_2 Av - b_{21}Au - b_{22}Av - N(v), \varphi - v\rangle \geq 0 \text{ for all } \varphi \in K, \end{cases} \tag{2.13}$$

respectively. We will actually obtain bifurcation of (2.13) "with fixed d_1" in the following sense:

Definition 2.1. A parameter d_2 is a *bifurcation point* of (2.13) *with fixed* d_1 if in any neighborhood of $(d_2, 0, 0)$ in $\mathbb{R} \times \mathbb{H} \times \mathbb{H}$ there is $(\widetilde{d}_2, u, v)$ with $(u, v) \neq (0, 0)$ such that $(d_1, \widetilde{d}_2, u, v)$ satisfies (2.13). We call d_2 a *critical point* of (2.12) (with fixed d_1) if (2.12) has a solution $(u, v) \neq (0, 0)$.

Remark 2.1. Every bifurcation point (with fixed d_1) is a critical point, see e.g. [2].

Notation 2.1. Let us denote by $0 = \kappa_0 < \kappa_1 \leq \kappa_2 \leq \cdots$ the eigenvalues of $-\Delta$ with Neumann boundary conditions, counted according to multiplicity, and let e_k $(k = 0, 1, \dots)$ be a corresponding orthonormal system of eigenvectors in $\mathbb{H}$. With each κ_k $(k = 1, 2, \dots)$, we associate the hyperbola segments

$$C_k := \Big\{d = (d_1, d_2) \in \mathbb{R}^2_+ : d_2 = \frac{b_{12}b_{21}/\kappa_k^2}{d_1 - b_{11}/\kappa_k} + \frac{b_{22}}{\kappa_k}\Big\}.$$

We denote by C_E the envelope of C_k $(k = 1, 2, \dots)$ and define the domain of stability

$$D_S := \{d \in \mathbb{R}^2_+ : d \text{ lies to the right from } C_E, \text{ i.e. from all } C_k,\ k = 1, 2, \dots\}$$

and the domain of instability

$$D_U := \{d \in \mathbb{R}^2_+ : d \text{ lies to the left from } C_E, \text{ i.e. from at least one } C_k\}$$

(see Figure 1).

For any $j = 1, 2, \dots,$ we will denote by $a_k := \frac{b_{11}}{\kappa_k}$ the d_1-coordinate of the vertical asymptote of C_k.

Remark 2.2. The above definition of the domain D_S (and D_U) of stability and instability indeed corresponds to the domains for which $(0, 0)$ is a linearly stable (or unstable) solution of (1.1), (1.4). Actually, for $(d_1, d_2) \in D_S$, the solution $(0, 0)$ of (1.1), (1.4) is even exponentially asymptotically stable in $\mathbb{H} \times \mathbb{H}$, see e.g. [24].

Remark 2.3. The eigenvalues of the operator A from (2.3), are of the form $\lambda_k = 1/(1+\kappa_k)$ for $j = 0, 1, \dots,$ and the corresponding eigenspaces are the eigenspaces of $-\Delta$ with Neumann boundary conditions to the eigenvalues κ_k.

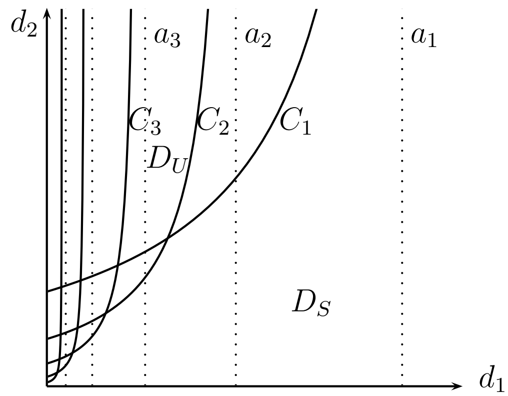

FIGURE 1. The system of hyperbolas C_k, their asymptotes a_k, domains of stability D_S (to the right from the envelope C_E) and instability D_U (to the left from C_E).

3. Main Result

In this section we will consider a general Hilbert space $\mathbb{H}$ with the scalar product $\langle \cdot, \cdot \rangle$ and a closed convex cone K with its vertex at the origin in $\mathbb{H}$. We will discuss the variational inequalities (2.12) and (2.13) with general operators $A, N \colon \mathbb{H} \to \mathbb{H}$ satisfying (2.5), (2.6), (2.7) with N having a potential G_N, i.e. (2.8) holds. The condition (1.8) will be always assumed.

Let $1 = \lambda_0 > \lambda_1 \geq \ldots > 0$ be the eigenvalues of A, counted according to multiplicity, and let $e_0, e_1, \ldots$ be a corresponding orthonormal system of eigenfunctions. In accordance with Remark 2.3, we use the notation $\kappa_k := \lambda_k^{-1} - 1$. For d_1 from

$$D_1 := \{d_1 > 0 \mid d_1 \neq a_k = b_{11}/\kappa_k \text{ for all } k = 1, 2, \ldots\}$$

let us define the auxiliary functions

$$c_k(d_1) := \frac{1}{1+\kappa_k}\left(\frac{b_{12}b_{21}}{\kappa_k d_1 - b_{11}} + b_{22}\right) = \frac{b_{22}\kappa_k d_1 - \det B}{(1+\kappa_k)(\kappa_k d_1 - b_{11})}.$$

Remark 3.1. Clearly, $c_0(d_1) = \det B/b_{11} > 0$ is actually independent of d_1, $c_k(d_1) < 0$ if $d_1 > a_k$, and $c_k(d_1) > 0$ if $d_1 < a_k$.

Theorem 3.1. *Let $e_0 \notin K \cup (-K)$, and $d_1 \in D_1$. Then* (2.12) *has a critical point $d_2 > 0$ if and only if*

$$\text{there is } v \in K \text{ with } \sum_{k=0}^{\infty} c_k(d_1)\,|\langle v, e_k\rangle|^2 > 0. \tag{3.1}$$

Moreover, in this case

$$d_2^{\max} := \max_{v\in K\setminus\{0\}} \frac{\sum_{k=0}^{\infty} c_k(d_1)\,|\langle v, e_k\rangle|^2}{\langle (I-A)v, v\rangle} = \max_{\substack{v\in K\\ \langle (I-A)v,v\rangle = 1}} \sum_{k=0}^{\infty} c_k(d_1)\,|\langle v, e_k\rangle|^2 \in (0,\infty)$$

is the maximal critical point of (2.12) *and simultaneously the maximal bifurcation point of* (2.13) *with fixed d_1.*

We will see from the considerations in Section 4 by setting $N = 0$ that for $d_2 = d_2^{\max}$ and a corresponding maximizer v there is a uniquely determined u such that (u, v) is a nontrivial solution of (2.12), and conversely, for $d_2 = d_2^{\max}$ all solutions of (2.12) are of this form (up to a scalar multiple).

Let us note that $d_2^{\max}$ is in fact $\max \langle Sv, v\rangle$ over all $v \in K$ with $\langle (I-A)v, v\rangle = 1$ where S is a symmetric operator which we will use to reduce our problem to a variational setting (see Lemma 4.1).

We postpone the proof of Theorem 3.1 to Section 4.

Proposition 3.1. *For $d_1 > a_1 = b_{11}/\kappa_1$ the condition (3.1) is satisfied if and only if there is $u \in (K+e_0) \cup (K-e_0)$ with $\langle u, e_0\rangle = 0$ such that*

$$\sum_{k=1}^{\infty} |c_k(d_1)|\, |\langle u, e_k\rangle|^2 < c_0(d_1) = \frac{\det B}{b_{11}}. \tag{3.2}$$

Proof. By Remark 3.1 we have $c_k(d_1) < 0 < c_0(d_1)$ for all $k \neq 0$, and so if v is as in (3.1), we must necessarily have $\langle v, e_0\rangle \neq 0$. By scaling, we can assume without loss of generality that $|\langle v, e_0\rangle| = 1$. Hence, $v = u + e_0$ or $v = u - e_0$ with a uniquely determined $u \in \mathbb{H}$ satisfying $\langle u, e_0\rangle = 0$. Now, (3.2) follows from (3.1). Conversely, if u is as in the assumptions of the opposite implication, then $v = u + e_0$ or $v = u - e_0$ satisfies the inequality in (3.1). □

Due to Theorem 3.1, we are only interested in the case $e_0 \notin K \cup (-K)$. In this case, we cannot choose $u = 0$ in Proposition 3.1, and it is a question of the interplay of the geometry of K and of the matrix $B = (b_{ij})$ (which determines the values $c_j(d_1)$) whether a parameter $d_1 > b_{11}/\kappa_1$ satisfies (3.1). If one is only interested in the existence of *some* (large) d_1 satisfying (3.1) (i.e. in the existence of some critical point or bifurcation point in D_S), the following result gives an exhaustive answer.

Proposition 3.2. *The following assertions are equivalent:*

(1) *There is $d_1 > b_{11}/\kappa_1$ satisfying (3.1).*
(2) *Every sufficiently large d_1 satisfies (3.1).*
(3) *There is $u \in (K+e_0) \cup (K-e_0)$ with $\langle u, e_0\rangle = 0$ and*

$$\sum_{k=1}^{\infty} \frac{1}{1+\kappa_k} |\langle u, e_k\rangle|^2 < \frac{b_{12}b_{21}}{b_{11}b_{22}} - 1. \tag{3.3}$$

Here, (3.3) can equivalently be replaced by

$$\langle Au, u\rangle < \frac{b_{12}b_{21}}{b_{11}b_{22}} - 1. \tag{3.4}$$

The conditions (1)–(3) are satisfied, in particular, if there is $u \in (K+e_0) \cup (K-e_0)$ with $\langle u, e_0\rangle = 0$ and

$$\frac{1}{1+\kappa_1} \|u\|^2 < \frac{b_{12}b_{21}}{b_{11}b_{22}} - 1. \tag{3.5}$$

Proof. Clearly, (2) implies (1). If (1) holds, that is, if there is $d_1 > b_{11}/\kappa_1$ satisfying (3.1), let u be the function from Proposition 3.1. Inserting the estimate $|c_k(d_1)| > -b_{22}/(1+\kappa_k)$ for $k = 1, 2, \ldots$ into (3.2), we obtain (3.3), and thus (3) holds. Conversely, if u satisfies (3.3) then u satisfies also (3.2) for all large d_1, because the difference

$$\left| |c_k(d_1)| - \frac{-b_{22}}{1+\kappa_k} \right| < \left| \frac{b_{12}b_{21}}{\kappa_1 d_1 - b_{11}} \right|$$

tends to zero as $d_1 \to \infty$, uniformly in $k = 1, 2, \ldots$. Hence, we have shown that (3) implies (2). The equivalence of (3.3) and (3.4) follows from the equality

$$\langle Au, u\rangle = \sum_{k=0}^{\infty} \lambda_k |\langle u, e_k\rangle|^2 = |\langle u, e_0\rangle|^2 + \sum_{k=1}^{\infty} \frac{1}{1+\kappa_k} |\langle u, e_k\rangle|^2. \tag{3.6}$$

Indeed, since (e_k) forms a complete orthonormal basis, we can write every $u \in \mathbb{H}$ as a series $u = \sum_{k=0}^{\infty} \mu_k e_k$ with $\mu_k = \langle u, e_k \rangle$; using that $Ae_k = \lambda_k e_k$, we obtain (3.6).

For the last assertion, we apply Bessel's inequality and the estimate $\kappa_k \geq \kappa_1$ in (3.5) and obtain (3.3). □

Since the right-hand side of (3.3) and (3.4) can become arbitrarily large or arbitrarily small for an appropriate choice of $B = (b_{ij})$, we find for every cone K with $e_0 \notin K \cup (-K)$ a matrix B such that the equivalent conditions from Proposition 3.2 are satisfied, and typically one will also find another matrix B such that they are not satisfied.

As an example, we consider the problem of Section 1. Note that in this example e_0 is the eigenfunction of $-\Delta$ to the eigenvalue $\kappa_0 = 0$, that is, constant. Hence, the hypothesis $e_0 \notin K \cup (-K)$ of Theorem 3.1 holds in view of (1.3).

If one is actually interested to estimate the quantity $d_2^{\max}$ from Theorem 3.1, it is worth to note that in the situation of the operator and the cone from (2.3) and (2.11) we have

$$\langle (I - A)v, v \rangle = \int_\Omega |\nabla v|^2 \, dx. \tag{3.7}$$

Since the conditions in Theorem 3.1 and Propositions 3.1 and 3.2 are not only sufficient for the conclusions but also necessary, it can happen that for all $d_1 > b_{11}/\kappa_1$ there are no critical points of (2.12). However, for every location of $\Gamma_\pm$ there *are* matrices B such that Theorem 3.1 applies with every large d_1. The conditions are cumbersome but can be verified numerically in particular situations.

Example 3.1. Let us consider the problem (1.5), (1.2) in the 1-dimensional case $d = 1$, $\Omega = (-\ell, \ell)$ with some $\ell > 0$, and with $\Gamma_\pm = \{\pm \ell\}$, that means in fact (2.13) with A, N and K from (2.3), (2.4) and (2.11). In this case $\kappa_k = (k\pi/(2\ell))^2$, and we can assume $e_0(x) \equiv (2\ell)^{-1/2}$. Choosing the particular function $u(x) = (2\ell)^{-1/2} \sin \frac{\pi x}{2\ell}$, we find $u \in K - e_0$, $\langle u, e_0 \rangle = 0$, and $\|u\|^2 = \frac{\pi^2 + 4\ell^2}{8\ell^2}$. Using the last assertion of Proposition 3.2, we thus obtain that if

$$\frac{1}{1 + \kappa_1} \|u\|^2 = \frac{1}{2} < \frac{b_{12} b_{21}}{b_{11} b_{22}} - 1, \tag{3.8}$$

then all sufficiently large d_1 satisfy (3.1). In particular, Theorem 3.1 then implies the existence of a bifurcation point $d_2 > 0$ of (1.5), (1.2) with fixed d_1 if $d_1 > 0$ is large enough.

Example 3.2. Let $d = 2$, $\Omega = (-\ell, \ell) \times (0, h)$ with some $\ell, h > 0$, and $\Gamma_\pm = \{\pm \ell\} \times [0, h]$. In this case we can assume $e_0(x) \equiv (2\ell)^{-1/2}$. Choosing the particular function $u(x, y) = (2\ell)^{-1/2} \sin \frac{\pi x}{2\ell}$, we find $u \in K - e_0$, $\langle u, e_0 \rangle = 0$, and $\|u\|^2 = \frac{\pi^2 + 4\ell^2}{8\ell^2} h$.

In case $2\ell \geq h$, we have $\kappa_1 = \frac{\pi^2}{4\ell^2}$ and thus obtain by using Proposition 3.2 that if

$$\frac{1}{1 + \kappa_1} \|u\|^2 = \frac{h}{2} < \frac{b_{12} b_{21}}{b_{11} b_{22}} - 1, \tag{3.9}$$

then all sufficiently large d_1 satisfy (3.1). In view of (1.8), (3.9) holds whenever $h > 0$ is sufficiently small.

In case $2\ell \leq h$, we have $\kappa_1 = \frac{\pi^2}{h^2}$ and thus we must replace (3.9) by

$$\frac{(\pi^2 + 4\ell^2) h^3}{(h^2 + \pi^2) 8\ell^2} < \frac{b_{12} b_{21}}{b_{11} b_{22}} - 1. \tag{3.10}$$

Example 3.3. Let $\Omega = (-\ell, \ell) \times (0, h)$ with $\ell, h > 0$ again, but let us consider now unilateral obstacles describing sources in the interior of Ω. Let $\Omega_\pm \subseteq \Omega$ be measurable subsets such that $\operatorname{mes}(\Omega_\pm) > 0$ (the d-dimensional Lebesgue measure), $\overline{\Omega}_+ \cap \overline{\Omega}_- = \varnothing$ and (for simplicity) $\overline{\Omega}_\pm \cap \partial\Omega = \varnothing$. We consider now the problem

$$\begin{aligned} & d_1\Delta u + b_{11}u + b_{12}v = 0 && \text{in } \Omega, \\ & d_2\Delta v + b_{21}u + b_{22}v + n(v) = 0 && \text{in } \Omega \setminus (\Omega_+ \cup \Omega_-) \\ \pm\big(& d_2\Delta v + b_{21}u + b_{22}v + n(v)\big) \le 0, \quad \pm v \ge 0 && \text{in } \Omega_\pm, \\ \big(& d_2\Delta v + b_{21}u + b_{22}v + n(v)\big)v = 0 && \text{in } \Omega_\pm \end{aligned} \tag{3.11}$$

with Neumann boundary conditions (1.4). The weak formulation is again (2.13) with the same operators as above but with the cone

$$K := \{v \in K : v|_{\Omega_+} \ge 0 \text{ and } v|_{\Omega_-} \le 0\}.$$

Suppose that $\Omega_- \subseteq (-\ell, 0) \times (0, h)$ and $\Omega_+ \subseteq (0, \ell) \times (0, h)$. Using the same functions e_0 and u as in Example 3.2, we have again $u \in K - e_0$. Hence, we get a result analogous to Example 3.2. More precisely, if $2\ell \ge h$, assume that the inequality in (3.9) holds, and if $2\ell \le h$, assume that (3.10) holds. Then (3.1) is fulfilled for all sufficiently large d_1. In particular, Theorem 3.1 implies the existence of a bifurcation point $d_2 > 0$ of (3.11), (1.4) with fixed d_1, if $d_1 > 0$ is large enough.

Of course, Example 3.1 can be also modified in a similar way for $\Omega_- \subseteq (-\ell, 0)$ and $\Omega_+ \subseteq (0, \ell)$ (or vice versa).

4. Proof of Theorem 3.1

Let $\sigma(A)$ denote the spectrum of A. Since A is compact this means that $\sigma(A)$ consists of all eigenvalues of A and of the value 0. For fixed $d_1 \in D_1$, we define the auxiliary function $f\colon \sigma(A) \to \mathbb{R}$ by

$$f(\lambda) := \frac{b_{12}b_{21}\lambda^2}{d_1 - \big(b_{11} + d_1\big)\lambda} + b_{22}\lambda.$$

Note that we have

$$f(0) = 0 \quad \text{and} \quad f(\lambda_k) = c_k(d_1) \quad \text{for } k = 0, 1, \dots \tag{4.1}$$

Since A is a symmetric (compact) operator in $\mathbb{H}$, we can define a selfadjoint operator $S := f(A)$ in the usual way by means of spectral calculus of symmetric operators.

Lemma 4.1. *For $d_1 \in D_1$ and $S = f(A)$ the variational inequality* (2.13) *is equivalent to*

$$v \in K,\ \langle d_2(I - A)v - Sv - N(v), \varphi - v\rangle \ge 0 \text{ for all } \varphi \in K, \tag{4.2}$$

$$u = (d_1(I - A) - b_{11}A)^{-1} b_{12}Av. \tag{4.3}$$

Similarly, (2.12) *is equivalent to* (4.3) *and*

$$v \in K,\ \langle d_2(I - A)v - Sv, \varphi - v\rangle \ge 0 \text{ for all } \varphi \in K. \tag{4.4}$$

Proof. The hypothesis $d_1 \in D_1$ means that the operator $d_1(I - A) - b_{11}A$ is invertible, and so for every $v \in \mathbb{H}$ the first equation of (2.13) has a unique solution given by (4.3). Inserting this formula into the inequality in (2.13), we find the assertion. □

For the rest of this section, we keep $d_1 \in D_1$ fixed and put $S = f(A)$ as above.

Lemma 4.2. *For every $v \in \mathbb{H}$ we have*

$$\langle Sv, v\rangle = \sum_{k=0}^{\infty} c_k(d_1)\, |\langle v, e_k\rangle|^2.$$

Proof. Since (e_k) form a complete orthonormal system, we can write the Fourier expansion $v = \sum_{k=0}^{\infty} \mu_k e_k$ with $\mu_k := \langle v, e_k \rangle$. The spectral calculus implies

$$\langle Sv, v \rangle = \sum_{k=0}^{\infty} \langle f(\lambda_k) \mu_k e_k, v \rangle = \sum_{k=0}^{\infty} f(\lambda_k) \left| \mu_k \right|^2 ,$$

so that the assertion follows from (4.1). □

Lemma 4.3. *If $e_0 \notin K \cup (-K)$ then there is some $c > 0$ with*

$$c \left\| v \right\|^2 \leq \langle (I - A)v, v \rangle \leq \left\| v \right\|^2 \quad \textit{for all } v \in K, \tag{4.5}$$

and

$$\sup_{v \in K \setminus \{0\}} \frac{\langle Sv, v \rangle}{\langle (I - A)v, v \rangle} = \sup_{v \in K \setminus \{0\}} \frac{\sum_{k=0}^{\infty} c_k(d_1) \left| \langle v, e_k \rangle \right|^2}{\langle (I - A)v, v \rangle} < \infty. \tag{4.6}$$

Proof. Since $\sigma(A) \subseteq [0, 1]$, we have for all $v \in \mathbb{H}$ with $\|v\| = 1$ that $\langle Av, v \rangle \in [0, 1]$, and so $\langle (I - A)v, v \rangle \in [0, 1]$. Hence, if (4.5) fails there is a sequence $v_n \in K$ with $\|v_n\| = 1$ and $1 - \langle Av_n, v_n \rangle = \langle (I - A)v_n, v_n \rangle \to 0$. Passing to a subsequence if necessary, we can assume $v_n \rightharpoonup v$. Then $Av_n \to Av$ and thus $\langle Av_n, v_n \rangle \to \langle Av, v \rangle$. In particular, $\langle Av, v \rangle = 1$ which implies $\|v\| \geq 1$. From $v_n \rightharpoonup v$ and $\|v_n\| = 1 \leq \|v\|$, we thus obtain by a standard Hilbert space argument that $v_n \to v$. Since $\langle Av, v \rangle = 1$ and $\|v\| = 1$, and since 1 is the largest eigenvalue of A with a simple eigenvector e_0, we obtain $v_n \to v \in \{\pm e_0\}$ which is a contradiction, because K is closed, $v_n \in K$, and $e_0 \notin K \cup (-K)$. Hence, (4.5) is established. The equality (4.6) follows from Lemma 4.2, and the finiteness of (4.6) follows from the boundedness of S and (4.5). □

In the following, we identify $\mathbb{H}$ with its dual by means of the scalar product. In this sense, the derivative of a functional $\Phi \colon \mathbb{H} \to \mathbb{R}$ becomes a function $\Phi' \colon \mathbb{H} \to \mathbb{H}$.

The following proof uses some ideas from [26, Section 64.5]. However, we cannot use the corresponding [26, Theorem 64.4], since the bilinear form $a(u, v) := \langle (I - A)u, v \rangle$ fails to be positive definite on $\mathbb{H}$ in our situation.

Replacing G_N by $G_N - G_N(0)$ if necessary, we assume from now on without loss of generality that $G_N(0) = 0$.

Lemma 4.4. *Let $e_0 \notin K \cup (-K)$, and suppose that the quantity from (4.6) is positive. Then the two suprema in (4.6) are maxima, hence, they are equal to $d_2^{\max}$ from Theorem 3.1. Moreover:*

(1) *For each sufficiently small $r > 0$ the maximum*

$$d_{2,r} := \frac{1}{r^2} \max_{\substack{v \in K \\ \langle (I - A)v, v \rangle = r^2}} \left(\langle Sv, v \rangle + G_N(v) \right) \tag{4.7}$$

exists, and $d_{2,r} \to d_2^{\max} > 0$ as $r \to 0^+$.

(2) *If v_r is a maximizer of (4.7) then there is a unique d_{2,r,v_r} such that*

$$v_r \in K, \quad d_{2,r,v_r} \langle (I - A)v_r, \varphi - v_r \rangle \geq \langle Sv_r + N(v_r), \varphi - v_r \rangle \quad \textit{for all } \varphi \in K, \tag{4.8}$$

and $d_{2,r,v_r} \to d_2^{\max}$ as $r \to 0^+$ (independent of the choice of v_r).

Proof. The set $K_r := \{v \in K : \langle (I - A)v, v \rangle \leq r^2\}$ is convex, closed, and bounded in view of (4.5). The functionals $\Phi_1(v) = \langle Sv, v \rangle$ and $\Phi_2(v) = \Phi_1(v) + G_N(v)$ have compact Fréchet derivatives and thus are weakly sequentially continuous by e.g. [25, Corollary 41.9]. Hence, the two maxima

$$m_{i,r} = \max_{v \in K_r} \Phi_i(v)$$

exist, see e.g. [25, Corollary 38.8 and 38.9]. Let $v_{i,r}$ be a corresponding maximizer. In view of $d_2^{\max} > 0$, it follows that $\Phi_1(v_{1,r}) = \langle Sv_{1,r}, v_{1,r}\rangle > 0$, and thus by homogeneity of Φ_1, we have

$$v_{1,r} \in B_r := \{v \in K : \langle (I-A)v, v\rangle = r^2\}\,.$$

Hence, the maximum of the first term in (4.6) is attained at $v_{1,r}/r$, and $m_{1,r} = r^2 d_2^{\max}$.

Let us prove that $m_{2,r}/r^2 \to d_2^{\max}$ as $r \to 0$. We note first that $N(0) = 0$ and $N'(0) = 0$ imply

$$\lim_{r\to 0}\ \sup_{\|v\|\le r} \frac{\|N(v)\|}{r} = 0,$$

hence, it follows by using (4.5) that

$$\lim_{r\to 0}\ \sup_{v\in K_r} \frac{|\langle N(v), v\rangle|}{r^2} = 0. \tag{4.9}$$

Applying the classical mean value theorem to the function $t \mapsto G_N(tu)$ on $[0,1]$, we obtain in view of $G_N(0) = 0$, $G_N' = N$, and since K_r is convex with $0 \in K_r$, that

$$\lim_{r\to 0}\ \sup_{v\in K_r} \frac{|G_N(v)|}{r^2} = 0. \tag{4.10}$$

The definition of $v_{i,r}$ implies

$$\langle Sv_{1,r}, v_{1,r}\rangle = \Phi_1(v_{1,r}) = m_{1,r} \ge \Phi_1(v_{2,r}) = \langle Sv_{2,r}, v_{2,r}\rangle\,,$$

and

$$\langle Sv_{2,r}, v_{2,r}\rangle + G_N(v_{2,r}) = \Phi_2(v_{2,r}) = m_{2,r} \ge \Phi_2(v_{1,r}) = \langle Sv_{1,r}, v_{1,r}\rangle + G_N(v_{1,r}).$$

Adding the term $G_N(v_{2,r})$ to the first inequality and using the second one we get

$$\langle Sv_{1,r}, v_{1,r}\rangle + G_N(v_{2,r}) \ge \langle Sv_{2,r}, v_{2,r}\rangle + G_N(v_{2,r}) \ge \langle Sv_{1,r}, v_{1,r}\rangle + G_N(v_{1,r}).$$

Since $m_{1,r} = r^2 d_2^{\max}$, we obtain in view of (4.10) that

$$\lim_{r\to 0} \frac{\langle Sv_{2,r}, v_{2,r}\rangle}{r^2} = \lim_{r\to 0} \frac{\langle Sv_{1,r}, v_{1,r}\rangle}{r^2} = \lim_{r\to 0} \frac{m_{1,r}}{r^2} = d_2^{\max}.$$

Using (4.10), we get the assertion $d_{2,r} = m_{2,r}/r^2 \to d_2^{\max}$.

Let us show now that $v_{2,r} \in B_r$ if r is small enough. By using $\langle \Phi_2'(v), v\rangle = \langle Sv, v\rangle + \langle N(v), v\rangle$, we obtain in view of (4.9) also

$$\lim_{r\to 0} \frac{\langle \Phi_2'(v_{2,r}), v_{2,r}\rangle}{r^2} = d_2^{\max} > 0. \tag{4.11}$$

Assuming by contradiction that $v_{2,r} \notin B_r$ holds for infinitely many $r = r_n \to 0$, we find for each $r = r_n$ that $(1+t)v_{2,r} \in K_r$ for all small $t > 0$ and thus $\Phi_2((1+t)v_{2,r}) \le m_{2,r} = \Phi_2(v_{2,r})$. Letting $t \to 0^+$, we obtain $\langle \Phi_2'(v_{2,r}), v_{2,r}\rangle \le 0$ for every $r = r_n$ which in view of $r_n \to 0$ contradicts (4.11).

The first assertion of (2) follows from the Lagrange Multiplier Rule on cones (see e.g. [26, Proposition 64.3] with $F(v) := \langle (I-A)v, v\rangle$ and $G(v) := \langle Sv, v\rangle + G_N(v)$ and the cone $C := K$). Setting $\varphi = 0$ and $\varphi = 2v_r$ in (4.8), we find

$$d_{2,r,v_r} = \frac{\langle Sv_r, v_r\rangle + \langle N(v_r), v_r\rangle}{\langle (I-A)v_r, v_r\rangle} = d_{2,r} + \frac{\langle N(v_r), v_r\rangle - G_N(v_r)}{r^2}.$$

Using (4.9), (4.10) and $d_{2,r} \to d_2^{\max}$, we find indeed $d_{2,r,v_r} \to d_2^{\max}$ as $r \to 0$. □

Proof of Theorem 3.1. Let us note that the equality (4.6) in Lemma 4.3 together with Lemma 4.4 imply that (3.1) is equivalent to the assertion that the quantities in (4.6) are positive.

Assume first that (2.12) has a critical point $d_2 > 0$. By Lemma 4.1, we find some $v \neq 0$ satisfying (4.4). Setting $\varphi = 0$ in (4.4), we obtain $d_2^{\max} \geq d_2 > 0$, in particular, (3.1) is satisfied.

Conversely, if the quantities from (4.6) are positive then Lemma 4.4 implies that they are equal to $d_2^{\max}$, and by the above argument $d_2^{\max} \geq d_2$ for any critical point d_2.

It remains to show that $d_2^{\max}$ is a bifurcation point with fixed d_1 (and thus a critical point). Due to Lemma 4.4, for any $r > 0$ small enough there are v_r, d_{2,r,v_r} satisfying (4.8) with $\langle (I - A)v_r, v_r \rangle = r^2$, $d_{2,r,v_r} \to d_2^{\max}$, and (4.5) in Lemma 4.3 gives $\|v_r\| \to 0$ as $r \to 0$. Lemma 4.1 implies that $(d_1, d_{2,r,v_r}, u_r, v_r)$ with u_r defined by (4.3) (with $v = v_r$) satisfies (2.13). Hence, $d_2^{\max}$ is a bifurcation point of (2.13) with fixed d_1. □

References

[1] Baltaev, J. I., Kučera, M., and Väth, M., *A variational approach to bifurcation in reaction-diffusion systems with Signorini type boundary conditions*, Applications of Math. **57** (2012), no. 2, 143–165.

[2] Drábek, P., Kučera, M., and Míková, M., *Bifurcation points of reaction-diffusion systems with unilateral conditions*, Czechoslovak Math. J. **35** (1985), 639–660.

[3] Drábek, P., Kufner, A., and Nicolosi, F., *Quasilinear elliptic equations with degenerations and singularities*, De Gruyter, Berlin, New York, 1997.

[4] Edelstein-Keshet, L., *Mathematical models in biology*, McGraw-Hill, Boston, 1988.

[5] Eisner, J. and Kučera, M., *Spatial patterning in reaction-diffusion systems with nonstandard boundary conditions*, Fields Institute Communications **25** (2000), 239–256.

[6] Eisner, J., Kučera, M., and Recke, L., *Direction and stability of bifurcating branches for variational inequalities*, J. Math. Anal. Appl. **301** (2005), 276–294.

[7] ———, *Smooth bifurcation branches of solutions for a Signorini problem*, Nonlinear Anal. **74** (2011), no. 5, 1853–1877.

[8] Eisner, J., Kučera, M., and Väth, M., *Global bifurcation of a reaction-diffusion system with inclusions*, J. Anal. Appl. **28** (2009), no. 4, 373–409.

[9] ———, *Bifurcation points for a reaction-diffusion system with two inequalities*, J. Math. Anal. Appl. **365** (2010), 176–194.

[10] Eisner, J. and Väth, M., *Location of bifurcation points for a reaction-diffusion system with Neumann-Signorini conditions*, Advanced Nonlinear Studies **11** (2011), 809–836.

[11] Fučík, S. and Kufner, A., *Nonlinear differential equations*, Elsevier, Amsterdam, Oxford, New York, 1980.

[12] Jones, D. S. and Sleeman, B. D., *Differential equations and mathematical biology*, Chapman & Hall/CRC, Boca Raton, London, Now York, Washington, 2003.

[13] Kučera, M., *Stability and bifurcation problems for reaction-diffusion systems with unilateral conditions*, Proceedings of Equadiff 6 (Brno) (Vosmanský, J. and Zlámal, M., eds.), University J. E. Purkyně, 1986, 227–234.

[14] ———, *Reaction-diffusion systems: Stabilizing effect of conditions described by quasivariational inequalities*, Czechoslovak Math. J. **47** (1997), no. 122, 469–486.

[15] Kučera, M., Recke, L., and Eisner, J., *Smooth bifurcation for variational inequalities and reaction-diffusion systems*, Progresses in Analysis (Singapore, New Jersey, London, Hong Kong) (Begehr, H. G. W., Gilbert, R. P., and Wong, M. W., eds.), vol. 2, World Scientific Publ., 2001, 1125–1133.

[16] Kučera, M. and Väth, M., *Bifurcation for a reaction-diffusion system with unilateral and Neumann boundary conditions*, J. Differential Equations **252** (2012), 2951–2982.

[17] Mimura, M., Nishiura, Y., and Yamaguti, M., *Some diffusive prey and predator systems and their bifurcation problems*, Ann. N. Y. Acad. Sci. **316** (1979), 490–510.

[18] Murray, J. D., *Mathematical biology*, Springer, New York, 1993.

[19] Nishiura, Y., *Global structure of bifurcating solutions of some reaction-diffusion systems and their stability problem*, Proceedings of the Fifth Int. Symp. on Computing Methods in Appl. Sciences and

Engineering, Versailles, France, 1981 (Amsterdam, New York, Oxford) (Glowinski, R. and Lions, J. L., eds.), North-Holland, 1982.

[20] Quittner, P., *Bifurcation points and eigenvalues of inequalities of reaction-diffusion type*, J. Reine Angew. Math. **380** (1987), no. 2, 1–13.

[21] Recke, L., Eisner, J., and Kučera, M., *Smooth bifurcation for variational inequalities based on the implicit function theorem*, J. Math. Anal. Appl. **275** (2002), 615–641.

[22] Väth, M., *A disc-cutting theorem and two-dimensional bifurcation*, CUBO **10** (2008), no. 4, 85–100.

[23] ———, *New beams of global bifurcation points for a reaction-diffusion system with inequalities or inclusions*, J. Differential Equations **247** (2009), 3040–3069.

[24] ———, *Instability of Turing type for a reaction-diffusion system with unilateral obstacles modeled by variational inequalities*, Math. Bohem. (Prague, 2013), Proceedings of Equadiff 13, 2013.

[25] Zeidler, E., *Nonlinear functional analysis and its applications*, vol. III, Springer, New York, Berlin, Heidelberg, 1985.

[26] ———, *Nonlinear functional analysis and its applications*, vol. IV, Springer, New York, Berlin, Heidelberg, 1988.

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