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Abstract. It is well known that the full Navier-Stokes-Fourier system does not possess a strong solution in three dimensions which causes problems in applications. However, when modeling the flow of a fluid in a thin long pipe, the influence of the cross section can be neglected and the flow is basically one-dimensional. This allows us to deal with strong solutions which are more convenient for numerical computations. The goal of this paper is to provide a rigorous justification of this approach. Namely, we prove that any suitable weak solution to the three-dimensional NSF system tends to a strong solution to the one-dimensional system as the thickness of the pipe tends to zero.

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1. Introduction

As introduced in [8, Chapter 1], governing equations for a flow of a general compressible viscous heat conducting fluid in a domain of $\mathbb{R}^3$ read as

$$\partial_t \rho + \mathrm{div}_x(\rho \mathbf{u}) = 0, \quad (1.1)$$

$$\partial_t(\rho \mathbf{u}) + \mathrm{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\rho, \theta) - \mathrm{div}_x \mathbb{S}(\theta, \nabla_x \mathbf{u}) = 0, \quad (1.2)$$

$$\partial_t(\rho s(\rho, \theta)) + \mathrm{div}_x(\rho s(\rho, \theta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}(\theta, \nabla_x \theta)}{\theta}\right) = \sigma, \quad (1.3)$$

where $(\rho, \mathbf{u}, \theta)$ stand for the unknown fluid mass density, the velocity field and the temperature respectively, p is the pressure, s is the entropy, $\mathbf{q}$ is the heat flux, σ is the entropy production rate and $\mathbb{S}$ represents the stress tensor.

We consider a family of shrinking domains Ω_ε of the form

$$\Omega_\varepsilon = Q_\varepsilon \times (0,1), \quad Q_\varepsilon = \varepsilon Q,$$

where Q is an open rectangular domain in $\mathbb{R}^2$ and $\varepsilon > 0$. Under suitable conditions on the initial data it is natural to expect that weak solutions $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon)$ of (1.1)–(1.3) on Ω_ε tend, as $\varepsilon \to 0$, to a classical solution $(\tilde\rho, \tilde u, \tilde\theta)$ of the one-dimensional system

$$\partial_t \tilde\rho + \partial_y(\tilde\rho \tilde u) = 0, \tag{1.4}$$

$$\partial_t(\tilde\rho \tilde u) + \partial_y(\tilde\rho \tilde u^2) + \partial_y p(\tilde\rho, \tilde\theta) - \partial_y[\tilde S(\tilde\theta, \partial_y \tilde u)] = 0, \tag{1.5}$$

$$\partial_t(\tilde\rho s(\tilde\rho, \tilde\theta)) + \partial_y(\tilde\rho s(\tilde\rho, \tilde\theta)\tilde u) + \partial_y\left(\frac{q(\tilde\theta, \partial_y \tilde\theta)}{\tilde\theta}\right) =$$
$$\frac{1}{\tilde\theta}\left(\tilde S(\tilde\theta, \partial_y \tilde u)\partial_y \tilde u - \frac{q(\tilde\theta, \partial_y \tilde\theta)\partial_y \tilde\theta}{\tilde\theta}\right), \tag{1.6}$$

where $\tilde S(\tilde\theta, \partial_y \tilde u)$ is naturally related to the three-dimensional stress tensor $\mathbb{S}$ and similarly q to the heat flux vector $\mathbf{q}$, see (2.19)–(2.21). Hereinafter we use the notation $x = (x_h, y) \in \mathbb{R}^3$, $x_h = (x_1, x_2) \in \mathbb{R}^2$, $y = x_3 \in \mathbb{R}$ and denote the derivative in x_3 by ∂_y. In this paper we give a rigorous justification of the convergence $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon) \to (\tilde\rho, (0,0,\tilde u), \tilde\theta)$.

As far as we know, the limit passage for heat conductive fluids has not yet been rigorously investigated and there is only a handful of results on related problems. Since incompressibility in one dimension does not allow any movement, such limit makes a little sense for 1D incompressible flows. However, dimension reduction to 2D-planar flows was examined in [9], [12], [13], [14] – see also references given therein. The case of a compressible barotropic fluid was studied by Vodák [17] and later by Bella et. al. [2].

The paper is organized as follows. In Section 2 we introduce detailed description of our problem. In Section 3 we present the concepts of a weak and a classical solution for 3D and 1D system, respectively, and discuss their existence. The main result is stated in Section 4. Section 5 contains preliminary calculations which are later used in Sections 6 and 7 in order to establish the proof of the main theorem from Section 4.

2. Setting of the problem

2.1. Structural hypothesis for the 3D problem

For given $\varepsilon > 0$ the system (1.1)–(1.3) on Ω_ε is complemented by the initial conditions

$$\rho_\varepsilon(0, \cdot) = \rho_{0,\varepsilon}, \ \rho_\varepsilon \mathbf{u}_\varepsilon(0, \cdot) = (\rho\mathbf{u})_{0,\varepsilon} \text{ and } \theta_\varepsilon(0, \cdot) = \theta_{0,\varepsilon},$$

such that the integral averages over Q_ε converge weakly (with respect to y) in $L^1(0,1)$ to some limit as $\varepsilon \to 0$. For the precise conditions see the statement of Theorem 4.1.

We suppose that the viscous stress tensor $\mathbb{S}$ is a linear function of the velocity gradient and therefore described by the Newton's law

$$\mathbb{S}(\theta, \nabla_x \mathbf{u}) = \mu(\theta)\left(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta(\theta)\mathrm{div}_x \mathbf{u}\mathbb{I}, \qquad (2.1)$$

with the shear viscosity coefficient $\mu(\theta) > 0$ and the bulk viscosity coefficient $\eta(\theta) \geq 0$ satisfying

$$\mu(\theta) = \mu_0 + \mu_1\theta, \quad \eta(\theta) = \eta_0 + \eta_1\theta, \quad \mu_0, \mu_1 > 0, \quad \eta_0, \eta_1 \geq 0. \qquad (2.2)$$

The heat flux $\mathbf{q}$ satisfies the Fourier's law

$$\mathbf{q}(\theta, \nabla_x \theta) = -\kappa(\theta)\nabla_x \theta, \qquad (2.3)$$

where we assume the following form of $\kappa(\theta)$

$$\kappa(\theta) = \kappa_0 + \kappa_2\theta^2 + \kappa_3\theta^3, \quad \kappa_i > 0, \ i = 0, 2, 3. \qquad (2.4)$$

The system of equations (1.1)–(1.3) with the constitutive relations (2.1) and (2.3) is called the Navier-Stokes-Fourier system.

Equations (1.1)–(1.3) are supplemented with the conservative boundary condition

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \qquad (2.5)$$

and the complete slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \quad [\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \qquad (2.6)$$

where the symbol $\mathbf{n}$ denotes the outer normal vector. It is worth pointing out that the complete slip boundary conditions are suitable for a dimension-reduction as the no-slip boundary conditions yield only a trivial solution in the asymptotic limit $\varepsilon \to 0$.

The concept of a weak solution to the Navier-Stokes-Fourier system based on the Second law of thermodynamics was introduced in [5]. The weak solutions satisfy the field equations (1.1)–(1.3) in the sense of distributions where the entropy production rate σ is a non-negative measure,

$$\sigma \geq \frac{1}{\theta}\left(\mathbb{S}(\theta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\theta, \nabla_x \theta)\nabla_x \theta}{\theta}\right). \qquad (2.7)$$

In order to compensate for the lack of information resulting from the inequality sign in (2.7) the system is supplemented with the total energy balance,

$$\frac{\partial}{\partial t}\int_{\Omega_\varepsilon}\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e(\rho, \theta)\right)dx = 0,$$

where $e = e(\rho, \theta)$ is the (specific) internal energy. Under these circumstances it can be shown (see [8, Chapter 3]) that any sufficiently smooth weak solution of (1.1)–(1.3) satisfies the standard relation

$$\sigma = \frac{1}{\theta}\left(\mathbb{S}(\theta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\theta, \nabla_x \theta)\nabla_x \theta}{\theta}\right).$$

The proof of our main theorem is based on the method of the relative entropy (see [3], [4], [15]), represented by the quantity

$$\mathcal{E}(\rho, \theta | r, \Theta) = H^\Theta(\rho, \theta) - \partial_\rho H^\Theta(r, \Theta)(\rho - r) - H^\Theta(r, \Theta), \qquad (2.8)$$

where $H^\Theta(\rho, \theta)$ is the thermodynamic potential called the ballistic free energy

$$H^\Theta(\rho, \theta) = \rho e(\rho, \theta) - \Theta \rho s(\rho, \theta),$$

introduced by Gibbs and discussed more recently by Ericksen [6].

We assume that the thermodynamic functions p, e and s are interrelated through the Gibbs' equation

$$\theta Ds(\rho, \theta) = De(\rho, \theta) + p(\rho, \theta)D\left(\frac{1}{\rho}\right). \qquad (2.9)$$

The subsequent analysis leans essentially on the thermodynamic stability of the fluid expressed through

$$\frac{\partial p(\rho, \theta)}{\partial \rho} > 0, \quad \frac{\partial e(\rho, \theta)}{\partial \theta} > 0 \text{ for all } \rho, \theta > 0. \qquad (2.10)$$

Motivated by the existence theory developed in [8, Chapter 3] we assume that the pressure $p = p(\rho, \theta)$ can be written in the form

$$p(\rho, \theta) = \theta^{\frac{5}{2}} P\left(\frac{\rho}{\theta^{\frac{3}{2}}}\right) + \frac{a}{3}\theta^4, \quad a > 0, \qquad (2.11)$$

where

$$P \in C^1[0, \infty), \ P(0) = 0, \ P'(Z) > 0 \text{ for all } Z \geq 0. \qquad (2.12)$$

In agreement with Gibbs' relation (2.9), the specific internal energy can be taken as

$$e(\rho, \theta) = \frac{3}{2}\frac{\theta^{\frac{5}{2}}}{\rho}P\left(\frac{\rho}{\theta^{\frac{3}{2}}}\right) + a\frac{\theta^4}{\rho}. \qquad (2.13)$$

Furthermore, by virtue of the second inequality in the thermodynamic stability hypotheses (2.10), we have

$$0 < \frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} < c, \quad \text{for all } Z > 0. \qquad (2.14)$$

In particular, (2.14) implies that the function $Z \mapsto \frac{P(Z)}{Z^{\frac{5}{3}}}$ is decreasing and we suppose that

$$\lim_{Z \to \infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = P_\infty > 0. \tag{2.15}$$

Finally, the formula for the (specific) entropy reads from the Gibbs' equation (2.9) as

$$s(\rho, \theta) = S\left(\frac{\rho}{\theta^{\frac{3}{2}}}\right) + \frac{4a}{3}\frac{\theta^3}{\rho}, \tag{2.16}$$

where, in accordance with the Third law of thermodynamics,

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} < 0. \tag{2.17}$$

2.2. Structural hypothesis for the 1D problem

Since we are interested in smooth solutions of the 1D equations, we complement the system of equations (1.4)-(1.6) with the initial conditions

$$\tilde{\rho}(0, \cdot) = \tilde{\rho}_0, \ \tilde{u}(0, \cdot) = \tilde{u}_0, \ \tilde{\theta}(0, \cdot) = \tilde{\theta}_0, \tag{2.18}$$

with $\tilde{\rho}_0 \geq c > 0$, $\tilde{\theta}_0 \geq c > 0$ and $\tilde{u}_0$ being smooth functions.

The form of the stress tensor $\mathbb{S}$ and the heat flux $\mathbf{q}$ naturally yields their one-dimensional counterparts, namely we expect that in the limit we recover

$$\tilde{S}(\tilde{\theta}, \partial_y \tilde{u}) = (\nu_0 + \nu_1 \tilde{\theta})\partial_y \tilde{u}, \tag{2.19}$$

with the viscosity coefficients

$$\nu_i = \frac{4}{3}\mu_i + \eta_i, \qquad i = 0, 1, \tag{2.20}$$

and

$$q(\tilde{\theta}, \partial_y \tilde{\theta}) = -\kappa(\tilde{\theta})\partial_y \tilde{\theta}. \tag{2.21}$$

3. Concepts of solutions

3.1. Weak solutions to the 3D-system

Let $T > 0$ and $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. We say that a triple $(\rho, \mathbf{u}, \theta)$ is a weak solution to the Navier-Stokes-Fourier system (1.1)–(1.3) with (2.1)–(2.7) in $(0, T) \times \Omega$ emanating from the initial data

$$\rho(0, \cdot) = \rho_0, \quad \rho\mathbf{u}(0, \cdot) = (\rho\mathbf{u})_0, \quad \rho s(\rho, \theta)(0, \cdot) = \rho_0 s(\rho_0, \theta_0), \ \rho_0 \geq 0, \ \theta_0 > 0,$$

if:

- the density and the absolute temperature satisfy $\rho(t, x) \geq 0$, $\theta(t, x) > 0$ for almost all $(t, x) \in (0, T) \times \Omega$, $\rho \in C_{weak}([0, T]; L^{\frac{5}{3}}(\Omega))$, $\rho\mathbf{u} \in C_{weak}([0, T]; L^{\frac{5}{4}}(\Omega; \mathbb{R}^3))$, $\theta \in L^\infty(0, T; L^4(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega))$ and $\mathbf{u} \in L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3))$, $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$;

- equation (1.1) is replaced by a family of integral identities

$$\int_\Omega \rho(\tau, \cdot)\varphi(\tau, \cdot)dx - \int_\Omega \rho_0\varphi(0, \cdot)dx = \int_0^\tau \int_\Omega (\rho\partial_t\varphi + \rho\mathbf{u} \cdot \nabla_x\varphi)dxdt,$$

  for any $\varphi \in C^1([0, T] \times \overline{\Omega})$ and any $\tau \in [0, T]$;
- the momentum equation (1.2) is satisfied in the sense of distributions, specifically,

$$\int_\Omega \rho\mathbf{u}(\tau, \cdot)\varphi(\tau, \cdot)dx - \int_\Omega (\rho\mathbf{u})_0\varphi(0, \cdot)dx$$

$$= \int_0^\tau \int_\Omega (\rho\mathbf{u}\partial_t\varphi + \rho\mathbf{u} \otimes \mathbf{u} : \nabla_x\varphi + p(\rho, \theta)\mathrm{div}_x\varphi - \mathbb{S}(\theta, \nabla_x\mathbf{u}) : \nabla_x\varphi)dxdt,$$

  for any $\varphi \in C^1([0, T] \times \overline{\Omega}; \mathbb{R}^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$ and any $\tau \in [0, T]$;
- the entropy balance (1.3), (2.7) is replaced by a family of integral inequalities

$$\int_\Omega \rho_0 s(\rho_0, \theta_0)\varphi(0, \cdot)dx - \int_\Omega \rho s(\rho, \theta)(\tau, \cdot)\varphi(\tau, \cdot)dx$$

$$+ \int_0^\tau \int_\Omega \frac{1}{\theta} \left(\mathbb{S}(\theta, \nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \frac{\mathbf{q}(\theta, \nabla_x\theta) \cdot \nabla_x\theta}{\theta} \right) \varphi dxdt$$

$$\leq - \int_0^\tau \int_\Omega \left(\rho s(\rho, \theta)\partial_t\varphi + \rho s(\rho, \theta)\mathbf{u} \cdot \nabla_x\varphi + \frac{\mathbf{q}(\theta, \nabla_x\theta) \cdot \nabla_x\varphi}{\theta} \right) dxdt,$$

  for any $\varphi \in C^1([0, T] \times \overline{\Omega})$, $\varphi \geq 0$ and almost all $\tau \in [0, T]$;
- the total energy is conserved

$$\int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e(\rho, \theta) \right) (\tau, \cdot)dx = \int_\Omega \left(\frac{1}{2\rho_0}|(\rho\mathbf{u})_0|^2 + \rho_0 e(\rho_0, \theta_0) \right) dx,$$

  for almost all $\tau \in [0, T]$.

The existence of global-in-time weak solutions to the three dimensional Navier-Stokes-Fourier system was established in [8, Theorem 3.1]. It reads as follows.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$, $\nu \in (0, 1)$. Assume that $\rho_0 \in L^{5/3}(\Omega)$, $\int_\Omega \rho_0 = M_0 > 0$; $(\rho\mathbf{u})_0 \in L^1(\Omega)$, $(\rho\mathbf{u})_0 = 0$ almost everywhere on the set $\{x \in \Omega, \rho_0(x) = 0\}$; $\theta_0 > 0$ a.e. in Ω is such that $\rho_0 s(\rho_0, \theta_0) \in L^1(\Omega)$ and the initial energy of the system satisfies*

$$\int_\Omega \left(\frac{1}{2\rho_0}|(\rho\mathbf{u})_0|^2 + \rho_0 e(\rho_0, \theta_0) \right) dx < \infty.$$

Assume that the structural hypotheses (2.1)–(2.4), (2.7), (2.9)–(2.17) hold. Then for any $T > 0$ the Navier-Stokes-Fourier system admits a weak solution $(\rho, \mathbf{u}, \theta)$ on $(0, T) \times \Omega$ in the sense specified above.

Moreover, the authors also proved that every weak solution satisfies the total dissipation balance [8, Equation (2.51)], i.e.,

$$\int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + H^{\bar\theta}(\rho, \theta) \right)(\tau)dx + \bar\theta\sigma\left[[0,\tau] \times \overline{\Omega}\right]$$

$$= \int_\Omega \left(\frac{1}{2}\rho_0|\mathbf{u}_0|^2 + H^{\bar\theta}(\rho_0, \theta_0) \right)dx, \quad (3.1)$$

for almost all $\tau \in [0, T]$ and for every positive constant $\bar\theta$.

Remark 3.2. It may seem that Theorem 3.1 is not suitable for our problem since the domain under consideration is of the form $(a, b) \times (c, d) \times (0, 1)$ and hence not of class $C^{2,\nu}$. We overcome this issue by the following consideration. The smoothness of the domain is used in the proof of the existence of weak solutions to ensure the smoothness of Galerkin approximations. In our case we may use the special structure of the spatial domain together with the boundary conditions to extend any solution from Ω appropriately (as an even or an odd function) to create a solution with periodic boundary conditions on a larger box where no restrictions on the smoothness of the boundary are necessary.

3.2. Classical solutions to the 1D-system

As $\varepsilon \to 0$ we observe that $\Omega_\varepsilon \to (0, 1)$. Moreover, we expect the solutions $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon)$ of (1.1)–(1.3) on Ω_ε to converge to a classical solution $(\tilde\rho, \tilde u, \tilde\theta)$ of (1.4)–(1.6). The boundary conditions (2.5) and (2.6) naturally lead to the no-slip boundary conditions for the velocity and the heat flux, i.e.,

$$\tilde u(\cdot, 0) = \tilde u(\cdot, 1) = 0, \quad q(\tilde\theta, \partial_y\tilde\theta)(\cdot, 0) = q(\tilde\theta, \partial_y\tilde\theta)(\cdot, 1) = 0. \quad (3.2)$$

There has been published a lot of papers about the one-dimensional Navier-Stokes-Fourier system (1.4)–(1.6) with the no-slip boundary conditions. We refer the reader to [1], [10] and [16] where the existence of solutions was provided under more restrictive assumptions on the pressure, viscosity, energy, etc. However, as far as we know, the global in time existence result for the system in its full generality has not been proven yet. Nevertheless, the local in time existence for any smooth initial data or global in time existence for small data can be expected to hold from the classical results on the topic. Since the existence of a classical one-dimensional solution is not a goal of this paper, we simply assume its existence without any proof. Let $T > 0$. We assume that there exists a trio

$$(\tilde\rho, \tilde u, \tilde\theta) : [0, T] \times [0, 1] \mapsto (0, \infty) \times \mathbb{R} \times (0, \infty),$$

of smooth functions that is the solution to (1.4)-(1.6) and (3.2) on $[0, T] \times (0, 1)$ satisfying

$$\tilde\rho \geq c > 0, \quad \tilde\theta \geq c > 0,$$

with the initial conditions (2.18).

4. Main Result

The main result of this paper reads as follows.

Theorem 4.1. *Let $Q \subset \mathbb{R}^2$ be an open rectangular domain and let $\Omega_\varepsilon = \varepsilon Q \times (0,1)$ for $\varepsilon > 0$. Suppose that the structural hypotheses (2.1)–(2.4) and (2.7) for the viscous stress tensor $\mathbb{S}$ and the heat flux $\mathbf{q}$ are satisfied together with (2.9)–(2.17) for the thermodynamic functions p, e and s.*

Assume that the initial data $(\rho_{0,\varepsilon}, (\rho\mathbf{u})_{0,\varepsilon}, \theta_{0,\varepsilon})$ satisfy all the assumptions of Theorem 3.1 on domains Ω_ε and denote $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon)$ the corresponding sequence of weak solutions to the 3D Navier-Stokes-Fourier system on $(0,T) \times \Omega_\varepsilon$ emanating from the initial data $(\rho_{0,\varepsilon}, (\rho\mathbf{u})_{0,\varepsilon}, \theta_{0,\varepsilon})$.

Let $(\tilde{\rho}_0, \tilde{u}_0, \tilde{\theta}_0)$ be smooth functions such that there exists the classical solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ to the 1D Navier-Stokes-Fourier system on $(0,T) \times (0,1)$ emanating from $(\tilde{\rho}_0, \tilde{u}_0, \tilde{\theta}_0)$. Define $\tilde{\mathbf{u}}_0 = [0,0,\tilde{u}_0]$ and $\tilde{\mathbf{u}} = [0,0,\tilde{u}]$.

Let moreover

$$\frac{1}{|Q_\varepsilon|} \int_{Q_\varepsilon} \rho_{0,\varepsilon}(x_h, \cdot) dx_h \to \tilde{\rho}_0, \qquad \frac{1}{|Q_\varepsilon|} \int_{Q_\varepsilon} (\rho\mathbf{u})_{0,\varepsilon}(x_h, \cdot) dx_h \to \tilde{\rho}_0 \tilde{\mathbf{u}}_0,$$

$$\frac{1}{|Q_\varepsilon|} \int_{Q_\varepsilon} \rho_{0,\varepsilon} s(\rho_{\varepsilon,0}, \theta_{\varepsilon,0}) dx_h \to \tilde{\rho}_0 s(\tilde{\rho}_0, \tilde{\theta}_0), \tag{4.1}$$

weakly in $L^1(0,1)$ and let

$$\frac{1}{|Q_\varepsilon|} \int_{\Omega_\varepsilon} \left[\frac{1}{2\rho_{0,\varepsilon}} |(\rho\mathbf{u})_{0,\varepsilon}|^2 + \rho_{\varepsilon,0} e(\rho_{\varepsilon,0}, \theta_{\varepsilon,0}) \right] dx$$

$$\to \int_0^1 \left[\frac{1}{2} \tilde{\rho}_0 |\tilde{u}_0|^2 + \tilde{\rho}_0 e(\tilde{\rho}_0, \tilde{\theta}_0) \right] dy. \tag{4.2}$$

Then

$$esssup_{t \in (0,T)} \frac{1}{|Q_\varepsilon|} \|\rho_\varepsilon - \tilde{\rho}\|_{L^{\frac{5}{3}}(\Omega_\varepsilon)}^{\frac{5}{3}} \to 0,$$

$$esssup_{t \in (0,T)} \frac{1}{|Q_\varepsilon|} \|\theta_\varepsilon - \tilde{\theta}\|_{L^2(\Omega_\varepsilon)}^2 \to 0,$$

and

$$\frac{1}{|Q_\varepsilon|} \|\mathbf{u}_\varepsilon - \tilde{\mathbf{u}}\|_{L^r((0,T) \times \Omega_\varepsilon)}^r \to 0,$$

for every $r \in [1,2)$ as $\varepsilon \to 0$.

Remark 4.2. Our result can be viewed as an extension of the dimension reduction for the barotropic Navier-Stokes system achieved in [2] to the full Navier-Stokes-Fourier system. The basic strategy of using the relative entropy inequality is the same. However, the presence of the temperature raises new obstacles.

In elasticity theory, the analysis of dimension reduction problems depends on the use of Korn's inequality which controls the gradient of velocity by its symmetric part, i.e.,

$$\|\nabla_x\mathbf{v}\|_{L^2(\Omega_\varepsilon)} \leq c(\varepsilon)\|\nabla_x\mathbf{v} + \nabla_x^T\mathbf{v}\|_{L^2(\Omega_\varepsilon)}, \quad \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0.$$

There are two problems that arise with respect to above inequality. Firstly, validity even for a fixed $\varepsilon > 0$ requires certain restrictions on the shape of the cross-section Q. Secondly, even "properly" shaped Q might not stop the constant $c(\varepsilon)$ from blowing up as $\varepsilon \to 0$.

In [2] authors obtain their result for a regular planar domain Q since they avoid the use of Korn's inequality by exploring the structural stability of the family of solutions of the barotropic Navier-Stokes system. In our case, the approach of [2] is disrupted by the temperature.

Therefore our result leans on the validity of stronger Korn's like inequality appropriate for compressible fluids, namely, we use

$$\|\nabla_x\mathbf{v}\|_{L^2(\Omega_\varepsilon)}^2 \leq \left\|\nabla_x\mathbf{v} + \nabla_x^T\mathbf{v} - \frac{2}{3}\text{div}_x\mathbf{v}\mathbb{I}\right\|_{L^2(\Omega_\varepsilon)}^2, \quad \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0.$$

To get that we assume that Q is an open rectangular domain in $\mathbb{R}^2$, i.e., Q can be written as

$$Q = (a,b) \times (c,d), \quad a < b, \ c < d, \ a,b,c,d \in \mathbb{R}.$$

5. Preliminary calculations

In this section we introduce the estimates which will be used repeatedly in the subsequent calculations. Hereinafter, we assume that $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ and $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon)$ are a classical and a weak solution to the respective problem satisfying the assumptions introduced in Theorem 4.1. For clarity, we omit the suffix ε where no confusion occurs and we write $(\rho, \mathbf{u}, \theta)$ instead of $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon)$.

Following [8, Chapters 4,5] we introduce essential and residual components based on ρ and θ. To begin, we choose positive constants $\underline{\rho}, \overline{\rho}, \underline{\theta}, \overline{\theta}$ fulfilling

$$0 < \underline{\rho} \leq \frac{1}{2}\min_{(\tau,y)\in[0,T]\times[0,1]}\tilde{\rho}(\tau,y) \leq 2\max_{(\tau,y)\in[0,T]\times[0,1]}\tilde{\rho}(\tau,y) \leq \overline{\rho},$$

$$0 < \underline{\theta} \leq \frac{1}{2}\min_{(\tau,y)\in[0,T]\times[0,1]}\tilde{\theta}(\tau,y) \leq 2\max_{(\tau,y)\in[0,T]\times[0,1]}\tilde{\theta}(\tau,y) \leq \overline{\theta}.$$

According to Lemma 5.1 in [8] there exists a constant $c > 0$ such that

$$\mathcal{E}(\rho,\theta|\tilde{\rho},\tilde{\theta}) \geq c \begin{cases} |\rho - \tilde{\rho}|^2 + |\theta - \tilde{\theta}|^2 & \text{if } (\rho,\theta) \in [\underline{\rho},\overline{\rho}] \times [\underline{\theta},\overline{\theta}], \\ 1 + |\rho s(\rho,\theta)| + \rho e(\rho,\theta) & \text{otherwise.} \end{cases} \tag{5.1}$$

It is worth pointing out that c is independent of ε.

Each measurable function h can be written as

$$h = h_{ess} + h_{res},$$

where

$$h_{ess}(t,x) = \begin{cases} h(t,x) & \text{if } (\rho(t,x), \theta(t,x)) \in [\underline{\rho}, \overline{\rho}] \times [\underline{\theta}, \overline{\theta}], \\ \\ 0 & \text{otherwise.} \end{cases}$$

5.1. Estimates on ρ and θ

We immediately see from (5.1) that

$$\|[\theta - \tilde{\theta}]_{ess}\|^s_{L^s(\Omega_\varepsilon)} + \|[\rho - \tilde{\rho}]_{ess}\|^s_{L^s(\Omega_\varepsilon)} \le c \int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) dx, \qquad (5.2)$$

for $s \ge 2$. We show that similar estimates hold also for the residual parts. Firstly,

$$\rho e(\rho, \theta) \ge c(\rho^{\frac{5}{3}} + \theta^4).$$

Since $\dfrac{P(z)}{z^{\frac{5}{3}}}$ is decreasing and $\lim\limits_{z \to \infty} \dfrac{P(z)}{z^{\frac{5}{3}}} = P_\infty$ we obtain

$$P(z) \ge P_\infty z^{\frac{5}{3}}.$$

Using the above estimate we get

$$\rho e(\rho, \theta) \ge \frac{3}{2} \theta^{\frac{5}{2}} P_\infty \left(\frac{\rho}{\theta^{\frac{3}{2}}} \right)^{\frac{5}{3}} + a\theta^4 \ge c(\rho^{\frac{5}{3}} + \theta^4).$$

Now we can estimate the residual parts. We have for $1 \le q \le 4$ that

$$\int_{\Omega_\varepsilon} |[\theta - \tilde{\theta}]_{res}|^q dx \le c \int_{\Omega_\varepsilon} (|[\theta]_{res}|^q + 1_{res}) dx,$$

and by the use of Hölder and Young inequalities together with (5.1) we obtain

$$\le c \left(\int_{\Omega_\varepsilon} |[\theta]_{res}|^4 dx \right)^{\frac{q}{4}} \left(\int_{\Omega_\varepsilon} 1_{res} dx \right)^{1 - \frac{q}{4}} + c \int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) dx$$

$$\le c \int_{\Omega_\varepsilon} |[\theta]_{res}|^4 dx + c \int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}).$$

Thus

$$\int_{\Omega_\varepsilon} |[\theta - \tilde{\theta}]_{res}|^q dx \le c \int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) dx, \qquad (5.3)$$

where $1 \le q \le 4$. Combining (5.2) and (5.3) we obtain

$$\|\theta - \tilde{\theta}\|^2_{L^2(\Omega_\varepsilon)} \le c \int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) dx. \qquad (5.4)$$

As for the density, we get analogously as above that for $1 \le p \le \frac{5}{3}$ it holds

$$\int_{\Omega_\varepsilon} |[\rho - \tilde{\rho}]_{res}|^p dx \le c \int_{\Omega_\varepsilon} (|[\rho]_{res}|^p + 1_{res}) \, dx$$

$$\le c \left(\int_{\Omega_\varepsilon} |[\rho]_{res}|^{\frac{5}{3}} dx \right)^{\frac{3p}{5}} \left(\int_{\Omega_\varepsilon} 1_{res} dx \right)^{1 - \frac{3p}{5}} + c \int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) dx$$

$$\le c \int_{\Omega_\varepsilon} |[\rho]_{res}|^{\frac{5}{3}} dx + c \int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) dx.$$

Hence, we have

$$\int_{\Omega_\varepsilon} |[\rho - \tilde{\rho}]_{res}|^p dx \le c \int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) dx, \tag{5.5}$$

for $1 \le p \le \frac{5}{3}$.

5.2. Korn and Poincaré inequalities

Although both inequalities are very well known, we need the estimates which are independent of ε. Note that this goal cannot be reached by a simple rescaling argument as the domain shrinks only in two dimensions. For more details see Remark 4.2.

For clarity, in this section we prefer the notation x_3 and ∂_{x_3} instead of y and ∂_y.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^3$ be a rectangular domain, i.e., $\Omega = (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$ and $\mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^3)$ be such that $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Then*

$$\|\nabla_x \mathbf{u}\|_{L^2(\Omega)}^2 \le \left\| \nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} \right\|_{L^2(\Omega)}^2,$$

$$\|\nabla_x \mathbf{u}\|_{L^2(\Omega)}^2 \le \left\| \nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right\|_{L^2(\Omega)}^2,$$

$$\|\nabla_x \mathbf{u}\|_{L^2(\Omega)}^2 \le \int_\Omega (\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I}) : \nabla_x \mathbf{u} dx.$$

Proof. Since smooth functions are dense in $W^{1,2}(\Omega, \mathbb{R}^3)$, we prove the lemma only for $\mathbf{u} \in C^2(\Omega)$. Denote $\mathbf{u} = [u^1, u^2, u^3]$. We split $\partial\Omega$ into three parts as follows:

$$\partial\Omega_1 = \{a_1\} \times [a_2, b_2] \times [a_3, b_3] \cup \{b_1\} \times [a_2, b_2] \times [a_3, b_3],$$

$$\partial\Omega_2 = [a_1, b_1] \times \{a_2\} \times [a_3, b_3] \cup [a_1, b_1] \times \{b_2\} \times [a_3, b_3],$$

$$\partial\Omega_3 = [a_1, b_1] \times [a_2, b_2] \times \{a_3\} \cup [a_1, b_1] \times [a_2, b_2] \times \{b_3\}.$$

Therefore, boundary conditions on $\mathbf{u}$ imply:

$$u^1|_{\partial\Omega_1} = 0, \quad u^2|_{\partial\Omega_2} = 0, \quad u^3|_{\partial\Omega_3} = 0. \tag{5.6}$$

Since

$$(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}) = \begin{pmatrix} 2\partial_{x_1} u^1 & \partial_{x_2} u^1 + \partial_{x_1} u^2 & \partial_{x_3} u^1 + \partial_{x_1} u^3 \\ \partial_{x_1} u^2 + \partial_{x_2} u^1 & 2\partial_{x_2} u^2 & \partial_{x_3} u^2 + \partial_{x_2} u^3 \\ \partial_{x_1} u^3 + \partial_{x_3} u^1 & \partial_{x_2} u^3 + \partial_{x_3} u^2 & 2\partial_{x_3} u^3 \end{pmatrix},$$

we calculate

$$(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}) : (\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}) = |\nabla_x \mathbf{u}|^2 + 3\left[(\partial_{x_1} u^1)^2 + (\partial_{x_2} u^2)^2 + (\partial_{x_3} u^3)^2\right]$$

$$+4\left[\partial_{x_2} u^1 \partial_{x_1} u^2 + \partial_{x_3} u^1 \partial_{x_1} u^3 + \partial_{x_3} u^2 \partial_{x_2} u^3\right].$$

We integrate by parts to modify the last terms.

First, we integrate by parts in x_2,

$$\int_\Omega \partial_{x_2} u^1 \partial_{x_1} u^2 dx = \int_{a_1}^{b_1} \int_{a_3}^{b_3} \left[u^1 \partial_{x_1} u^2\right]_{a_2}^{b_2} dx_3 dx_1 - \int_\Omega u^1 \partial_{x_2} \partial_{x_1} u^2 dx.$$

From the boundary condition $u^2|_{\partial\Omega_2} = 0$ we get $\partial_{x_1} u^2|_{\partial\Omega_2} = 0$ and hence the boundary term above disappears. Next, we integrate by parts in x_1,

$$-\int_\Omega u^1 \partial_{x_1} \partial_{x_2} u^2 dx = \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left[u^1 \partial_{x_2} u^2\right]_{a_1}^{b_1} dx_3 dx_2 + \int_\Omega \partial_{x_1} u^1 \partial_{x_2} u^2 dx.$$

The boundary term above disappears due to $u^1|_{\partial\Omega_1} = 0$ and we end up with

$$\int_\Omega \partial_{x_2} u^1 \partial_{x_1} u^2 dx = \int_\Omega \partial_{x_1} u^1 \partial_{x_2} u^2 dx.$$

The rest of the terms is treated analogously and we get

$$\int_\Omega \partial_{x_3} u^1 \partial_{x_1} u^3 dx = \int_\Omega \partial_{x_1} u^1 \partial_{x_3} u^3 dx,$$

and

$$\int_\Omega \partial_{x_3} u^2 \partial_{x_2} u^3 dx = \int_\Omega \partial_{x_2} u^2 \partial_{x_3} u^3 dx.$$

Finally,

$$\int_\Omega (\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}) : (\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}) dx = \int_\Omega |\nabla_x \mathbf{u}|^2 dx$$

$$+ 3\int_\Omega \left[(\partial_{x_1} u^1)^2 + (\partial_{x_2} u^2)^2 + (\partial_{x_3} u^3)^2\right] dx$$

$$+ 4\int_\Omega \left[\partial_{x_1} u^1 \partial_{x_2} u^2 + \partial_{x_1} u^1 \partial_{x_3} u^3 + \partial_{x_2} u^2 \partial_{x_3} u^3\right] dx,$$

and since

$$(\operatorname{div}_x \mathbf{u})^2 = \sum_{i=1}^3 (\partial_{x_i} u^i)^2 + 2[\partial_{x_1} u^1 \partial_{x_2} u^2 + \partial_{x_1} u^1 \partial_{x_3} u^3 + \partial_{x_2} u^2 \partial_{x_3} u^3],$$

we get that

$$\int_\Omega (\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}) : (\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u})dx = \int_\Omega |\nabla_x \mathbf{u}|^2 dx + 2\int_\Omega (\mathrm{div}_x \mathbf{u})^2 dx$$

$$+ \int_\Omega \left[(\partial_{x_1} u^1)^2 + (\partial_{x_2} u^2)^2 + (\partial_{x_3} u^3)^2\right]dx.$$

Thus the first inequality is proven. The second inequality follows easily since

$$(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}) : (\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I})$$

$$= (\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}) : (\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}) - \frac{4}{3}(\mathrm{div}_x \mathbf{u})^2.$$

We use the integration by parts in the same way as before in order to get

$$\int_\Omega (\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}) : \nabla_x \mathbf{u}\, dx = \int_\Omega |\nabla_x \mathbf{u}|^2 dx + \frac{1}{3}\int_\Omega (\mathrm{div}_x \mathbf{u})^2 dx,$$

which directly implies the last desired inequality. $\qquad\square$

We introduce the following notation. For a set $M \subset \mathbb{R}^d$, $d \in \mathbb{N}$ and a function $f \in L^1(M)$ we denote by $(f)_M$ its integral average, i.e.,

$$(f)_M = \frac{1}{|M|}\int_M f(x)dx.$$

Lemma 5.2. *There exists a constant $c > 0$ independent of ε such that for every $f \in W^{1,2}(\Omega_\varepsilon)$ fulfilling $f(\cdot, 0) = f(\cdot, 1) = 0$ on Q_ε it holds that*

$$\int_0^1 \int_{Q_\varepsilon} |f(x_h, y) - (f)_{Q_\varepsilon}(y)|^4 dx_h dy = \|f - (f)_{Q_\varepsilon}\|_{L^4(\Omega_\varepsilon)}^4 \le c\|\nabla f\|_{L^2(\Omega_\varepsilon)}^4.$$

Proof. From Ladyzhenskaya's inequality (see [11]) we have

$$\|f - (f)_{Q_\varepsilon}\|_{L^4(Q_\varepsilon)}^4 \le c\|\nabla_{x_h} f\|_{L^2(Q_\varepsilon)}^2 \|f - (f)_{Q_\varepsilon}\|_{L^2(Q_\varepsilon)}^2.$$

with the constant c independent of ε. Indeed, as this is a 2D inequality, the independence can be shown by a simple rescaling argument. Further,

$$\|f - (f)_{Q_\varepsilon}\|_{L^4(\Omega_\varepsilon)}^4 = \int_0^1 \|f - (f)_{Q_\varepsilon}\|_{L^4(Q_\varepsilon)}^4 dy$$

$$\le c\int_0^1 \|\nabla_{x_h} f\|_{L^2(Q_\varepsilon)}^2 \|f - (f)_{Q_\varepsilon}\|_{L^2(Q_\varepsilon)}^2 dy$$

$$\le c\|f - (f)_{Q_\varepsilon}\|_{L^\infty((0,1),L^2(Q_\varepsilon))}^2 \|\nabla f\|_{L^2(\Omega_\varepsilon)}^2 \le c\|\nabla f\|_{L^2(\Omega_\varepsilon)}^4,$$

where we used the fact that, due to Hölder inequality,

$$\|f - (f)_{Q_\varepsilon}\|_{L^2(Q_\varepsilon)}^2 \leq \int_{Q_\varepsilon} \left| f(x_h) - \frac{1}{|Q_\varepsilon|} \int_{Q_\varepsilon} f(z_h) dz_h \right|^2 dx_h$$

$$\leq \int_{Q_\varepsilon} \frac{1}{|Q_\varepsilon|} \int_{Q_\varepsilon} |f(x_h) - f(z_h)|^2 dz_h dx_h$$

$$\leq c \frac{1}{|Q_\varepsilon|} \left(\int_{Q_\varepsilon} \int_{Q_\varepsilon} |f(x_h)|^2 dx_h dz_h + \int_{Q_\varepsilon} \int_{Q_\varepsilon} |f(z_h)|^2 dx_h dz_h \right)$$

$$\leq c\|f\|_{L^2(Q_\varepsilon)}^2,$$

for a.a. $y \in (0,1)$ and thus by Sobolev-Poincaré inequality we finally obtain

$$\|f - (f)_{Q_\varepsilon}\|_{L^\infty((0,1),L^2(Q_\varepsilon))}^2 \leq c\|f\|_{L^\infty((0,1),L^2(Q_\varepsilon))}^2$$

$$\leq c\|\partial_{x_3} f\|_{L^2((0,1),L^2(Q_\varepsilon))}^2 \leq c\|\nabla f\|_{L^2(\Omega_\varepsilon)}^2.$$

$\square$

6. Relative entropy balance

Recall that we omit the index ε for the functions $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon)$ and simply write $(\rho, \mathbf{u}, \theta)$ instead. Following the calculations in [7, Section 3] we obtain for each $\varepsilon > 0$ the following relative entropy inequality satisfied by any weak solution $(\rho, \mathbf{u}, \theta)$ to the Navier-Stokes-Fourier system on Ω_ε and any trio $(r, \mathbf{U}, \Theta)$ of smooth functions, r and Θ bounded below away from zero in $[0,T] \times \Omega_\varepsilon$ and $\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0$.

$$\frac{1}{|Q_\varepsilon|} \int_{\Omega_\varepsilon} \left(\frac{1}{2}\rho|\mathbf{u} - \mathbf{U}|^2 + \mathcal{E}(\rho, \theta|r, \Theta) \right)(\tau, \cdot) dx$$

$$+ \frac{1}{|Q_\varepsilon|} \int_0^\tau \int_{\Omega_\varepsilon} \frac{\Theta}{\theta} \left(\mathbb{S}(\theta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\theta, \nabla_x \theta) \cdot \nabla_x \theta}{\theta} \right) dx dt$$

$$\leq \frac{1}{|Q_\varepsilon|} \int_{\Omega_\varepsilon} \left(\frac{1}{2\rho_{0,\varepsilon}} |(\rho\mathbf{u})_{0,\varepsilon} - \rho_{0,\varepsilon} \mathbf{U}(0, \cdot)|^2 + \mathcal{E}(\rho_{\varepsilon,0}, \theta_{\varepsilon,0}|r(0, \cdot), \Theta(0, \cdot)) \right) dx$$

$$+ \frac{1}{|Q_\varepsilon|} \int_0^\tau \int_{\Omega_\varepsilon} \rho(\mathbf{u} - \mathbf{U}) \cdot \nabla_x \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) dx dt$$

$$+ \frac{1}{|Q_\varepsilon|} \int_0^\tau \int_{\Omega_\varepsilon} \rho(s(\rho, \theta) - s(r, \Theta))(\mathbf{U} - \mathbf{u}) \cdot \nabla_x \Theta dx dt$$

$$+ \frac{1}{|Q_\varepsilon|} \int_0^\tau \int_{\Omega_\varepsilon} (\rho(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U}) \cdot (\mathbf{U} - \mathbf{u}) - p(\rho, \theta) \operatorname{div}_x \mathbf{U} + \mathbb{S}(\theta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U}) dx dt$$

$$- \frac{1}{|Q_\varepsilon|} \int_0^\tau \int_{\Omega_\varepsilon} \left(\rho(s(\rho, \theta) - s(r, \Theta))(\partial_t \Theta + \mathbf{U} \cdot \nabla_x \Theta) + \frac{\mathbf{q}(\theta, \nabla_x \theta)}{\theta} \cdot \nabla_x \Theta \right) dx dt$$

$$+\frac{1}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega_\varepsilon}\left(\left(1-\frac{\rho}{r}\right)\partial_t p(r,\Theta)-\frac{\rho}{r}\mathbf{u}\cdot\nabla_x p(r,\Theta)\right)dxdt, \qquad (6.1)$$

for almost all $\tau\in[0,T]$, where $\mathcal{E}$ was introduced in (2.8).

To prove Theorem 4.1 we take

$$r=\tilde{\rho}(t,y),\quad \Theta=\tilde{\theta}(t,y),\quad \mathbf{U}=\tilde{\mathbf{u}}(t,y)=\begin{bmatrix}0\\0\\\tilde{u}(t,y)\end{bmatrix},$$

in (6.1) to obtain

$$\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\left(\frac{1}{2}\rho|\mathbf{u}-\tilde{\mathbf{u}}|^2+\mathcal{E}(\rho,\theta|\tilde{\rho},\tilde{\theta})\right)(\tau,\cdot)dx$$

$$+\frac{1}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega_\varepsilon}\frac{\tilde{\theta}}{\theta}\left(\mathbb{S}(\theta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u}-\frac{\mathbf{q}(\theta,\nabla_x\theta)\cdot\nabla_x\theta}{\theta}\right)dxdt$$

$$\leq\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\left(\frac{1}{2\rho_{0,\varepsilon}}|(\rho\mathbf{u})_{0,\varepsilon}-\rho_{0,\varepsilon}\tilde{\mathbf{u}}_0|^2+\mathcal{E}(\rho_{0,\varepsilon},\theta_{0,\varepsilon}|\tilde{\rho}_0,\tilde{\theta}_0)\right)dx$$

$$+\frac{1}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega_\varepsilon}\rho|u^3-\tilde{u}|^2|\partial_y\tilde{u}|dxdt+\frac{1}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega_\varepsilon}\rho(s(\rho,\theta)-s(\tilde{\rho},\tilde{\theta}))(\tilde{u}-u^3)\cdot\partial_y\tilde{\theta}dxdt$$

$$+\frac{1}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega_\varepsilon}\left(\rho(\partial_t\tilde{u}+\tilde{u}\partial_y\tilde{u})(\tilde{u}-u^3)-p(\rho,\theta)\partial_y\tilde{u}+\mathbb{S}(\theta,\nabla_x\mathbf{u}):\nabla_x\tilde{\mathbf{u}}\right)dxdt$$

$$-\frac{1}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega_\varepsilon}\left(\rho(s(\rho,\theta)-s(\tilde{\rho},\tilde{\theta}))(\partial_t\tilde{\theta}+\tilde{u}\partial_y\tilde{\theta})+\frac{\mathbf{q}(\theta,\nabla_x\theta)}{\theta}\cdot\nabla_x\tilde{\theta}\right)dxdt$$

$$+\frac{1}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega_\varepsilon}\left(\left(1-\frac{\rho}{\tilde{\rho}}\right)\partial_t p(\tilde{\rho},\tilde{\theta})-\frac{\rho}{\tilde{\rho}}\mathbf{u}\cdot\nabla_x p(\tilde{\rho},\tilde{\theta})\right)dxdt, \qquad (6.2)$$

for almost all $\tau\in[0,T]$.

In order to handle the integrals on the right-hand side of (6.2) we proceed in several steps:

Step 0 Observe that by (4.1) and (4.2) we get

$$\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\left(\frac{1}{2\rho_{0,\varepsilon}}|(\rho\mathbf{u})_{0,\varepsilon}-\rho_{0,\varepsilon}\tilde{\mathbf{u}}_0|^2+\mathcal{E}(\rho_{0,\varepsilon},\theta_{0,\varepsilon}|\tilde{\rho}_0,\tilde{\theta}_0)\right)dx\to 0, \qquad (6.3)$$

as $\varepsilon\to 0$. From now on, we include this term in $\Gamma(\varepsilon)$ where $\Gamma(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Step 1

$$\int_{\Omega_\varepsilon} \rho|u^3 - \tilde{u}|^2 |\partial_y \tilde{u}| dx \le 2\|\partial_y \tilde{u}\|_{L^\infty(0,1)} \int_{\Omega_\varepsilon} \frac{1}{2}\rho|u^3 - \tilde{u}|^2 dx. \qquad (6.4)$$

Step 2

$$\left| \int_{\Omega_\varepsilon} \rho \left(s(\rho, \theta) - s(\tilde{\rho}, \tilde{\theta}) \right) (\tilde{u} - u^3) \partial_y \tilde{\theta} dx \right|$$

$$\le \|\partial_y \tilde{\theta}\|_{L^\infty(0,1)} \left[\overline{\rho} \int_{\Omega_\varepsilon} \left| \left[s(\rho, \theta) - s(\tilde{\rho}, \tilde{\theta}) \right]_{ess} \right| |u^3 - \tilde{u}| dx \right.$$

$$\left. + \int_{\Omega_\varepsilon} \left| \left[\rho \left(s(\rho, \theta) - s(\tilde{\rho}, \tilde{\theta}) \right) \right]_{res} \right| |u^3 - \tilde{u}| dx \right].$$

First, we estimate the essential part. Since

$$\left| \left[s(\rho, \theta) - s(\tilde{\rho}, \tilde{\theta}) \right]_{ess} \right| = \left| \left[S\left(\frac{\rho}{\theta^{\frac{3}{2}}} \right) + \frac{4a}{3} \frac{\theta^3}{\rho} - S\left(\frac{\tilde{\rho}}{\tilde{\theta}^{\frac{3}{2}}} \right) - \frac{4a}{3} \frac{\tilde{\theta}^3}{\tilde{\rho}} \right]_{ess} \right|$$

$$\le \left| \left[S\left(\frac{\rho}{\theta^{\frac{3}{2}}} \right) - S\left(\frac{\tilde{\rho}}{\tilde{\theta}^{\frac{3}{2}}} \right) \right]_{ess} \right| + \frac{4a}{3} \left| \left[\frac{\theta^3}{\rho} - \frac{\tilde{\theta}^3}{\tilde{\rho}} \right]_{ess} \right|$$

$$\le |S'(\xi)| \left| \left[\frac{\rho}{\theta^{\frac{3}{2}}} - \frac{\tilde{\rho}}{\tilde{\theta}^{\frac{3}{2}}} \right]_{ess} \right| + \frac{4a}{3} \left| \left[\frac{\theta^3}{\rho} - \frac{\tilde{\theta}^3}{\tilde{\rho}} \right]_{ess} \right| \le c[\rho - \tilde{\rho}]_{ess} + c[\theta - \tilde{\theta}]_{ess},$$

thus,

$$\int_{\Omega_\varepsilon} \left| \left[s(\rho, \theta) - s(\tilde{\rho}, \tilde{\theta}) \right]_{ess} \right| |u^3 - \tilde{u}| dx$$

$$\le K(\cdot) \int_{\Omega_\varepsilon} \left(\frac{1}{2}\rho|u^3 - \tilde{u}|^2 + \mathcal{E}(\rho, \theta|\tilde{\rho}, \tilde{\theta}) \right) dx.$$

Here and hereafter, $K(\cdot)$ is a generic constant depending on $\tilde{\rho}$, $\tilde{u}$, $\tilde{\theta}$, $\underline{\rho}$, $\underline{\theta}$ through its respective norms. It is independent of ε.

Next we treat the residual part. We compute

$$\int_{\Omega_\varepsilon} \left| \left[\rho(s(\rho, \theta) - s(\tilde{\rho}, \tilde{\theta})) \right]_{res} \right| |\tilde{u} - u^3| dx$$

$$\le \int_{\Omega_\varepsilon} \left| \left[\rho(s(\rho, \theta) - s(\tilde{\rho}, \tilde{\theta})) \right]_{res} \right| |\tilde{u} - (u^3)_{Q_\varepsilon}| dx$$

$$+ \int_{\Omega_\varepsilon} \left| \left[\rho(s(\rho, \theta) - s(\tilde{\rho}, \tilde{\theta})) \right]_{res} \right| |(u^3)_{Q_\varepsilon} - u^3| dx = \mathcal{I}_1 + \mathcal{I}_2.$$

It holds

$$\mathcal{I}_1 \le \|\tilde{u} - (u^3)_{Q_\varepsilon}\|_{L^\infty(\Omega_\varepsilon)} \|[\rho(s(\rho, \theta) - s(\tilde{\rho}, \tilde{\theta}))]_{res}\|_{L^{\frac{4}{3}}(\Omega_\varepsilon)} |Q_\varepsilon|^{\frac{1}{4}}.$$

Further,

$$|[\rho(s(\rho,\theta) - s(\tilde{\rho},\tilde{\theta})]_{res}| \le c[\rho + \rho s(\rho,\theta)]_{res}$$
$$\le c[\rho + \theta^3 + \rho[\log\theta]^+ + \rho|\log\rho|]_{res}, \quad (6.5)$$

(cf. Equation (3.39) in [8]). Each term on the right hand side of (6.5) can be estimated using (5.3) and (5.5) as follows:

$$\left(\int_{\Omega_\varepsilon} \rho_{res}^{\frac{4}{3}} dx\right)^{\frac{3}{4}} \le \left(\int_{\Omega_\varepsilon} \rho_{res}^{\frac{5}{3}} dx\right)^{\frac{3}{5}} |Q_\varepsilon|^{\frac{3}{20}} \le \left(\frac{1}{|Q_\varepsilon|}\int \mathcal{E}(\rho,\theta|\tilde{\rho},\tilde{\theta})dx\right)^{\frac{3}{5}} |Q_\varepsilon|^{\frac{3}{4}},$$

$$\left(\int_{\Omega_\varepsilon} \theta_{res}^4 dx\right)^{\frac{3}{4}} \le \left(\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon} \mathcal{E}(\rho,\theta|\tilde{\rho},\tilde{\theta})dx\right)^{\frac{3}{4}} |Q_\varepsilon|^{\frac{3}{4}},$$

$$\left(\int_\varepsilon (\rho[\log\theta]_{res}^+)^{\frac{4}{3}} dx\right)^{\frac{3}{4}} \le c\left(\int_{\Omega_\varepsilon} \rho_{res}^{\frac{4}{3}}\theta_{res}^{\frac{4}{5}} dx\right)^{\frac{3}{4}}$$
$$\le c\left(\int_{\Omega_\varepsilon} \rho_{res}^{\frac{5}{3}} dx\right)^{\frac{3}{5}} \left(\int_{\Omega_\varepsilon} \theta_{res}^4 dx\right)^{\frac{3}{20}}$$
$$\le \left(\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon} \mathcal{E}(\rho,\theta|\tilde{\rho},\tilde{\theta})dx\right)^{\frac{3}{4}} |Q_\varepsilon|^{\frac{3}{4}},$$

$$\left(\int_{\Omega_\varepsilon} (|\rho\log\rho|_{res})^{\frac{4}{3}} dx\right)^{\frac{3}{4}} \le c\left(\int_{\Omega_\varepsilon} (\rho^{\frac{4}{3}} + \rho^{\frac{5}{3}})dx\right)^{\frac{3}{4}}$$
$$\le \left(\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon} \mathcal{E}(\rho,\theta|\tilde{\rho},\tilde{\theta})dx\right)^{\frac{3}{4}} |Q_\varepsilon|^{\frac{3}{4}}.$$

From (3.1) and (4.1),(4.2) we deduce that

$$\left(\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon} \mathcal{E}(\rho,\theta|\tilde{\rho},\tilde{\theta})dx\right) \in L^\infty(0,T),$$

with a bound independent of ε (cf. Equation (2.52) in [8]). Thus

$$\mathcal{I}_1 \le \|\tilde{u} - (u^3)_{Q_\varepsilon}\|_{L^\infty(\Omega_\varepsilon)} \left(\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon} \mathcal{E}(\rho,\theta|\tilde{\rho},\tilde{\theta})dx\right)^{\frac{1}{2}} |Q_\varepsilon|.$$

As far as $W^{1,1}(0,1) \hookrightarrow L^\infty(0,1)$ we have

$$\|\tilde{u} - (u^3)_{Q_\varepsilon}\|_{L^\infty(\Omega_\varepsilon)} = \|\tilde{u} - (u^3)_{Q_\varepsilon}\|_{L^\infty(0,1)} \le \|\partial_y(\tilde{u} - (u^3)_{Q_\varepsilon})\|_{L^1(0,1)}$$
$$\le \frac{1}{|Q_\varepsilon|}\|\partial_y(\tilde{u} - u^3)\|_{L^1(\Omega_\varepsilon)} \le \frac{1}{|Q_\varepsilon|^{\frac{1}{2}}}\|\partial_y(\tilde{u} - u^3)\|_{L^2(\Omega_\varepsilon)}.$$

This implies

$$\mathcal{I}_1 \leq \|\partial_y(\tilde{u} - u^3)\|_{L^2(\Omega_\varepsilon)} \left(\frac{1}{|Q_\varepsilon|} \int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) dx\right)^{\frac{1}{2}} |Q_\varepsilon|^{\frac{1}{2}}$$

$$\leq \delta \|\partial_y(\tilde{u} - u^3)\|^2_{L^2(\Omega_\varepsilon)} + K(\delta) \int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) dx.$$

Summarizing the calculations above we get

$$\|[\rho(s(\rho, \theta) - s(\tilde{\rho}, \tilde{\theta}))]_{res}\|_{L^{\frac{4}{3}}(\Omega_\varepsilon)} \leq \left(\frac{1}{|Q_\varepsilon|} \int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) dx\right)^{\frac{1}{2}} |Q_\varepsilon|^{\frac{3}{4}},$$

and thus

$$\mathcal{I}_2 \leq \|(u^3)_{Q_\varepsilon} - u^3\|_{L^4(\Omega_\varepsilon)} \left(\int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) dx\right)^{\frac{1}{2}} |Q_\varepsilon|^{\frac{1}{4}}$$

$$\leq c|Q_\varepsilon|^{\frac{1}{4}} \|\nabla \mathbf{u}\|_{L^2(\Omega_\varepsilon)} \left(\int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) dx\right)^{\frac{1}{2}}$$

$$\leq \varepsilon \|\nabla \mathbf{u}\|^2_{L^2(\Omega_\varepsilon)} + c \int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) dx,$$

where we used the result of Lemma 5.2.

We conclude that

$$\left|\int_{\Omega_\varepsilon} \rho\left(s(\rho, \theta) - s(\tilde{\rho}, \tilde{\theta})\right)(\tilde{u} - u^3)\partial_y\tilde{\theta}dx\right|$$

$$\leq \delta \|\partial_y(\tilde{u} - u^3)\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon \|\nabla \mathbf{u}\|^2_{L^2(\Omega_\varepsilon)}$$

$$+ K(\delta, \cdot) \int_{\Omega_\varepsilon} \left(\frac{1}{2}\rho|u^3 - \tilde{u}|^2 + \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta})\right) dx, \quad (6.6)$$

for any $\delta > 0$. Here and hereafter, $K(\delta, \cdot)$ is a generic constant depending on δ, $\tilde{\rho}$, $\tilde{u}$, $\tilde{\theta}$, $\underline{\rho}$, $\underline{\theta}$ through its respective norms. It is independent of ε.

Step 3 Using (1.4) and (1.5) we get

$$\int_{\Omega_\varepsilon} \rho(\partial_t\tilde{u} + \tilde{u}\partial_y\tilde{u})(\tilde{u} - u^3)dx$$

$$= \int_{\Omega_\varepsilon} \frac{\rho}{\tilde{\rho}}(\tilde{u} - u^3)\left(\partial_y[(\nu_0 + \nu_1\tilde{\theta})\partial_y\tilde{u}] - \partial_y p(\tilde{\rho}, \tilde{\theta})\right) dx$$

$$= \int_{\Omega_\varepsilon} \frac{1}{\tilde{\rho}}(\rho - \tilde{\rho})(\tilde{u} - u^3)\left(\partial_y[(\nu_0 + \nu_1\tilde{\theta})\partial_y\tilde{u}] - \partial_y p(\tilde{\rho}, \tilde{\theta})\right) dx$$

$$+ \int_{\Omega_\varepsilon} (\tilde{u} - u^3)\left(\partial_y[(\nu_0 + \nu_1\tilde{\theta})\partial_y\tilde{u}] - \partial_y p(\tilde{\rho}, \tilde{\theta})\right) dx. \quad (6.7)$$

Due to the regularity of $(\tilde{u}, \tilde{\rho}, \tilde{\theta})$ it follows that

$$\int_{\Omega_\varepsilon} \frac{1}{\tilde{\rho}}(\rho - \tilde{\rho})(\tilde{u} - u^3) \left(\partial_y[(\nu_0 + \nu_1\tilde{\theta})\partial_y\tilde{u}] - \partial_y p(\tilde{\rho}, \tilde{\theta}) \right) dx$$

$$\leq c \int_{\Omega_\varepsilon} |(\rho - \tilde{\rho})(\tilde{u} - u^3)| dx.$$

To estimate $\int_{\Omega_\varepsilon} |(\rho - \tilde{\rho})(\tilde{u} - u^3)| dx$ we introduce the following proposition.

Proposition 6.1. *There exists $c > 0$ independent of ρ and θ such that*

$$\begin{aligned}
\mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) &\geq c|\rho - \tilde{\rho}|^2 & \text{for } \rho \in [\underline{\rho}, \overline{\rho}], \\
\mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) &\geq c|\rho - \tilde{\rho}| & \text{for } \rho < \underline{\rho}, \\
\mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) &\geq c\rho & \text{for } \rho > \overline{\rho}.
\end{aligned} \qquad (6.8)$$

Proof. We use the notation from the proof of Lemma 5.1 in [8], i.e.,

$$\mathcal{F}(\rho) = H^{\tilde{\theta}}(\rho, \tilde{\theta}) - \partial_\rho H^{\tilde{\theta}}(\tilde{\rho}, \tilde{\theta})(\rho - \tilde{\rho}) - H^{\tilde{\theta}}(\tilde{\rho}, \tilde{\theta}),$$

and

$$\mathcal{G}(\rho, \theta) = H^{\tilde{\theta}}(\rho, \theta) - H^{\tilde{\theta}}(\rho, \tilde{\theta}).$$

According to Section 2.2.3 in [8] it holds

$$\mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) = \mathcal{F}(\rho) + \mathcal{G}(\rho, \theta) \geq \mathcal{F}(\rho) + \mathcal{G}(\rho, \tilde{\theta}) \geq \mathcal{F}(\rho),$$

and the function $\mathcal{F}(\rho)$ is strictly convex attaining its minimum 0 when $\rho = \tilde{\rho}$. This immediately implies the first inequality in (6.8).

As far as $\partial_\rho \mathcal{F}(r) \leq -c < 0$ for $r \in \left(0, \frac{3}{2}\rho\right)$, we have

$$\mathcal{F}(\rho) = \mathcal{F}(\rho) - \mathcal{F}\left(\frac{3}{2}\rho\right) + \mathcal{F}\left(\frac{3}{2}\rho\right) - \mathcal{F}(\tilde{\rho}).$$

Since $\mathcal{F}(\rho) - \mathcal{F}\left(\frac{3}{2}\rho\right)$ as well as $\mathcal{F}\left(\frac{3}{2}\rho\right) - \mathcal{F}(\tilde{\rho})$ are positive, we may proceed as follows

$$\mathcal{F}(\rho) = \left| \mathcal{F}(\rho) - \mathcal{F}\left(\frac{3}{2}\rho\right) \right| + \left| \mathcal{F}\left(\frac{3}{2}\rho\right) - \mathcal{F}(\tilde{\rho}) \right| \geq c \left| \rho - \frac{3}{2}\rho \right| \geq |\rho - \tilde{\rho}|.$$

Similarly, $\partial_\rho \mathcal{F}(r) \geq c > 0$ for $r \in (\frac{2}{3}\overline{\rho}, \infty)$ and we deduce

$$\mathcal{F}(\rho) = \mathcal{F}(\rho) - \mathcal{F}(\tilde{\rho}) = \mathcal{F}(\rho) - \mathcal{F}\left(\frac{2}{3}\overline{\rho}\right) + \mathcal{F}\left(\frac{2}{3}\overline{\rho}\right) - \mathcal{F}(\tilde{\rho})$$

$$\geq c\left(\rho - \frac{2}{3}\overline{\rho}\right) + c\left(\frac{2}{3}\overline{\rho} - \tilde{\rho}\right)^2 \geq \frac{c}{3}\rho.$$

$\square$

Now we estimate $\int_{\Omega_\varepsilon} |(\rho - \tilde{\rho})(\tilde{u} - u^3)|dx$. First, we split it into three parts:

$$\int_{\Omega_\varepsilon} |(\rho - \tilde{\rho})(\tilde{u} - u^3)|dx = \int_{\{\underline{\rho} \leq \rho \leq \overline{\rho}\}} |(\rho - \tilde{\rho})(\tilde{u} - u^3)|dx$$

$$+ \int_{\{\rho < \underline{\rho}\}} |(\rho - \tilde{\rho})(\tilde{u} - u^3)|dx + \int_{\{\rho > \overline{\rho}\}} |(\rho - \tilde{\rho})(\tilde{u} - u^3)|dx.$$

Second, using (6.8) we estimate each term as follows:

$$\int_{\{\underline{\rho} \leq \rho \leq \overline{\rho}\}} |(\rho - \tilde{\rho})(\tilde{u} - u^3)|dx \leq \delta \|\tilde{u} - u^3\|^2_{L^2(\Omega_\varepsilon)} + K(\delta, \cdot) \int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta})dx,$$

$$\int_{\{\rho < \underline{\rho}\}} |(\rho - \tilde{\rho})(\tilde{u} - u^3)|dx \leq \delta \int_{\Omega_\varepsilon} |\tilde{u} - u^3|^2 dx + K(\delta, \cdot) \int_{\{\rho < \underline{\rho}\}} |\rho - \tilde{\rho}|^2 \, dx$$

$$\leq \delta \|\tilde{u} - u^3\|^2_{L^2(\Omega_\varepsilon)} + K(\delta, \cdot) \int_{\{\rho < \underline{\rho}\}} |\rho - \tilde{\rho}| \, dx$$

$$\leq \delta \|\tilde{u} - u^3\|^2_{L^2(\Omega_\varepsilon)} + K(\delta, \cdot) \int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta})dx,$$

for any $\delta > 0$ and

$$\int_{\{\rho > \overline{\rho}\}} \frac{|\rho - \tilde{\rho}|}{\sqrt{\rho}} \sqrt{\rho}|\tilde{u} - u^3|dx \leq \int_{\Omega_\varepsilon} \rho|\tilde{u} - u^3|^2 dx + \int_{\{\rho > \overline{\rho}\}} \frac{|\rho - \tilde{\rho}|^2}{\rho} dx$$

$$\leq \int_{\Omega_\varepsilon} \rho|\tilde{u} - u^3|^2 + K(\cdot) \int_{\{\rho > \overline{\rho}\}} \rho dx \leq \int_{\Omega_\varepsilon} \rho|\tilde{u} - u^3|^2 dx + K(\cdot) \int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta})dx.$$

The second integral on the right-hand side of (6.7) is handled by integration by parts as

$$\int_{\Omega_\varepsilon} (\tilde{u} - u^3)\left(\partial_y[(\nu_0 + \nu_1\tilde{\theta})\partial_y\tilde{u}] - \partial_y p(\tilde{\rho}, \tilde{\theta})\right) dx$$

$$= \int_{\Omega_\varepsilon} (\nu_0 + \nu_1\tilde{\theta})\partial_y\tilde{u}\partial_y(u^3 - \tilde{u}) + p(\tilde{\rho}, \tilde{\theta})\partial_y(\tilde{u} - u^3)dx.$$

Thus we arrive at

$$\int_{\Omega_\varepsilon} \rho(\partial_t\tilde{u} + \tilde{u}\partial_y\tilde{u})(\tilde{u} - u^3)dx \leq \int_{\Omega_\varepsilon} (\nu_0 + \nu_1\tilde{\theta})\partial_y\tilde{u}\partial_y(u^3 - \tilde{u}) + p(\tilde{\rho}, \tilde{\theta})\partial_y(\tilde{u} - u^3)dx$$

$$+ \delta\|u^3 - \tilde{u}\|^2_{L^2(\Omega_\varepsilon)} + K(\delta, \cdot) \int_{\Omega_\varepsilon} \left(\frac{1}{2}\rho|\tilde{u} - u^3|^2 + \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta})\right) dx, \qquad (6.9)$$

for any $\delta > 0$.

Step 4 We calculate

$$\int_{\Omega_\varepsilon} \rho\left(s(\rho,\theta) - s(\tilde\rho,\tilde\theta)\right)(\partial_t\tilde\theta + \tilde u\partial_y\tilde\theta)dx$$

$$= \int_{\Omega_\varepsilon} \rho\Big[s(\rho,\theta) - s(\tilde\rho,\tilde\theta)\Big]_{ess}(\partial_t\tilde\theta + \tilde u\partial_y\tilde\theta)dx + \int_{\Omega_\varepsilon} \rho\Big[s(\rho,\theta) - s(\tilde\rho,\tilde\theta)\Big]_{res}(\partial_t\tilde\theta + \tilde u\partial_y\tilde\theta)dx.$$

By a combination of (5.1) and (5.5) we get

$$\left|\int_{\Omega_\varepsilon} \rho\Big[s(\rho,\theta) - s(\tilde\rho,\tilde\theta)\Big]_{res}(\partial_t\tilde\theta + \tilde u\partial_y\tilde\theta)dx\right| \le K(\cdot)\int_{\Omega_\varepsilon} \mathcal{E}(\rho,\theta|\tilde\rho,\tilde\theta)dx,$$

while

$$\int_{\Omega_\varepsilon} \rho\Big[s(\rho,\theta) - s(\tilde\rho,\tilde\theta)\Big]_{ess}(\partial_t\tilde\theta + \tilde u\partial_y\tilde\theta)dx = \int_{\Omega_\varepsilon} (\rho - \tilde\rho)\Big[s(\rho,\theta) - s(\tilde\rho,\tilde\theta)\Big]_{ess}(\partial_t\tilde\theta + \tilde u\partial_y\tilde\theta)dx$$

$$+ \int_{\Omega_\varepsilon} \tilde\rho\Big[s(\rho,\theta) - s(\tilde\rho,\tilde\theta)\Big]_{ess}(\partial_t\tilde\theta + \tilde u\partial_y\tilde\theta)dx,$$

where with the help of the Taylor-Lagrange formula and (5.2),

$$\left|\int_{\Omega_\varepsilon} (\rho - \tilde\rho)\Big[s(\rho,\theta) - s(\tilde\rho,\tilde\theta)\Big]_{ess}(\partial_t\tilde\theta + \tilde u\partial_y\tilde\theta)dx\right|$$

$$\le \left(\sup_{(\rho,\theta)\in[\underline\rho,\overline\rho]\times[\underline\theta,\overline\theta]} |\partial_\rho s(\rho,\theta)| + \sup_{(\rho,\theta)\in[\underline\rho,\overline\rho]\times[\underline\theta,\overline\theta]} |\partial_\theta s(\rho,\theta)|\right)\|\partial_t\tilde\theta + \tilde u\partial_y\tilde\theta\|_{L^\infty(0,1)}$$

$$\times \int_{\Omega_\varepsilon} |[\rho - \tilde\rho]_{ess}|(|[\rho - \tilde\rho]_{ess}| + |[\theta - \tilde\theta]_{ess}|)dx \le K(\cdot)\int_{\Omega_\varepsilon} \mathcal{E}(\rho,\theta|\tilde\rho,\tilde\theta)dx. \quad (6.10)$$

Finally, we write

$$\int_{\Omega_\varepsilon} \tilde\rho\Big[s(\rho,\theta) - s(\tilde\rho,\tilde\theta)\Big]_{ess}(\partial_t\tilde\theta + \tilde u\partial_y\tilde\theta)dx$$

$$= \int_{\Omega_\varepsilon} \tilde\rho\Big[s(\rho,\theta) - \partial_\rho s(\tilde\rho,\tilde\theta)(\rho - \tilde\rho) - \partial_\theta s(\tilde\rho,\tilde\theta)(\theta - \tilde\theta) - s(\tilde\rho,\tilde\theta)\Big]_{ess}(\partial_t\tilde\theta + \tilde u\partial_y\tilde\theta)dx$$

$$- \int_{\Omega_\varepsilon} \tilde\rho\Big[\partial_\rho s(\tilde\rho,\tilde\theta)(\rho - \tilde\rho) + \partial_\theta s(\tilde\rho,\tilde\theta)(\theta - \tilde\theta)\Big]_{res}(\partial_t\tilde\theta + \tilde u\partial_y\tilde\theta)dx$$

$$+ \int_{\Omega_\varepsilon} \tilde\rho\left(\partial_\rho s(\tilde\rho,\tilde\theta)(\rho - \tilde\rho) + \partial_\theta s(\tilde\rho,\tilde\theta)(\theta - \tilde\theta)\right)(\partial_t\tilde\theta + \tilde u\partial_y\tilde\theta)dx,$$

where the first integral on the right-hand side can be estimated using the Taylor-Lagrange formula of the second order and (5.2) (similarly to (6.10)) and the second integral on the right-hand side can be estimated by (5.3) and (5.5).

Thus we conclude that

$$-\int_{\Omega_\varepsilon} \rho \left(s(\rho, \theta) - s(\tilde\rho, \tilde\theta) \right) (\partial_t \tilde\theta + \tilde u \partial_y \tilde\theta) dx \le K(\cdot) \int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde\rho, \tilde\theta) dx$$

$$-\int_{\Omega_\varepsilon} \tilde\rho \left(\partial_\rho s(\tilde\rho, \tilde\theta)(\rho - \tilde\rho) + \partial_\theta s(\tilde\rho, \tilde\theta)(\theta - \tilde\theta) \right) (\partial_t \tilde\theta + \tilde u \partial_y \tilde\theta) dx. \qquad (6.11)$$

Step 5 By the integration by parts

$$\int_{\Omega_\varepsilon} \left(\left(1 - \frac{\rho}{\tilde\rho} \right) \partial_t p(\tilde\rho, \tilde\theta) - \frac{\rho}{\tilde\rho} \mathbf{u} \cdot \nabla_x p(\tilde\rho, \tilde\theta) \right) dx$$

$$= \int_{\Omega_\varepsilon} \left(\left(1 - \frac{\rho}{\tilde\rho} \right) \partial_t p(\tilde\rho, \tilde\theta) - \frac{\rho}{\tilde\rho} u^3 \partial_y p(\tilde\rho, \tilde\theta) \right) dx$$

$$= \int_{\Omega_\varepsilon} (\tilde\rho - \rho) \frac{1}{\tilde\rho} \left(\partial_t p(\tilde\rho, \tilde\theta) + \tilde u \partial_y p(\tilde\rho, \tilde\theta) \right) dx + \int_{\Omega_\varepsilon} p(\tilde\rho, \tilde\theta) \partial_y u^3 dx$$

$$+ \int_{\Omega_\varepsilon} (\tilde\rho - \rho) \frac{1}{\tilde\rho} \partial_y p(\tilde\rho, \tilde\theta)(u^3 - \tilde u) dx,$$

where, by means of the same arguments as in Step 3,

$$\left| \int_{\Omega_\varepsilon} (\tilde\rho - \rho) \frac{1}{\tilde\rho} \partial_y p(\tilde\rho, \tilde\theta)(u^3 - \tilde u) dx \right|$$

$$\le \delta \| u^3 - \tilde u \|_{L^2(\Omega_\varepsilon)}^2 + K(\delta, \cdot) \int_{\Omega_\varepsilon} \left(\frac{1}{2} \rho |u^3 - \tilde u|^2 + \mathcal{E}(\rho, \theta | \tilde\rho, \tilde\theta) \right) dx,$$

for any $\delta > 0$. Using this estimate we get

$$\int_{\Omega_\varepsilon} \left(\left(1 - \frac{\rho}{\tilde\rho} \right) \partial_t p(\tilde\rho, \tilde\theta) - \frac{\rho}{\tilde\rho} \mathbf{u} \cdot \nabla_x p(\tilde\rho, \tilde\theta) \right) dx$$

$$\le \int_{\Omega_\varepsilon} (\tilde\rho - \rho) \frac{1}{\tilde\rho} \left(\partial_t p(\tilde\rho, \tilde\theta) + \tilde u \partial_y p(\tilde\rho, \tilde\theta) \right) dx + \int_{\Omega_\varepsilon} p(\tilde\rho, \tilde\theta) \partial_y u^3 dx$$

$$+ \delta \| u^3 - \tilde u \|_{L^2(\Omega_\varepsilon)}^2 + K(\delta, \cdot) \int_{\Omega_\varepsilon} \left(\frac{1}{2} \rho |u^3 - \tilde u|^2 + \mathcal{E}(\rho, \theta | \tilde\rho, \tilde\theta) \right) dx, \quad (6.12)$$

for any $\delta > 0$.

Step 6 Summing up the estimates (6.3), (6.4), (6.6), (6.9), (6.11) and (6.12) we can rewrite the relative entropy inequality (6.2) in the form

$$\frac{1}{|Q_\varepsilon|} \int_{\Omega_\varepsilon} \left(\frac{1}{2} \rho |\mathbf{u} - \tilde{\mathbf{u}}|^2 + \mathcal{E}(\rho, \theta | \tilde\rho, \tilde\theta) \right) (\tau, \cdot) dx$$

$$+ \frac{1}{|Q_\varepsilon|} \int_0^\tau \int_{\Omega_\varepsilon} \left(\frac{\tilde\theta}{\theta} \mathbb{S}(\theta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - (\nu_0 + \nu_1 \tilde\theta) \partial_y \tilde u \partial_y (u^3 - \tilde u) - \mathbb{S}(\theta, \nabla_x \mathbf{u}) : \nabla_x \tilde{\mathbf{u}} \right) dx dt$$

$$+\frac{1}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega_\varepsilon}\left(\frac{\mathbf{q}(\theta,\nabla_x\theta)}{\theta}\cdot\nabla_x\tilde\theta-\frac{\tilde\theta}{\theta}\frac{\mathbf{q}(\theta,\nabla_x\theta)\cdot\nabla_x\theta}{\theta}\right)dxdt\le\Gamma(\varepsilon)$$

$$+\frac{1}{|Q_\varepsilon|}\int_0^\tau\left[\delta\|u^3-\tilde u\|^2_{L^2(\Omega_\varepsilon)}+K(\delta,\cdot)\int_{\Omega_\varepsilon}\left(\frac12\rho|u^3-\tilde u|^2+\mathcal{E}(\rho,\theta|\tilde\rho,\tilde\theta)\right)dx\right]dt$$

$$+\frac{1}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega_\varepsilon}\left(\delta\|\partial_y(u^3-\tilde u)\|^2_{L^2(\Omega_\varepsilon)}+\varepsilon\|\nabla\mathbf{u}\|^2_{L^2(\Omega_\varepsilon)}\right)dxdt$$

$$+\frac{1}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega_\varepsilon}\left(p(\tilde\rho,\tilde\theta)-p(\rho,\theta)\right)\partial_y\tilde u\,dxdt$$

$$+\frac{1}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega_\varepsilon}(\tilde\rho-\rho)\frac1{\tilde\rho}\left(\partial_tp(\tilde\rho,\tilde\theta)+\tilde u\partial_yp(\tilde\rho,\tilde\theta)\right)dxdt$$

$$-\frac{1}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega_\varepsilon}\tilde\rho\left(\partial_\rho s(\tilde\rho,\tilde\theta)(\rho-\tilde\rho)+\partial_\theta s(\tilde\rho,\tilde\theta)(\theta-\tilde\theta)\right)(\partial_t\tilde\theta+\tilde u\partial_y\tilde\theta)dxdt,$$

$$(6.13)$$

for almost all $\tau\in[0,T]$ and any $\delta>0$. Recall that $\Gamma(\varepsilon)\to0$ as $\varepsilon\to0$.

Step 7 Our next goal is to control the last three integrals on the right-hand side of (6.13). To this end, we recall a useful identity that follows directly from the Gibbs' equation (2.9):

$$r\partial_\rho s(r,\Theta)=-\frac1r\partial_\theta p(r,\Theta).\qquad(6.14)$$

Using (6.14) we obtain

$$\int_{\Omega_\varepsilon}(\tilde\rho-\rho)\frac1{\tilde\rho}\left(\partial_tp(\tilde\rho,\tilde\theta)+\tilde u\partial_yp(\tilde\rho,\tilde\theta)\right)dx$$

$$-\int_{\Omega_\varepsilon}\tilde\rho\left(\partial_\rho s(\tilde\rho,\tilde\theta)(\rho-\tilde\rho)+\partial_\theta s(\tilde\rho,\tilde\theta)(\theta-\tilde\theta)\right)(\partial_t\tilde\theta+\tilde u\partial_y\tilde\theta)dx$$

$$=\int_{\Omega_\varepsilon}\tilde\rho(\tilde\theta-\theta)\partial_\theta s(\tilde\rho,\tilde\theta)(\partial_t\tilde\theta+\tilde u\partial_y\tilde\theta)dx+\int_{\Omega_\varepsilon}(\tilde\rho-\rho)\frac1{\tilde\rho}\partial_\rho p(\tilde\rho,\tilde\theta)(\partial_t\tilde\rho+\tilde u\partial_y\tilde\rho)dx.$$

Since $\tilde\rho$ and $\tilde u$ satisfy the equation of continuity (1.4) we get

$$\int_{\Omega_\varepsilon}(\tilde\rho-\rho)\frac1{\tilde\rho}\partial_\rho p(\tilde\rho,\tilde\theta)(\partial_t\tilde\rho+\tilde u\partial_y\tilde\rho)dx=-\int_{\Omega_\varepsilon}(\tilde\rho-\rho)\partial_\rho p(\tilde\rho,\tilde\theta)\partial_y\tilde u\,dx.$$

By using (6.14) and (1.4) once more, followed by the use of the entropy equation (1.6) we deduce that

$$\int_{\Omega_\varepsilon}\tilde\rho(\tilde\theta-\theta)\partial_\theta s(\tilde\rho,\tilde\theta)(\partial_t\tilde\theta+\tilde u\partial_y\tilde\theta)dx$$

$$= \int_{\Omega_\varepsilon} \tilde{\rho}(\tilde{\theta} - \theta) \left[\partial_t s(\tilde{\rho}, \tilde{\theta}) + \tilde{u}\partial_y s(\tilde{\rho}, \tilde{\theta}) \right] dx - \int_{\Omega_\varepsilon} (\tilde{\theta} - \theta)\partial_\theta p(\tilde{\rho}, \tilde{\theta})\partial_y \tilde{u} dx$$

$$= \int_{\Omega_\varepsilon} (\tilde{\theta} - \theta) \left[\frac{1}{\tilde{\theta}} \left((\nu_0 + \nu_1\tilde{\theta})(\partial_y \tilde{u})^2 - \frac{q(\tilde{\theta}, \partial_y\tilde{\theta})\partial_y\tilde{\theta}}{\tilde{\theta}} \right) - \partial_y \left(\frac{q(\tilde{\theta}, \partial_y\tilde{\theta})}{\tilde{\theta}} \right) \right] dx$$

$$- \int_{\Omega_\varepsilon} (\tilde{\theta} - \theta)\partial_\theta p(\tilde{\rho}, \tilde{\theta})\partial_y \tilde{u} dx.$$

We use the Taylor-Lagrange formula together with (5.2) on the essential part; (5.1) with $p(\rho,\theta) \leq c\rho e(\rho,\theta)$ and (5.3), (5.5) on the residual part to obtain

$$\left| \int_{\Omega_\varepsilon} \left(p(\tilde{\rho}, \tilde{\theta}) - \partial_\rho p(\tilde{\rho}, \tilde{\theta})(\tilde{\rho} - \rho) - \partial_\theta p(\tilde{\rho}, \tilde{\theta})(\tilde{\theta} - \theta) - p(\rho, \theta) \right) \partial_y \tilde{u} dx \right|$$

$$\leq K(\cdot) \int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) dx.$$

Finally, the integration by parts with (3.2) allows us to rewrite (6.13) in the form

$$\frac{1}{|Q_\varepsilon|} \int_{\Omega_\varepsilon} \left(\frac{1}{2}\rho|\mathbf{u} - \tilde{\mathbf{u}}|^2 + \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) \right) (\tau, \cdot) dx$$

$$+ \frac{1}{|Q_\varepsilon|} \int_0^\tau \int_{\Omega_\varepsilon} \left(\frac{\tilde{\theta}}{\theta}\mathbb{S}(\theta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - (\nu_0 + \nu_1\tilde{\theta})\partial_y \tilde{u}\partial_y(u^3 - \tilde{u}) - \mathbb{S}(\theta, \nabla_x \mathbf{u}) : \nabla_x \tilde{\mathbf{u}} \right.$$

$$\left. - \frac{\tilde{\theta} - \theta}{\tilde{\theta}}(\nu_0 + \nu_1\tilde{\theta})(\partial_y\tilde{u})^2 \right) dx dt$$

$$+ \frac{1}{|Q_\varepsilon|} \int_0^\tau \int_{\Omega_\varepsilon} \left(\frac{\mathbf{q}(\theta, \nabla_x\theta) \cdot \nabla_x\tilde{\theta}}{\theta} - \frac{\tilde{\theta}}{\theta}\frac{\mathbf{q}(\theta, \nabla_x\theta) \cdot \nabla_x\theta}{\theta} \right.$$

$$\left. + (\tilde{\theta} - \theta)\frac{q(\tilde{\theta}, \partial_y\tilde{\theta})\partial_y\tilde{\theta}}{\tilde{\theta}^2} + \partial_y(\theta - \tilde{\theta})\frac{q(\tilde{\theta}, \partial_y\tilde{\theta})}{\tilde{\theta}} \right) dx dt$$

$$\leq \Gamma(\varepsilon) + \frac{1}{|Q_\varepsilon|} \int_0^\tau \left[\delta\|u^3 - \tilde{u}\|_{L^2(\Omega_\varepsilon)}^2 + K(\delta, \cdot) \int_{\Omega_\varepsilon} \left(\frac{1}{2}\rho|u^3 - \tilde{u}|^2 + \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) \right) dx \right] dt$$

$$+ \frac{1}{|Q_\varepsilon|} \int_0^\tau \int_{\Omega_\varepsilon} \left(\delta\|\partial_y(u^3 - \tilde{u})\|_{L^2(\Omega_\varepsilon)}^2 + \varepsilon\|\nabla\mathbf{u}\|_{L^2(\Omega_\varepsilon)}^2 \right) dx dt, \qquad (6.15)$$

for almost all $\tau \in [0, T]$ and any $\delta > 0$.

7. Dissipative terms

The goal of this section is to show that the "dissipative" terms on the left-hand side of (6.15) containing $\nabla_x \mathbf{u}$ are strong enough to control the velocity terms on the right-hand side.

7.1. Viscosity

In accordance with hypotheses (2.1) and (2.2) we have

$$\mathbb{S}(\theta, \nabla_x \mathbf{v}) = \mathbb{S}^0(\nabla_x \mathbf{v}) + \theta \mathbb{S}^1(\nabla_x \mathbf{v}),$$

with

$$\mathbb{S}^i(\nabla_x \mathbf{v}) = \mu_i(\nabla_x \mathbf{v} + \nabla_x^T \mathbf{v} - \frac{2}{3}\mathrm{div}_x \mathbf{v} \mathbb{I}) + \eta_i \mathrm{div}_x \mathbf{v} \mathbb{I}, \quad i = 0, 1.$$

Using Lemma 5.1 we immediately obtain

$$\int_{\Omega_\varepsilon} \mathbb{S}^i(\nabla_x \mathbf{v}) : \nabla_x \mathbf{v}\, dx \geq \int_{\Omega_\varepsilon} \mu_i \left|\nabla_x \mathbf{v}\right|^2 + \eta_i \left|\mathrm{div}_x \mathbf{v}\right|^2 \, dx, \qquad (7.1)$$

for $i = 0, 1$.

7.2. Viscosity terms

The terms to be dealt with are

$$\int_{\Omega_\varepsilon} \left(\frac{\tilde{\theta}}{\theta}\mathbb{S}(\theta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - (\nu_0 + \nu_1 \tilde{\theta})\partial_y \tilde{u}\partial_y(u^3 - \tilde{u}) \right.$$

$$\left. - \mathbb{S}(\theta, \nabla_x \mathbf{u}) : \nabla_x \tilde{\mathbf{u}} - \frac{\tilde{\theta} - \theta}{\tilde{\theta}}(\nu_0 + \nu_1 \tilde{\theta})(\partial_y \tilde{u})^2 \right) dx.$$

First we deal with the "$\theta \mathbb{S}^1(\nabla_x \mathbf{u})$" part:

$$\frac{\tilde{\theta}}{\theta}\theta \mathbb{S}^1(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \tilde{\theta}\mathbb{S}^1(\nabla_x \tilde{\mathbf{u}}) : (\nabla_x \mathbf{u} - \nabla_x \tilde{\mathbf{u}}) - \theta \mathbb{S}^1(\nabla_x \mathbf{u}) : \nabla_x \tilde{\mathbf{u}} - \frac{\tilde{\theta} - \theta}{\tilde{\theta}}\tilde{\theta}\mathbb{S}^1(\nabla_x \tilde{\mathbf{u}}) : \nabla_x \tilde{\mathbf{u}}$$

$$= \tilde{\theta}\left(\mathbb{S}^1(\nabla_x \mathbf{u}) - \mathbb{S}^1(\nabla_x \tilde{\mathbf{u}}) \right) : (\nabla_x \mathbf{u} - \nabla_x \tilde{\mathbf{u}}) + (\tilde{\theta} - \theta)\left(\mathbb{S}^1(\nabla_x \mathbf{u}) - \mathbb{S}^1(\nabla_x \tilde{\mathbf{u}}) \right) : \nabla_x \tilde{\mathbf{u}}.$$

Using (7.1) we obtain

$$\int_{\Omega_\varepsilon} \tilde{\theta}\left(\mathbb{S}^1(\nabla_x \mathbf{u}) - \mathbb{S}^1(\nabla_x \tilde{\mathbf{u}}) \right) : (\nabla_x \mathbf{u} - \nabla_x \tilde{\mathbf{u}})dx \geq \mu_1 \underline{\theta} \int_{\Omega_\varepsilon} \left|\nabla_x(\mathbf{u} - \tilde{\mathbf{u}})\right|^2 dx.$$

Since

$$u^3(\cdot, y) = \tilde{u}(\cdot, y) = 0 \text{ for } y = 0, 1,$$

we see that

$$\int_{\Omega_\varepsilon} \tilde{\theta} \left(\mathbb{S}^1(\nabla_x \mathbf{u}) - \mathbb{S}^1(\nabla_x \tilde{\mathbf{u}}) \right) : (\nabla_x \mathbf{u} - \nabla_x \tilde{\mathbf{u}}) dx$$

$$\geq c \left(\|u^3 - \tilde{u}\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla_x(\mathbf{u} - \tilde{\mathbf{u}})\|^2_{L^2(\Omega_\varepsilon)} \right).$$

Similarly using (5.4) we can show that

$$\left| \int_{\Omega_\varepsilon} (\tilde{\theta} - \theta) \left(\mathbb{S}^1(\nabla_x \mathbf{u}) - \mathbb{S}^1(\nabla_x \tilde{\mathbf{u}}) \right) : \nabla_x \tilde{\mathbf{u}} dx \right|$$

$$\leq \delta \left(\|\partial_y(u^3 - \tilde{u})\|^2_{L^2(\Omega_\varepsilon)} + \|\mathrm{div}_x(\mathbf{u} - \tilde{\mathbf{u}})\|^2_{L^2(\Omega_\varepsilon)} \right) + K(\delta, \cdot)\|\theta - \tilde{\theta}\|^2_{L^2(\Omega_\varepsilon)}, \tag{7.2}$$

for any $\delta > 0$.

Since

$$\mathbb{S}^i(\nabla_x \tilde{\mathbf{u}}) : \nabla_x \tilde{\mathbf{u}} = \nu_i(\partial_y \tilde{u})^2, \ i = 0, 1, \tag{7.3}$$

we have that

$$\frac{\tilde{\theta} - \theta}{\tilde{\theta}} \tilde{\theta} \mathbb{S}^1(\nabla_x \tilde{\mathbf{u}}) : \nabla_x \tilde{\mathbf{u}} = \frac{\tilde{\theta} - \theta}{\tilde{\theta}} \nu_1 \tilde{\theta}(\partial_y \tilde{u})^2.$$

Finally, we estimate the difference

$$\int_{\Omega_\varepsilon} \left(\tilde{\theta} \mathbb{S}^1(\nabla_x \tilde{\mathbf{u}}) : (\nabla_x \mathbf{u} - \nabla_x \tilde{\mathbf{u}}) - \nu_1 \tilde{\theta} \partial_y \tilde{u} \partial_y(u^3 - \tilde{u}) \right) dx.$$

Proceeding with the integration by parts

$$\int_{\Omega_\varepsilon} \tilde{\theta} \mathbb{S}^1(\nabla_x \tilde{\mathbf{u}}) : (\nabla_x \mathbf{u} - \nabla_x \tilde{\mathbf{u}}) dx = -\int_{\Omega_\varepsilon} \mathrm{div}_x[\tilde{\theta} \mathbb{S}^1(\nabla_x \tilde{\mathbf{u}})] \cdot (\mathbf{u} - \tilde{\mathbf{u}}) dx$$

$$+ \int_{\partial\Omega_\varepsilon} \tilde{\theta} \mathbb{S}^1(\nabla_x \tilde{\mathbf{u}})(\mathbf{u} - \tilde{\mathbf{u}})\mathbf{n} dS.$$

Since

$$-\int_{\Omega_\varepsilon} \mathrm{div}_x[\tilde{\theta} \mathbb{S}^1(\nabla_x \tilde{\mathbf{u}})] \cdot (\mathbf{u} - \tilde{\mathbf{u}}) dx = -\int_{\Omega_\varepsilon} \partial_y[\tilde{\theta} \nu_1 \partial_y \tilde{u}](u^3 - \tilde{u}) dx$$

$$= \int_{\Omega_\varepsilon} \tilde{\theta} \nu_1 \partial_y \tilde{u} \partial_y(u^3 - \tilde{u}) dx,$$

is is only left to estimate the boundary integral. By the same arguments that lead to (7.5) later, we see that

$$\int_{\partial\Omega_\varepsilon} \tilde{\theta} \mathbb{S}^1(\nabla_x \tilde{\mathbf{u}})(\mathbf{u} - \tilde{\mathbf{u}})\mathbf{n} dS = 0.$$

Thus we get

$$\int_{\Omega_\varepsilon} \tilde{\theta} \mathbb{S}^1(\nabla_x \tilde{\mathbf{u}}) : (\nabla_x \mathbf{u} - \nabla_x \tilde{\mathbf{u}}) dx = \int_{\Omega_\varepsilon} \nu_1 \tilde{\theta} \partial_y \tilde{u} \partial_y(u^3 - \tilde{u}) dx.$$

Now we treat the "$\mathbb{S}^0(\nabla_x \mathbf{u})$" part: We first suppose that $\theta \geq \tilde{\theta}$ and calculate

$$\frac{\tilde{\theta}}{\theta} \mathbb{S}^0(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \mathbb{S}^0(\nabla_x \tilde{\mathbf{u}}) : (\nabla_x \mathbf{u} - \nabla_x \tilde{\mathbf{u}}) - \mathbb{S}^0(\nabla_x \mathbf{u}) : \nabla_x \tilde{\mathbf{u}}$$

$$-\frac{\tilde{\theta} - \theta}{\tilde{\theta}} \mathbb{S}^0(\nabla_x \tilde{\mathbf{u}}) : \nabla_x \tilde{\mathbf{u}} \geq \frac{\tilde{\theta}}{\theta}(\mathbb{S}^0(\nabla_x \mathbf{u}) - \mathbb{S}^0(\nabla_x \tilde{\mathbf{u}})) : \nabla_x(\mathbf{u} - \tilde{\mathbf{u}})$$

$$+\tilde{\theta}\left(\frac{1}{\theta} - \frac{1}{\tilde{\theta}}\right) \mathbb{S}^0(\nabla_x \tilde{\mathbf{u}}) : \nabla_x(\mathbf{u} - \tilde{\mathbf{u}}) + \frac{\tilde{\theta} - \theta}{\theta}(\mathbb{S}^0(\nabla_x \mathbf{u}) - \mathbb{S}^0(\nabla_x \tilde{\mathbf{u}})) : \nabla_x \tilde{\mathbf{u}}.$$

Since the function $\theta \mapsto \frac{1}{\theta}$ is Lipschitz on the set $\theta \geq \tilde{\theta}$, we conclude that

$$\int_{\{\theta \geq \tilde{\theta}\}} \left| \tilde{\theta}\left(\frac{1}{\theta} - \frac{1}{\tilde{\theta}}\right) \mathbb{S}^0(\nabla_x \tilde{\mathbf{u}}) : \nabla_x(\mathbf{u} - \tilde{\mathbf{u}}) \right| dx$$

$$\leq K(\cdot) \|\partial_y \tilde{u}\|_{L^\infty(0,1)} \frac{\|\tilde{\theta}\|_{L^\infty(0,1)}}{\underline{\theta}^2} \int_{\Omega_\varepsilon} |\theta - \tilde{\theta}|(|\partial_y(u^3 - \tilde{u})| + |\text{div}_x(\mathbf{u} - \tilde{\mathbf{u}})|) dx$$

$$\leq \delta \left(\|\partial_y(u^3 - \tilde{u})\|_{L^2(\Omega_\varepsilon)}^2 + \|\text{div}_x(\mathbf{u} - \tilde{\mathbf{u}})\|_{L^2(\Omega_\varepsilon)}^2 \right) + K(\delta, \cdot) \|\theta - \tilde{\theta}\|_{L^2(\Omega_\varepsilon)}^2,$$
$$(7.4)$$

for any $\delta > 0$. Analogously, we obtain a similar estimate for the last term,

$$\int_{\{\theta \geq \tilde{\theta}\}} \left| \frac{\tilde{\theta} - \theta}{\theta}(\mathbb{S}^0(\nabla_x \mathbf{u}) - \mathbb{S}^0(\nabla_x \tilde{\mathbf{u}})) : \nabla_x \tilde{\mathbf{u}} \right| dx$$

$$\leq \delta \left(\|\partial_y(u^3 - \tilde{u})\|_{L^2(\Omega_\varepsilon)}^2 + \|\text{div}_x(\mathbf{u} - \tilde{\mathbf{u}})\|_{L^2(\Omega_\varepsilon)}^2 \right) + K(\delta, \cdot) \|\theta - \tilde{\theta}\|_{L^2(\Omega_\varepsilon)}^2,$$

for any $\delta > 0$.

Next, if $0 < \theta \leq \tilde{\theta}$, we have

$$\frac{\tilde{\theta}}{\theta} \mathbb{S}^0(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \mathbb{S}^0(\nabla_x \tilde{\mathbf{u}}) : (\nabla_x \mathbf{u} - \nabla_x \tilde{\mathbf{u}}) - \mathbb{S}^0(\nabla_x \mathbf{u}) : \nabla_x \tilde{\mathbf{u}} - \frac{\tilde{\theta} - \theta}{\tilde{\theta}} \mathbb{S}^0(\nabla_x \tilde{\mathbf{u}}) : \nabla_x \tilde{\mathbf{u}}$$

$$\geq (\mathbb{S}^0(\nabla_x \mathbf{u}) - \mathbb{S}^0(\nabla_x \tilde{\mathbf{u}})) : \nabla_x(\mathbf{u} - \tilde{\mathbf{u}}) + \frac{\tilde{\theta} - \theta}{\tilde{\theta}}[\mathbb{S}^0(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \mathbb{S}^0(\nabla_x \tilde{\mathbf{u}}) : \nabla_x \tilde{\mathbf{u}}],$$

using $(\theta - \tilde{\theta})^2 \geq 0$ and whence, by means of convexity of the function $\nabla_x \mathbf{u} \mapsto \mathbb{S}^0(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}$ we get

$$\frac{\tilde{\theta} - \theta}{\tilde{\theta}}[\mathbb{S}^0(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \mathbb{S}^0(\nabla_x \tilde{\mathbf{u}}) : \nabla_x \tilde{\mathbf{u}}] \geq 2\frac{\tilde{\theta} - \theta}{\tilde{\theta}}\mathbb{S}^0(\nabla_x \tilde{\mathbf{u}}) : \nabla_x (\mathbf{u} - \tilde{\mathbf{u}}),$$

where similarly as in (7.4), we can estimate

$$\left| \int_{\Omega_\varepsilon} \frac{\tilde{\theta} - \theta}{\tilde{\theta}}\mathbb{S}^0(\nabla_x \tilde{\mathbf{u}}) : \nabla_x (\mathbf{u} - \tilde{\mathbf{u}})dx \right|$$

$$\leq \delta \left(\|\partial_y (u^3 - \tilde{u})\|^2_{L^2(\Omega_\varepsilon)} + \|\text{div}_x (\mathbf{u} - \tilde{\mathbf{u}})\|^2_{L^2(\Omega_\varepsilon)} \right) + K(\delta, \cdot)\|\theta - \tilde{\theta}\|^2_{L^2(\Omega_\varepsilon)},$$

for any $\delta > 0$.

Finally, we estimate the "extra terms". Thanks to (7.3) we immediately get that

$$\frac{\tilde{\theta} - \theta}{\tilde{\theta}}\mathbb{S}^0(\nabla_x \tilde{\mathbf{u}}) : \nabla_x \tilde{\mathbf{u}} = \frac{\tilde{\theta} - \theta}{\tilde{\theta}}\nu_0 (\partial_y \tilde{u})^2.$$

Using the integration by parts we calculate

$$\int_{\Omega_\varepsilon} \mathbb{S}^0(\nabla_x \tilde{\mathbf{u}}) : (\nabla_x \mathbf{u} - \nabla_x \tilde{\mathbf{u}})dx =$$

$$- \int_{\Omega_\varepsilon} \text{div}_x \mathbb{S}^0(\nabla_x \tilde{\mathbf{u}}) \cdot (\mathbf{u} - \tilde{\mathbf{u}})dx + \int_{\partial\Omega_\varepsilon} \mathbb{S}^0(\nabla_x \tilde{\mathbf{u}})(\mathbf{u} - \tilde{\mathbf{u}})\mathbf{n}dS.$$

Since

$$(\mathbf{u} - \tilde{\mathbf{u}}) \cdot \mathbb{S}^i(\nabla_x \tilde{\mathbf{u}})\mathbf{n} = \partial_y \tilde{u} \begin{pmatrix} (\eta_i - \frac{2}{3}\mu_i)n^1 \\ (\eta_i - \frac{2}{3}\mu_i)n^2 \\ \nu_i n^3 \end{pmatrix} \cdot \begin{pmatrix} u^1 \\ u^2 \\ u^3 - \tilde{u} \end{pmatrix}, \text{ with } \mathbf{n} = \begin{pmatrix} n^1 \\ n^2 \\ n^3 \end{pmatrix},$$

for $i = 0, 1$ and the fact that

$$\mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ n^3 \end{pmatrix},$$

on $Q_\varepsilon \times \{0\}$ and $Q_\varepsilon \times \{1\}$ and

$$\mathbf{n} = \begin{pmatrix} n^1 \\ n^2 \\ 0 \end{pmatrix},$$

on $\partial Q_\varepsilon \times (0, 1)$, we get from the boundary conditions $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0$ and $\tilde{u}(\cdot, 0) = \tilde{u}(\cdot, 1) = 0$ that

$$\int_{\partial\Omega_\varepsilon} \mathbb{S}^0(\nabla_x \tilde{\mathbf{u}})(\mathbf{u} - \tilde{\mathbf{u}})\mathbf{n}dS = 0. \tag{7.5}$$

Finally, it is straightforward to see that

$$-\int_{\Omega_\varepsilon} \mathrm{div}_x \mathbb{S}^0(\nabla_x\tilde{\mathbf{u}})\cdot(\mathbf{u}-\tilde{\mathbf{u}})dx = -\int_{\Omega_\varepsilon}\nu_0\partial_y^2\tilde{u}(u^3-\tilde{u})dx = \int_{\Omega_\varepsilon}\nu_0\partial_y\tilde{u}\partial_y(u^3-\tilde{u})dx.$$

Hence,

$$\int_{\Omega_\varepsilon}\mathbb{S}^0(\nabla_x\tilde{\mathbf{u}}):(\nabla_x\mathbf{u}-\nabla_x\tilde{\mathbf{u}})dx = \int_{\Omega_\varepsilon}\nu_0\partial_y\tilde{u}\partial_y(u^3-\tilde{u})dx.$$

Summing up the results of this section together with (5.4) we can rewrite the relation (6.15) as

$$\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\left(\frac{1}{2}\rho|\mathbf{u}-\tilde{\mathbf{u}}|^2 + \mathcal{E}(\rho,\theta|\tilde{\rho},\tilde{\theta})\right)(\tau,\cdot)dx$$

$$+c\frac{1}{|Q_\varepsilon|}\int_0^\tau\left(\|u^3-\tilde{u}\|_{L^2(\Omega_\varepsilon)}^2 + \|\nabla_x(\mathbf{u}-\tilde{\mathbf{u}})\|_{L^2(\Omega_\varepsilon)}^2\right)dt$$

$$+\frac{1}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega_\varepsilon}\left(\frac{\mathbf{q}(\theta,\nabla_x\theta)\cdot\nabla_x\tilde{\theta}}{\theta} - \frac{\tilde{\theta}}{\theta}\frac{\mathbf{q}(\theta,\nabla_x\theta)\cdot\nabla_x\theta}{\theta}\right.$$

$$\left.+(\tilde{\theta}-\theta)\frac{q(\tilde{\theta},\partial_y\tilde{\theta})\partial_y\tilde{\theta}}{\tilde{\theta}^2} + \partial_y(\theta-\tilde{\theta})\frac{q(\tilde{\theta},\partial_y\tilde{\theta})}{\tilde{\theta}}\right)dxdt$$

$$\leq \Gamma(\varepsilon) + \frac{1}{|Q_\varepsilon|}\int_0^\tau\left[\delta\left(\|u^3-\tilde{u}\|_{L^2(\Omega_\varepsilon)}^2 + \|\nabla_x(\mathbf{u}-\tilde{\mathbf{u}})\|_{L^2(\Omega_\varepsilon)}^2\right)\right.$$

$$+\varepsilon\|\nabla_x\mathbf{u}\|_{L^2(\Omega_\varepsilon)}^2 + K(\delta,\cdot)\int_{\Omega_\varepsilon}\left(\frac{1}{2}\rho|u^3-\tilde{u}|^2 + \mathcal{E}(\rho,\theta|\tilde{\rho},\tilde{\theta})\right)dx\bigg]dt, \quad (7.6)$$

for almost all $\tau \in [0,T]$ and any $\delta > 0$. At this moment we also point out that due to (3.1) and (4.1), (4.2) we have that

$$\frac{1}{|Q_\varepsilon|}\int_0^T\left\|\nabla_x\mathbf{u}+\nabla_x^T\mathbf{u}-\frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right\|_{L^2(\Omega_\varepsilon)}^2 dt \leq c,$$

with c independent of ε and thus by Lemma 5.1 the term

$$\varepsilon\frac{1}{|Q_\varepsilon|}\int_0^\tau\|\nabla_x\mathbf{u}\|_{L^2(\Omega_\varepsilon)}dt,$$

can be included in $\Gamma(\varepsilon)$ (cf. Equation (2.54) in [8]).

Now we may choose $\delta > 0$ so small that the inequality (7.6) takes the form

$$\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\left(\frac{1}{2}\rho|\mathbf{u}-\tilde{\mathbf{u}}|^2 + \mathcal{E}(\rho,\theta|\tilde{\rho},\tilde{\theta})\right)(\tau,\cdot)dx$$

$$+\frac{1}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega_\varepsilon}\left(\frac{\mathbf{q}(\theta,\nabla_x\theta)\cdot\nabla_x\tilde{\theta}}{\theta} - \frac{\tilde{\theta}}{\theta}\frac{\mathbf{q}(\theta,\nabla_x\theta)\cdot\nabla_x\theta}{\theta}\right.$$

$$+(\tilde{\theta} - \theta)\frac{q(\tilde{\theta}, \partial_y \tilde{\theta})\partial_y \tilde{\theta}}{\tilde{\theta}^2} + \partial_y(\theta - \tilde{\theta})\frac{q(\tilde{\theta}, \partial_y \tilde{\theta})}{\tilde{\theta}}\right) dxdt$$

$$\leq \Gamma(\varepsilon) + C\frac{1}{|Q_\varepsilon|}\int_0^\tau \int_{\Omega_\varepsilon} \left(\frac{1}{2}\rho|u^3 - \tilde{u}|^2 + \mathcal{E}(\rho, \theta|\tilde{\rho}, \tilde{\theta})\right) dxdt, \qquad (7.7)$$

for almost all $\tau \in [0, T]$.

7.3. Heat conductivity

In accordance with hypotheses (2.3) and (2.4)

$$\mathbf{q}(\theta, \nabla_x \theta) = -\kappa_0 \nabla_x \theta - \kappa_2 \theta^2 \nabla_x \theta - \kappa_3 \theta^3 \nabla_x \theta,$$

and thus

$$q(\tilde{\theta}, \partial_y \tilde{\theta}) = -\kappa_0 \partial_y \tilde{\theta} - \kappa_2 \tilde{\theta}^2 \partial_y \tilde{\theta} - \kappa_3 \tilde{\theta}^3 \partial_y \tilde{\theta}.$$

The terms to be dealt with are

$$\frac{\mathbf{q}(\theta, \nabla_x \theta) \cdot \nabla_x \tilde{\theta}}{\theta} - \frac{\tilde{\theta}}{\theta}\frac{\mathbf{q}(\theta, \nabla_x \theta) \cdot \nabla_x \theta}{\theta} + (\tilde{\theta} - \theta)\frac{q(\tilde{\theta}, \partial_y \tilde{\theta})\partial_y \tilde{\theta}}{\tilde{\theta}^2} + \partial_y(\theta - \tilde{\theta})\frac{q(\tilde{\theta}, \partial_y \tilde{\theta})}{\tilde{\theta}}.$$

We compute

$$\frac{\tilde{\theta}}{\theta}\frac{\kappa_0}{\theta}|\nabla_x \theta|^2 - \frac{\kappa_0}{\theta}\partial_y \theta \partial_y \tilde{\theta} + \frac{\theta - \tilde{\theta}}{\tilde{\theta}}\frac{\kappa_0}{\tilde{\theta}}|\partial_y \tilde{\theta}|^2 + \frac{\kappa_0}{\tilde{\theta}}\partial_y \tilde{\theta}\partial_y(\tilde{\theta} - \theta)$$

$$= \kappa_0 \left[\tilde{\theta}|\nabla_x \log(\theta)|^2 - \tilde{\theta}\partial_y \log(\theta)\partial_y \log(\tilde{\theta}) + (\theta - \tilde{\theta})|\partial_y \log(\tilde{\theta})|^2 + \partial_y \log(\tilde{\theta})\partial_y(\tilde{\theta} - \theta)\right]$$

$$= \kappa_0 \left[\tilde{\theta}|\nabla_x \log(\theta) - \nabla_x \log(\tilde{\theta})|^2 + (\theta - \tilde{\theta})|\partial_y \log(\tilde{\theta})|^2 + \partial_y \log(\tilde{\theta}) \cdot \partial_y(\tilde{\theta} - \theta)\right.$$

$$\left. + \tilde{\theta}\partial_y \log(\tilde{\theta})(\partial_y \log(\theta) - \partial_y \log(\tilde{\theta}))\right]$$

$$= \kappa_0 \left[\tilde{\theta}|\nabla_x \log(\theta) - \nabla_x \log(\tilde{\theta})|^2 + (\theta - \tilde{\theta})|\partial_y \log(\tilde{\theta})|^2 + (\tilde{\theta} - \theta)\partial_y \log(\tilde{\theta})\partial_y \log(\theta)\right]$$

$$= \kappa_0 \left[\tilde{\theta}|\nabla_x \log(\theta) - \nabla_x \log(\tilde{\theta})|^2 + (\theta - \tilde{\theta})\partial_y \log(\tilde{\theta})\partial_y \left(\log(\tilde{\theta}) - \log(\theta)\right)\right].$$
$$(7.8)$$

Similarly, we get

$$\kappa_2 \tilde{\theta}|\nabla_x \theta|^2 - \kappa_2 \theta \partial_y \theta \partial_y \tilde{\theta} + \kappa_2(\theta - \tilde{\theta})|\partial_y \tilde{\theta}|^2 + \kappa_2 \tilde{\theta}\partial_y \tilde{\theta}\partial_y(\tilde{\theta} - \theta)$$

$$= \kappa_2 \left[\tilde{\theta}|\nabla_x \theta - \nabla_x \tilde{\theta}|^2 - (\theta - \tilde{\theta})\partial_y \tilde{\theta}\partial_y(\theta - \tilde{\theta})\right]. \qquad (7.9)$$

Finally,

$$\kappa_3 \theta\tilde{\theta}|\nabla_x\theta|^2 - \kappa_3\theta^2\partial_y\theta\partial_y\tilde{\theta} + \kappa_3(\theta-\tilde{\theta})\tilde{\theta}|\partial_y\tilde{\theta}|^2 + \kappa_3\tilde{\theta}^2\partial_y\tilde{\theta}\partial_y(\tilde{\theta}-\theta)$$

$$= \kappa_3\left[\tilde{\theta}\theta\nabla_x\theta\cdot(\nabla_x\theta-\nabla_x\tilde{\theta}) + \tilde{\theta}\theta\partial_y\theta\partial_y\tilde{\theta} - \theta^2\partial_y\theta\partial_y\tilde{\theta} + (\theta-\tilde{\theta})\tilde{\theta}|\partial_y\tilde{\theta}|^2 + \tilde{\theta}^2\partial_y\tilde{\theta}\partial_y(\tilde{\theta}-\theta)\right]$$

$$= \kappa_3\left[\tilde{\theta}\theta|\nabla_x\theta-\nabla_x\tilde{\theta}|^2 + 2\tilde{\theta}\theta\partial_y\theta\partial_y\tilde{\theta} - \theta^2\partial_y\theta\partial_y\tilde{\theta} - \tilde{\theta}^2\partial_y\tilde{\theta}\partial_y\theta\right]$$

$$= \kappa_3\tilde{\theta}\theta|\nabla_x\theta-\nabla_x\tilde{\theta}|^2 - \kappa_3(\theta-\tilde{\theta})^2\partial_y\theta\partial_y\tilde{\theta}, \tag{7.10}$$

where

$$\kappa_3(\theta-\tilde{\theta})^2\partial_y\theta\partial_y\tilde{\theta} = \kappa_3|\partial_y\tilde{\theta}|^2(\theta-\tilde{\theta})^2 + \kappa_3\partial_y(\theta-\tilde{\theta})\partial_y\tilde{\theta}(\theta-\tilde{\theta})^2.$$

We conclude by observing that the terms on the right-hand side of (7.8)–(7.10) have either "good" sign or they can be "absorbed" by the remaining integrals in (7.7). Therefore, using (7.8)–(7.10) we can rewrite the inequality (7.7) in the form[1]

$$\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\left(\frac{1}{2}\rho|\mathbf{u}_\varepsilon-\tilde{\mathbf{u}}|^2 + \mathcal{E}(\rho_\varepsilon,\theta_\varepsilon|\tilde{\rho},\tilde{\theta})\right)(\tau,\cdot)dx$$

$$\leq \Gamma(\varepsilon) + c\frac{1}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega_\varepsilon}\left(\frac{1}{2}\rho|u_\varepsilon^3-\tilde{u}|^2 + \mathcal{E}(\rho_\varepsilon,\theta_\varepsilon|\tilde{\rho},\tilde{\theta})\right)dxdt,$$

for almost all $\tau\in[0,T]$ with $\Gamma(\varepsilon)\to 0$ as $\varepsilon\to 0$.

The Gronwall inequality yields

$$\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\left(\frac{1}{2}\rho|\mathbf{u}_\varepsilon-\tilde{\mathbf{u}}|^2 + \mathcal{E}(\rho_\varepsilon,\theta_\varepsilon|\tilde{\rho},\tilde{\theta})\right)(\tau,\cdot)dx \leq \Gamma(\varepsilon,T),$$

where $\Gamma(\varepsilon,T)\to 0$ as $\varepsilon\to 0$. Consequently, the main theorem is proven.

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