Dimension reduction for the full Navier-Stokes-Fourier system

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Problem formulation I

We study the full Navier-Stokes-Fourier system

$$
\begin{gathered}
\partial_t \rho + \operatorname{div}_x(\rho \mathbf{u}) = 0 \\
\partial_t(\rho \mathbf{u}) + \operatorname{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\rho, \theta) - \operatorname{div}_x \mathbb{S}(\theta, \nabla_x \mathbf{u}) = 0 \\
\partial_t(\rho s(\rho, \theta)) + \operatorname{div}_x(\rho s(\rho, \theta)\mathbf{u}) + \operatorname{div}_x \left(\frac{\mathbf{q}(\theta, \nabla_x \theta)}{\theta} \right) = \sigma
\end{gathered}
$$

on a family of shrinking domains Ω_ε of the form

$$
\Omega_\varepsilon = Q_\varepsilon \times (0, 1), \quad Q_\varepsilon = \varepsilon Q,
$$

where Q is an open rectangular domain in $\mathbb{R}^2$ and $\varepsilon > 0$.
Our aim is to prove that under suitable conditions the weak solutions $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon)$ on Ω_ε converge to a strong solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ of the 1D problem as $\varepsilon \to 0$.

Problem formulation II

The 1D equations on $(0,1)$ are as follows

$$
\begin{aligned}
&\partial_t\tilde\rho + \partial_y(\tilde\rho\tilde u) = 0\\
&\partial_t(\tilde\rho\tilde u) + \partial_y(\tilde\rho\tilde u^2) + \partial_y p(\tilde\rho,\tilde\theta) - \partial_y[\tilde S(\tilde\theta,\partial_y\tilde u)] = 0\\
&\partial_t(\tilde\rho s(\tilde\rho,\tilde\theta)) + \partial_y(\tilde\rho s(\tilde\rho,\tilde\theta)\tilde u) + \partial_y\left(\frac{q(\tilde\theta,\partial_y\tilde\theta)}{\tilde\theta}\right) =\\
&\qquad \frac{1}{\tilde\theta}\left(\tilde S(\tilde\theta,\partial_y\tilde u)\partial_y\tilde u - \frac{q(\tilde\theta,\partial_y\tilde\theta)\partial_y\tilde\theta}{\tilde\theta}\right),
\end{aligned}
$$

where $\tilde S(\tilde\theta,\partial_y\tilde u)$ is naturally related to the three-dimensional stress tensor $\mathbb{S}$ and similarly q to the heat flux vector $\mathbf{q}$.

Previous work

- Global existence of weak solutions to the 3D problem (with structural hypothesis which we use): Feireisl, Novotný
- 3D to 1D reduction for the barotropic case: Bella, Feireisl, Novotný
- Global existence of strong solutions to the 1D problem:
  - with other structural hypothesis: Valli, Kawohl, Antontsev, Kazhikhov, Monakhov, Jiang and others...
  - with our structural hypothesis: seems to be an open problem
- Local existence of strong solutions to the 1D problem: we did not find any reference either
- Main tool: Relative entropy inequality due to Feireisl and Novotný

Structural hypothesis I

Stress tensor:

$$\mathbb{S}(\theta, \nabla_x \mathbf{u}) = \mu(\theta) \left(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\theta) \mathrm{div}_x \mathbf{u} \mathbb{I},$$

with

$$\mu(\theta) = \mu_0 + \mu_1 \theta, \quad \eta(\theta) = \eta_0 + \eta_1 \theta, \quad \mu_0, \mu_1 > 0, \quad \eta_0, \eta_1 \geq 0.$$

Heat conductivity:

$$\mathbf{q}(\theta, \nabla_x \theta) = -\kappa(\theta) \nabla_x \theta,$$

with

$$\kappa(\theta) = \kappa_0 + \kappa_2 \theta^2 + \kappa_3 \theta^3, \quad \kappa_i > 0, \; i = 0, 2, 3.$$

Entropy production:

$$\sigma \geq \frac{1}{\theta} \left(\mathbb{S}(\theta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\theta, \nabla_x \theta) \nabla_x \theta}{\theta} \right).$$

Structural hypothesis II

Total energy balance:

$$\frac{d}{dt}\int_{\Omega_\varepsilon}\left(\frac{1}{2}\rho|\mathbf{u}|^2+\rho e(\rho,\theta)\right)dx=0$$

Boundary conditions:

$$\mathbf{q}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon}=0$$

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon}=0,\quad [\mathbb{S}(\nabla_x\mathbf{u})\cdot\mathbf{n}]\times\mathbf{n}|_{\partial\Omega_\varepsilon}=0.$$

Structural hypothesis III

Pressure:

$$p(\rho,\theta) = \theta^{\frac{5}{2}} P\left(\frac{\rho}{\theta^{\frac{3}{2}}}\right) + \frac{a}{3}\theta^4, \quad a > 0,$$

with

$$P \in C^1[0,\infty),\ P(0) = 0,\ P'(Z) > 0,\ P(Z) \approx Z^{5/3} \text{ for } Z >> 1$$

Internal energy:

$$e(\rho,\theta) = \frac{3}{2}\frac{\theta^{\frac{5}{2}}}{\rho} P\left(\frac{\rho}{\theta^{\frac{3}{2}}}\right) + a\frac{\theta^4}{\rho}$$

Entropy:

$$s(\rho,\theta) = S\left(\frac{\rho}{\theta^{\frac{3}{2}}}\right) + \frac{4a}{3}\frac{\theta^3}{\rho},$$

with

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} < 0$$

1D structure

Stress:

$$\tilde{S}(\tilde{\theta}, \partial_y \tilde{u}) = (\nu_0 + \nu_1 \tilde{\theta}) \partial_y \tilde{u},$$

with

$$\nu_i = \frac{4}{3}\mu_i + \eta_i, \qquad i = 0, 1$$

Heat flux:

$$q(\tilde{\theta}, \partial_y \tilde{\theta}) = -\kappa(\tilde{\theta}) \partial_y \tilde{\theta}$$

Boundary conditions:

$$\tilde{u}(\cdot, 0) = \tilde{u}(\cdot, 1) = 0, \quad q(\tilde{\theta}, \partial_y \tilde{\theta})(\cdot, 0) = q(\tilde{\theta}, \partial_y \tilde{\theta})(\cdot, 1) = 0$$

Weak solutions in 3D

Initial data:

$$\rho(0,\cdot) = \rho_0, \quad \rho\mathbf{u}(0,\cdot) = (\rho\mathbf{u})_0,$$

$$\rho s(\rho,\theta)(0,\cdot) = \rho_0 s(\rho_0,\theta_0),\ \rho_0 \geq 0,\ \theta_0 > 0,$$

Function spaces:

- $\rho(t,x) \geq 0$, $\theta(t,x) > 0$ for almost all $(t,x) \in (0,T) \times \Omega$
- $\rho \in C_{weak}([0,T]; L^{\frac{5}{3}}(\Omega))$
- $\rho\mathbf{u} \in C_{weak}([0,T]; L^{\frac{5}{4}}(\Omega;\mathbb{R}^3))$ and $\mathbf{u} \in L^2(0,T; W^{1,2}(\Omega;\mathbb{R}^3))$, $\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0$
- $\theta \in L^\infty(0,T; L^4(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega))$

Weak solutions in 3D II

(i) Continuity equation:

$$\int_\Omega \rho(\tau,\cdot)\varphi(\tau,\cdot)dx - \int_\Omega \rho_0\varphi(0,\cdot)dx = \int_0^\tau \int_\Omega (\rho\partial_t\varphi + \rho\mathbf{u}\cdot\nabla_x\varphi)dxdt,$$

for any $\varphi \in C^1([0,T]\times\overline{\Omega})$ and any $\tau \in [0,T]$

(ii) Momentum equation:

$$\int_\Omega \rho\mathbf{u}(\tau,\cdot)\cdot\varphi(\tau,\cdot)dx - \int_\Omega (\rho\mathbf{u})_0\cdot\varphi(0,\cdot)dx$$

$$= \int_0^\tau \int_\Omega (\rho\mathbf{u}\cdot\partial_t\varphi + \rho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p(\rho,\theta)\mathrm{div}_x\varphi - \mathbb{S}(\theta,\nabla_x\mathbf{u}) : \nabla_x\varphi)dxdt,$$

for any $\varphi \in C^1([0,T]\times\overline{\Omega};\mathbb{R}^3)$, $\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0$ and any $\tau \in [0,T]$

(iii) Entropy balance:

$$\int_\Omega \rho_0 s(\rho_0,\theta_0)\varphi(0,\cdot)dx - \int_\Omega \rho s(\rho,\theta)(\tau,\cdot)\varphi(\tau,\cdot)dx$$

$$+\int_0^\tau \int_\Omega \frac{1}{\theta}\left(\mathbb{S}(\theta,\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\theta,\nabla_x\theta)\cdot\nabla_x\theta}{\theta}\right)\varphi dxdt$$

$$\leq -\int_0^\tau \int_\Omega \left(\rho s(\rho,\theta)\partial_t\varphi + \rho s(\rho,\theta)\mathbf{u}\cdot\nabla_x\varphi + \frac{\mathbf{q}(\theta,\nabla_x\theta)\cdot\nabla_x\varphi}{\theta}\right)dxdt,$$

for any $\varphi \in C^1([0,T]\times\overline{\Omega})$, $\varphi \geq 0$ and almost all $\tau \in [0,T]$;

(iv) Total energy conservation:

$$\int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e(\rho,\theta)\right)(\tau,\cdot)dx = \int_\Omega \left(\frac{1}{2\rho_0}|(\rho\mathbf{u})_0|^2 + \rho_0 e(\rho_0,\theta_0)\right)dx,$$

for almost all $\tau \in [0,T]$.

We assume that there exists a trio

$$(\tilde{\rho}, \tilde{u}, \tilde{\theta}) : [0, T] \times [0, 1] \mapsto (0, \infty) \times \mathbb{R} \times (0, \infty),$$

of smooth functions that solves the 1D problem in a classical sense on $[0, T] \times (0, 1)$ satisfying

$$\tilde{\rho} \geq c > 0, \quad \tilde{\theta} \geq c > 0,$$

with the initial conditions

$$\tilde{\rho}(0, \cdot) = \tilde{\rho}_0, \ \tilde{u}(0, \cdot) = \tilde{u}_0, \ \tilde{\theta}(0, \cdot) = \tilde{\theta}_0,$$

(with $\tilde{\rho}_0 \geq c > 0$, $\tilde{\theta}_0 \geq c > 0$.)

Main theorem

Theorem 1

Let all the above hypothesis be satisfied. Denote by $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon)$ *a sequence of weak solutions to the 3D problem on* $(0,T) \times \Omega_\varepsilon$ *emanating from* $(\rho_{0,\varepsilon}, (\rho\mathbf{u})_{0,\varepsilon}, \theta_{0,\varepsilon})$*. Denote by* $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ *a classical solution to the 1D problem on* $(0,T) \times (0,1)$ *emanating from* $(\tilde{\rho}_0, \tilde{u}_0, \tilde{\theta}_0)$*. Define* $\tilde{\mathbf{u}}_0 = [0, 0, \tilde{u}_0]$ *and* $\tilde{\mathbf{u}} = [0, 0, \tilde{u}]$*.*
Let moreover

$$\frac{1}{|Q_\varepsilon|} \int_{Q_\varepsilon} \rho_{0,\varepsilon}(x_h, \cdot) dx_h \to \tilde{\rho}_0, \quad \frac{1}{|Q_\varepsilon|} \int_{Q_\varepsilon} (\rho\mathbf{u})_{0,\varepsilon}(x_h, \cdot) dx_h \to \tilde{\rho}_0 \tilde{\mathbf{u}}_0,$$

$$\frac{1}{|Q_\varepsilon|} \int_{Q_\varepsilon} \rho_{0,\varepsilon} s(\rho_{\varepsilon,0}, \theta_{\varepsilon,0}) dx_h \to \tilde{\rho}_0 s(\tilde{\rho}_0, \tilde{\theta}_0),$$

weakly in $L^1(0,1)$ *and ...*

Theorem 1 (cont.)

... and let

$$\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\left[\frac{1}{2\rho_{0,\varepsilon}}|(\rho\mathbf{u})_{0,\varepsilon}|^2+\rho_{\varepsilon,0}e(\rho_{\varepsilon,0},\theta_{\varepsilon,0})\right]dx$$
$$\to\int_0^1\left[\frac{1}{2}\tilde\rho_0|\tilde u_0|^2+\tilde\rho_0 e(\tilde\rho_0,\tilde\theta_0)\right]dy.$$

Then (as $\varepsilon\to 0$)

$$esssup_{t\in(0,T)}\frac{1}{|Q_\varepsilon|}\|\rho_\varepsilon-\tilde\rho\|^{\frac{5}{3}}_{L^{\frac{5}{3}}(\Omega_\varepsilon)}\to 0,$$

$$esssup_{t\in(0,T)}\frac{1}{|Q_\varepsilon|}\|\theta_\varepsilon-\tilde\theta\|^2_{L^2(\Omega_\varepsilon)}\to 0,$$

$$\frac{1}{|Q_\varepsilon|}\|\mathbf{u}_\varepsilon-\tilde{\mathbf{u}}\|^r_{L^r((0,T)\times\Omega_\varepsilon)}\to 0 \textit{ for every } r\in[1,2).$$

Some remarks

- The analysis of dimension reduction problems usually depends on the use of Korn's inequality

$$\|\nabla_x \mathbf{v}\|_{L^2(\Omega_\varepsilon)} \leq c(\varepsilon)\|\nabla_x \mathbf{v} + \nabla_x^T \mathbf{v}\|_{L^2(\Omega_\varepsilon)}, \quad \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0$$

- First problem: validity even for a fixed $\varepsilon > 0$ requires certain restrictions on the shape of the cross-section Q
- Second problem: even "properly" shaped Q might not stop the constant $c(\varepsilon)$ from blowing up as $\varepsilon \to 0$
- Barotropic case (Bella, Feireisl, Novotný): Authors avoid using Korn's inequality completely - not possible in our case
- Our result leans on the validity of stronger Korn's like inequality appropriate for compressible fluids:

$$\|\nabla_x \mathbf{v}\|^2_{L^2(\Omega_\varepsilon)} \leq \left\|\nabla_x \mathbf{v} + \nabla_x^T \mathbf{v} - \frac{2}{3}\mathrm{div}_x \mathbf{v}\mathbb{I}\right\|^2_{L^2(\Omega_\varepsilon)}, \ \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0.$$

- We are able to prove this only for $Q = (a,b)\times(c,d)$, $a<b$, $c<d$, $a,b,c,d \in \mathbb{R}$.

Korn's inequality

Lemma 2

Let $\Omega \subset \mathbb{R}^3$ *be a rectangular domain, i.e.,* $\Omega = (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$ *and* $\mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^3)$ *be such that* $\mathbf{u} \cdot \mathbf{n} = 0$ *on* $\partial\Omega$. *Then*

$$\|\nabla_x \mathbf{u}\|^2_{L^2(\Omega)} \leq \left\| \nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} \right\|^2_{L^2(\Omega)},$$

$$\|\nabla_x \mathbf{u}\|^2_{L^2(\Omega)} \leq \left\| \nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right\|^2_{L^2(\Omega)},$$

$$\|\nabla_x \mathbf{u}\|^2_{L^2(\Omega)} \leq \int_\Omega (\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I}) : \nabla_x \mathbf{u} dx.$$

Proof

Direct calculation:

$$(\nabla_x \mathbf{u}+\nabla_x^T \mathbf{u}) : (\nabla_x \mathbf{u}+\nabla_x^T \mathbf{u}) = |\nabla_x \mathbf{u}|^2+3\left[(\partial_{x_1} u^1)^2 + (\partial_{x_2} u^2)^2 + (\partial_{x_3} u^3)^2\right.$$

$$\left.+4\left[\partial_{x_2} u^1 \partial_{x_1} u^2 + \partial_{x_3} u^1 \partial_{x_1} u^3 + \partial_{x_3} u^2 \partial_{x_2} u^3\right]\right.$$

Using the boundary condition we can easily show

$$\int_\Omega \partial_{x_j} u^i \partial_{x_i} u^j \, dx = \int_\Omega \partial_{x_i} u^i \partial_{x_j} u^j \, dx.$$

and thus conclude with

$$\int_\Omega (\nabla_x \mathbf{u}+\nabla_x^T \mathbf{u}) : (\nabla_x \mathbf{u}+\nabla_x^T \mathbf{u}) dx = \int_\Omega |\nabla_x \mathbf{u}|^2 dx + 2\int_\Omega (\mathrm{div}_x \mathbf{u})^2 dx$$

$$+\int_\Omega \left[(\partial_{x_1} u^1)^2 + (\partial_{x_2} u^2)^2 + (\partial_{x_3} u^3)^2\right] dx.$$

Poincaré inequality

For a set $M \subset \mathbb{R}^d$, $d \in \mathbb{N}$ and a function $f \in L^1(M)$ we denote by $(f)_M$ its integral average, i.e.,

$$(f)_M = \frac{1}{|M|} \int_M f(x)dx.$$

Lemma 3

There exists a constant $c > 0$ independent of ε such that for every $f \in W^{1,2}(\Omega_\varepsilon)$ fulfilling $f(\cdot,0) = f(\cdot,1) = 0$ on Q_ε it holds that

$$\int_0^1 \int_{Q_\varepsilon} |f(x_h,y) - (f)_{Q_\varepsilon}(y)|^4 dx_h dy$$

$$= \|f - (f)_{Q_\varepsilon}\|^4_{L^4(\Omega_\varepsilon)} \leq c\|\nabla_x f\|^4_{L^2(\Omega_\varepsilon)}.$$

Essential and residual parts

Choose positive constants $\underline{\rho}, \overline{\rho}, \underline{\theta}, \overline{\theta}$ such that

$$0 < \underline{\rho} \leq \frac{1}{2} \min_{(\tau,y)\in[0,T]\times[0,1]} \tilde{\rho}(\tau, y) \leq 2 \max_{(\tau,y)\in[0,T]\times[0,1]} \tilde{\rho}(\tau, y) \leq \overline{\rho},$$

$$0 < \underline{\theta} \leq \frac{1}{2} \min_{(\tau,y)\in[0,T]\times[0,1]} \tilde{\theta}(\tau, y) \leq 2 \max_{(\tau,y)\in[0,T]\times[0,1]} \tilde{\theta}(\tau, y) \leq \overline{\theta}.$$

Each measurable function h can be written as

$$h = h_{ess} + h_{res},$$

where

$$h_{ess}(t,x) = \begin{cases} h(t,x) & \text{if } (\rho(t,x), \theta(t,x)) \in [\underline{\rho}, \overline{\rho}] \times [\underline{\theta}, \overline{\theta}], \\ 0 & \text{otherwise.} \end{cases}$$

Relative entropy

Define the ballistic free energy as

$$H^{\Theta}(\rho,\theta) = \rho e(\rho,\theta) - \Theta \rho s(\rho,\theta)$$

and the relative entropy as

$$\mathcal{E}(\rho,\theta|r,\Theta) = H^{\Theta}(\rho,\theta) - \partial_\rho H^{\Theta}(r,\Theta)(\rho - r) - H^{\Theta}(r,\Theta).$$

It holds:

$$\mathcal{E}(\rho,\theta|\tilde{\rho},\tilde{\theta}) \geq c \begin{cases} |\rho - \tilde{\rho}|^2 + |\theta - \tilde{\theta}|^2 & \text{if } (\rho,\theta) \in [\underline{\rho},\overline{\rho}] \times [\underline{\theta},\overline{\theta}], \\ 1 + |\rho s(\rho,\theta)| + \rho e(\rho,\theta) & \text{otherwise} \end{cases}$$

with c independent of ε.

Estimates on ρ and θ

We have

$$\|[\theta - \tilde{\theta}]_{ess}\|^s_{L^s(\Omega_\varepsilon)} + \|[\rho - \tilde{\rho}]_{ess}\|^s_{L^s(\Omega_\varepsilon)} \leq c \int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) dx$$

for $s \geq 2$.
Using

$$\rho e(\rho, \theta) \geq c(\rho^{\frac{5}{3}} + \theta^4)$$

and some easy calculations we get

$$\int_{\Omega_\varepsilon} |[\theta - \tilde{\theta}]_{res}|^q dx + \int_{\Omega_\varepsilon} |[\rho - \tilde{\rho}]_{res}|^p dx \leq c \int_{\Omega_\varepsilon} \mathcal{E}(\rho, \theta | \tilde{\rho}, \tilde{\theta}) dx$$

for $1 \leq q \leq 4$ and $1 \leq p \leq \frac{5}{3}$.

Relative entropy inequality I

We prove that weak solutions satisfy the following relative entropy inequality

$$\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\left(\frac{1}{2}\rho|\mathbf{u}-\mathbf{U}|^2+\mathcal{E}(\rho,\theta|r,\Theta)\right)(\tau,\cdot)dx$$

$$+\frac{1}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega\varepsilon}\frac{\Theta}{\theta}\left(\mathbb{S}(\theta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u}-\frac{\mathbf{q}(\theta,\nabla_x\theta)\cdot\nabla_x\theta}{\theta}\right)dxdt$$

$$\leq\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\left(\frac{1}{2\rho_{0,\varepsilon}}|(\rho\mathbf{u})_{0,\varepsilon}-\rho_{0,\varepsilon}\mathbf{U}(0)|^2+\mathcal{E}(\rho_{\varepsilon,0},\theta_{\varepsilon,0}|r(0),\Theta(0))\right)dx$$

$$+\frac{1}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega\varepsilon}\mathcal{R}(\rho,\mathbf{u},\theta,r,\mathbf{U},\Theta)dxdt$$

for any smooth and suitable "test" functions $(r,\mathbf{U},\Theta)$.

Relative entropy inequality II

The remainder term is given as follows

$$\mathcal{R}(\rho, \mathbf{u}, \theta, r, \mathbf{U}, \Theta) = \rho(\mathbf{u} - \mathbf{U}) \cdot \nabla_x \mathbf{U} \cdot (\mathbf{U} - \mathbf{u})$$

$$+\rho(s(\rho, \theta) - s(r, \Theta))(\mathbf{U} - \mathbf{u}) \cdot \nabla_x \Theta$$

$$+\rho(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U}) \cdot (\mathbf{U} - \mathbf{u}) - p(\rho, \theta) \mathrm{div}_x \mathbf{U} + \mathbb{S}(\theta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U}$$

$$-\rho(s(\rho, \theta) - s(r, \Theta))(\partial_t \Theta + \mathbf{U} \cdot \nabla_x \Theta) + \frac{\mathbf{q}(\theta, \nabla_x \theta)}{\theta} \cdot \nabla_x \Theta$$

$$+ \left(1 - \frac{\rho}{r}\right) \partial_t p(r, \Theta) - \frac{\rho}{r} \mathbf{u} \cdot \nabla_x p(r, \Theta)$$

In particular, in every term derivatives appear on the test functions.

Relative entropy inequality III

To prove our main theorem we set

$$r = \tilde{\rho}(t,y), \quad \Theta = \tilde{\theta}(t,y), \quad \mathbf{U} = \tilde{\mathbf{u}}(t,y) = \begin{bmatrix} 0 \\ 0 \\ \tilde{u}(t,y) \end{bmatrix}.$$

All space derivatives of test functions then reduce just to ∂_y.
Idea:

- create on the left hand side "viscous" terms in $\mathbf{u} - \mathbf{U}$ and $\theta - \Theta$ (simply add what is necessary)
- estimate all terms on the right hand side and use Gronwall

One example

We show how to estimate

$$\left|\int_{\Omega_\varepsilon} \rho\left(s(\rho,\theta)-s(\tilde\rho,\tilde\theta)\right)(\tilde u-u^3)\partial_y\tilde\theta dx\right|$$

$$\leq \|\partial_y\tilde\theta\|_{L^\infty(0,1)}\left|\int_{\Omega_\varepsilon} \rho\left(s(\rho,\theta)-s(\tilde\rho,\tilde\theta)\right)(\tilde u-u^3)dx\right|$$

Split into essential and residual part:
On the essential it holds

$$\left|\left[s(\rho,\theta)-s(\tilde\rho,\tilde\theta)\right]_{ess}\right| \leq c|[\rho-\tilde\rho]_{ess}| + c|[\theta-\tilde\theta]_{ess}|$$

and therefore

$$\int_{\Omega_\varepsilon}\left|\left[s(\rho,\theta)-s(\tilde\rho,\tilde\theta)\right]_{ess}\right||u^3-\tilde u|dx$$

$$\leq C\int_{\Omega_\varepsilon}\left(\frac{1}{2}\rho|u^3-\tilde u|^2+\mathcal{E}(\rho,\theta|\tilde\rho,\tilde\theta)\right)dx.$$

One example II

Residual part split further into

$$\int_{\Omega_\varepsilon} \left| \left[\rho(s(\rho,\theta) - s(\tilde{\rho},\tilde{\theta})) \right]_{res} \right| |\tilde{u} - (u^3)_{Q_\varepsilon}| dx$$

$$+ \int_{\Omega_\varepsilon} \left| \left[\rho(s(\rho,\theta) - s(\tilde{\rho},\tilde{\theta})) \right]_{res} \right| |(u^3)_{Q_\varepsilon} - u^3| dx = \mathcal{I}_1 + \mathcal{I}_2$$

Now we have

$$\mathcal{I}_1 \leq \|\tilde{u} - (u^3)_{Q_\varepsilon}\|_{L^\infty(\Omega_\varepsilon)} \|[\rho(s(\rho,\theta) - s(\tilde{\rho},\tilde{\theta}))]_{res}\|_{L^{\frac{4}{3}}(\Omega_\varepsilon)} |Q_\varepsilon|^{\frac{1}{4}}$$

$$\leq \|\tilde{u} - (u^3)_{Q_\varepsilon}\|_{L^\infty(\Omega_\varepsilon)} \left(\frac{1}{|Q_\varepsilon|} \int_{\Omega_\varepsilon} \mathcal{E}(\rho,\theta|\tilde{\rho},\tilde{\theta}) dx \right)^{\frac{1}{2}} |Q_\varepsilon|$$

Since $W^{1,1}(0,1) \hookrightarrow L^\infty(0,1)$ we have

$$\|\tilde{u}-(u^3)_{Q_\varepsilon}\|_{L^\infty(\Omega_\varepsilon)} = \|\tilde{u}-(u^3)_{Q_\varepsilon}\|_{L^\infty(0,1)} \leq \|\partial_y(\tilde{u}-(u^3)_{Q_\varepsilon})\|_{L^1(0,1)}$$
$$\leq \frac{1}{|Q_\varepsilon|}\|\partial_y(\tilde{u}-u^3)\|_{L^1(\Omega_\varepsilon)} \leq \frac{1}{|Q_\varepsilon|^{\frac{1}{2}}}\|\partial_y(\tilde{u}-u^3)\|_{L^2(\Omega_\varepsilon)}.$$

and we thus end up with

$$\mathcal{I}_1 \leq \|\partial_y(\tilde{u}-u^3)\|_{L^2(\Omega_\varepsilon)}\left(\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\mathcal{E}(\rho,\theta|\tilde{\rho},\tilde{\theta})dx\right)^{\frac{1}{2}}|Q_\varepsilon|^{\frac{1}{2}}$$
$$\leq \delta\|\partial_y(\tilde{u}-u^3)\|^2_{L^2(\Omega_\varepsilon)} + K(\delta)\int_{\Omega_\varepsilon}\mathcal{E}(\rho,\theta|\tilde{\rho},\tilde{\theta})dx.$$

One example IV

In the second integral we use the Poincaré type inequality to get

$$\begin{aligned}\mathcal{I}_2 &\leq \|(u^3)_{Q_\varepsilon} - u^3\|_{L^4(\Omega_\varepsilon)} \left(\int_{\Omega_\varepsilon} \mathcal{E}(\rho,\theta|\tilde{\rho},\tilde{\theta})dx\right)^{\frac{1}{2}} |Q_\varepsilon|^{\frac{1}{4}} \\ &\leq c|Q_\varepsilon|^{\frac{1}{4}} \|\nabla \mathbf{u}\|_{L^2(\Omega_\varepsilon)} \left(\int_{\Omega_\varepsilon} \mathcal{E}(\rho,\theta|\tilde{\rho},\tilde{\theta})dx\right)^{\frac{1}{2}} \\ &\leq \varepsilon\|\nabla \mathbf{u}\|^2_{L^2(\Omega_\varepsilon)} + c\int_{\Omega_\varepsilon} \mathcal{E}(\rho,\theta|\tilde{\rho},\tilde{\theta})dx\end{aligned}$$

and thus

$$\begin{aligned}&\left|\int_{\Omega_\varepsilon} \rho\left(s(\rho,\theta) - s(\tilde{\rho},\tilde{\theta})\right)(\tilde{u} - u^3)\partial_y\tilde{\theta}dx\right| \\ &\quad\leq \delta\|\partial_y(\tilde{u} - u^3)\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon\|\nabla \mathbf{u}\|^2_{L^2(\Omega_\varepsilon)} \\ &\quad + K(\delta,\cdot)\int_{\Omega_\varepsilon}\left(\frac{1}{2}\rho|u^3 - \tilde{u}|^2 + \mathcal{E}(\rho,\theta|\tilde{\rho},\tilde{\theta})\right)dx, \quad (1)\end{aligned}$$

Conclusion

After similar technical manipulation with all arising terms on the left hand side we are finally able to get the desired inequality

$$\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\left(\frac{1}{2}\rho|\mathbf{u}_\varepsilon-\tilde{\mathbf{u}}|^2+\mathcal{E}(\rho_\varepsilon,\theta_\varepsilon|\tilde{\rho},\tilde{\theta})\right)(\tau,\cdot)dx$$

$$\leq \Gamma(\varepsilon)+c\frac{1}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega_\varepsilon}\left(\frac{1}{2}\rho|u_\varepsilon^3-\tilde{u}|^2+\mathcal{E}(\rho_\varepsilon,\theta_\varepsilon|\tilde{\rho},\tilde{\theta})\right)dxdt,$$

for almost all $\tau\in[0,T]$ with $\Gamma(\varepsilon)\to 0$ as $\varepsilon\to 0$ and thus

$$\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\left(\frac{1}{2}\rho|\mathbf{u}_\varepsilon-\tilde{\mathbf{u}}|^2+\mathcal{E}(\rho_\varepsilon,\theta_\varepsilon|\tilde{\rho},\tilde{\theta})\right)(\tau,\cdot)dx\leq\Gamma(\varepsilon,T),$$

where $\Gamma(\varepsilon,T)\to 0$ as $\varepsilon\to 0$.
The theorem is proved.

Thank you

Thank you for your attention.