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On ρ-dilations of commuting operators

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Abstract

Let $n \geq 1$ and let $c_{F,G}$ be given real numbers defined for all pairs of disjoint subsets $F, G \subset \{1, \dots, n\}$. We characterize commuting n-tuples of operators $T = (T_1, \dots, T_n)$ acting on a Hilbert space H which have a commuting unitary dilation $U = (U_1, \dots, U_n) \in B(K)^n$, $K \supset H$ such that $P_H U^{*\beta} U^\alpha | H = c_{\operatorname{supp}\alpha, \operatorname{supp}\beta} T^{*\beta} T^\alpha$ for all $\alpha, \beta \in \mathbb{Z}_+^n$, $\operatorname{supp}\alpha \cap \operatorname{supp}\beta = \emptyset$. This unifies and generalizes the concepts of ρ-dilations of a single operator and of regular unitary dilations of commuting n-tuples. We discuss also other interesting cases.[1]

1 Introduction

There are many successful generalizations of the dilation theory of Hilbert space contractions.

Let $\rho > 0$. An operator T on a Hilbert space H is said to have a ρ-dilation if there exists a Hilbert space $K \supset H$ and a unitary operator $U \in B(K)$ such that $T^k = \rho P_H U^k | H$ for all $k \geq 1$, where P_H denotes the orthogonal projection onto H. It is known [4] that T has a ρ-dilation if and only if

$$\|h\|^2 + 2\Big(\frac{1}{\rho} - 1\Big) \operatorname{Re} \langle zTh, h \rangle + \Big(1 - \frac{2}{\rho}\Big) \|zTh\|^2 \geq 0$$

for all $h \in H$ and $z \in \mathbb{D}$.

The most important particular cases are $\rho = 1$ (which reduces to the classical dilation theory of Hilbert space contractions) and $\rho = 2$. An operator has a 2-dilation if and only if its numerical range is contained in the closed unit disc, see [1], [4].

Let $n \geq 1$ and let $T = (T_1, \dots, T_n) \in B(H)^n$ be a commuting n-tuple of operators. T is said to have a unitary dilation if there exists a Hilbert space $K \supset H$ and an n-tuple of commuting unitary operators $U = (U_1, \dots, U_n) \in B(K)^n$ such that $T^\alpha = P_H U^\alpha | H$ for all $\alpha \in \mathbb{Z}_+^n$. It is well known that every pair of commuting contractions has a unitary dilation (the Ando dilation). However, the Ando dilation is not unique, its structure is not clear and in general such a dilation does not exist for more than two commuting contractions. The main difficulty is that the values of compressions $P_H U^\alpha | H$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, $\min \alpha_j < 0$, $\max \alpha_j > 0$ are not prescribed and can be chosen arbitrarily.

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The theory of regular unitary dilations overcomes this difficulty by requiring that $P_H U^{*\beta} U^{\alpha}|H = T^{*\beta} T^{\alpha}$ for all $\alpha, \beta \in \mathbb{Z}_+^n$, $\operatorname{supp}\alpha \cap \operatorname{supp}\beta = \emptyset$. It is known that an n-tuple $T = (T_1, \dots, T_n)$ has a regular unitary dilation if and only if

$$\sum_{A \subset B} (-1)^{|A|} \Big\| \Big(\prod_{j \in A} T_j \Big) h \Big\|^2 \geq 0$$

for all $B \subset \{1, \dots, n\}$ and $h \in H$, see [4], [2].

The aim of this paper is to unify and generalize these two approaches.

Let $n \geq 1$ and let $c_{F,G}$ be a system of real numbers defined for pairs of disjoint subsets $F, G \subset \{1, \dots, n\}$ satisfying natural conditions $c_{\emptyset,\emptyset} = 1$ and $c_{G,F} = c_{F,G}$ for all F, G. We characterize the n-tuples of commuting operators $T = (T_1, \dots, T_n) \in B(H)^n$ which have a commuting unitary dilation $U = (U_1, \dots, U_n) \in B(K)^n$ satisfying

$$P_H U^{*\beta} U^{\alpha}|H = c_{\operatorname{supp}\alpha, \operatorname{supp}\beta} \cdot T^{*\beta} T^{\alpha}$$

for all $\alpha, \beta \in \mathbb{Z}_+^n$, $\operatorname{supp}\alpha \cap \operatorname{supp}\beta = \emptyset$. This includes the above described cases of ρ-dilations of a single operator and of regular unitary dilations. We describe also other interesting cases.

2 Notations

We denote by $\mathbb{Z}$ and $\mathbb{Z}_+$ the set of all integers and non-negative integers, respectively. Denote by $\mathbb{D}$ and $\mathbb{T}$ the open unit disc and the unit circle in the complex plane, respectively. Let $n \in \mathbb{N}$. We use the standard multiindex notation. For $\alpha, \beta \in \mathbb{Z}_+^n$ we write $\alpha \leq \beta$ if $\alpha_j \leq \beta_j$ for all $j = 1, \dots, n$, $|\alpha| = \sum_{j=1}^n \alpha_j$, $\alpha + \beta = (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$ and $\operatorname{supp}\alpha = \{j : \alpha_j \neq 0\}$. For $\alpha \in \mathbb{Z}^n$ write $\alpha_+ = (\max\{\alpha_1, 0\}, \dots, \max\{\alpha_n, 0\})$ and $\alpha_- = (\max\{-\alpha_1, 0\}, \dots, \max\{-\alpha_n, 0\})$.

For $F \subset \{1, \dots, n\}$ we define $e_F \in \mathbb{Z}_+^n$ by $(e_F)_j = 1 \quad (j \in F)$ and $(e_F)_j = 0 \quad (j \notin F)$. We denote by $|F|$ the cardinality of F.

Let $T = (T_1, \dots, T_n) \in B(H)^n$ be a commuting n-tuple of operators acting on a Hilbert space H. For $\alpha \in \mathbb{Z}_+^n$ we write $T^{\alpha} = \prod_{j=1}^n T_j^{\alpha_j}$. For $F \subset \{1, \dots, n\}$ write $T_F = \prod_{j \in F} T_j$. In particular, $T_{\emptyset} = I$, the identity operator on H.

Let $c_{F,G} \quad (F, G \subset \{1, \dots, n\}, F \cap G = \emptyset)$ be a system of real numbers such that

$$c_{\emptyset,\emptyset} = 1 \qquad \text{and} \qquad c_{G,F} = c_{F,G} \qquad \text{for all} \qquad F, G. \tag{1}$$

Let $T = (T_1, \dots, T_n) \in B(H)^n$ be a commuting system of operators. We say that T has a dilation determined by the system $(c_{F,G})$ if there exist a Hilbert space $K \supset H$ and an n-tuple $U = (U_1, \dots, U_n) \in B(K)^n$ of commuting unitary operators such that

$$\langle U^{\alpha} h, U^{\beta} g \rangle = c_{\operatorname{supp}\alpha, \operatorname{supp}\beta} \cdot \langle T^{\alpha} h, T^{\beta} g \rangle \tag{2}$$

for all $h, g \in H$ and $\alpha, \beta \in \mathbb{Z}_+^n$ with $\operatorname{supp}\alpha \cap \operatorname{supp}\beta = \emptyset$. In particular,

$$\langle U^{\alpha} h, g \rangle = c_{\operatorname{supp}\alpha, \emptyset} \langle T^{\alpha} h, g \rangle$$

for all $h, g \in H$ and $\alpha \in \mathbb{Z}_+^n$. Clearly (2) is equivalent to

$$P_H U^{\alpha - \beta}|H = c_{\operatorname{supp}\alpha, \operatorname{supp}\beta} T^{*\beta} T^{\alpha}$$

for all $\alpha, \beta \in \mathbb{Z}_+^n$, $\operatorname{supp}\alpha \cap \operatorname{supp}\beta = \emptyset$.

This definition includes the ρ-dilations of a single operator T_1 (for $n = 1$ and $c_{\{1\},\emptyset} = \rho^{-1}$) and the regular unitary dilations (for $c_{F,G} = 1$ for all F, G) of n-tuples of commuting operators.

If we assume the natural minimality condition $K = \bigvee_{\alpha\in\mathbb{Z}^n} U^\alpha H$ then it is easy to see that conditions (2) determine the dilation uniquely up to a unitary equivalence.

The aim of this paper is to characterize the n-tuples $T = (T_1, \dots, T_n)$ which have dilation determined by (2). This will generalize the cases of ρ-dilations of single operators as well as the case of regular unitary dilations of commuting contractions.

3 Necessary conditions

In this section we fix a Hilbert space H, an n-tuple $T = (T_1, \dots, T_n) \in B(H)^n$ of commuting operators, real numbers $c_{F,G}$ $(F, G \subset \{1, \dots, n\}, F \cap G = \emptyset)$ and a dilation $U = (U_1, \dots, U_n) \in B(K)^n$ satisfying (1) and (2).

For $A \subset \{1, \dots, n\}$ let $D_A : H \to K$ be defined by $D_A = \sum_{F\subset A}(-1)^{|F|}U_{A\setminus F}T_F$. Thus $D_\emptyset$ is the isometrical embedding of H into K and $D_{\{j\}} = U_j - T_j$ for all $j \in \{1, \dots, n\}$. If $j \in A$ then $D_A = U_j D_{A\setminus\{j\}} - D_{A\setminus\{j\}}T_j$.

Write for short $[1, n] = \{1, \dots, n\}$.

Note that in the classical dilation theory for $n = 1$ the space $\overline{(U_1 - T_1)H}$ plays an important role — it is a copy of the defect space $\overline{(I - T_1^*T_1)^{1/2}H}$ and it is a wandering subspace for the unitary dilation U_1. For ρ-dilations this space is not exactly wandering any more but it is "almost" wandering: $U_1^j\overline{(U_1 - T_1)H} \perp U_1^k\overline{(U_1 - T_1)H}$ if $|j-k| \geq 2$, see [3].

In our situation the space $\overline{D_{[1,n]}H}$ may be viewed on as an analogy of this defect space.

Note that if $\alpha, \beta \in \mathbb{Z}_+^n$, $\operatorname{supp}\alpha \cap \operatorname{supp}\beta = \emptyset$, $j \in [1, n]$, $\alpha_j \geq 2$ and $h, g \in H$ then

$$\begin{aligned}\langle U^\alpha h, U^\beta g\rangle &= c_{\operatorname{supp}\alpha,\operatorname{supp}\beta}\langle T^\alpha h, T^\beta g\rangle = c_{\operatorname{supp}\alpha,\operatorname{supp}\beta}\langle T^{\alpha-e_j}T_j h, T^\beta g\rangle \\ &= \langle U^{\alpha-e_j}T_j h, U^\beta\rangle.\end{aligned}$$

Consequently,

$$\left\langle U^\alpha D_{[1,n]\setminus\{j\}}h, U^\beta D_{[1,n]\setminus\{j\}}g\right\rangle = \left\langle U^{\alpha-e_j}D_{[1,n]\setminus\{j\}}T_j h, U^\beta D_{[1,n]\setminus\{j\}}g\right\rangle \tag{3}$$

if $\alpha_j \geq 2$.

The next proposition shows that in our situation the space $D_{[1,n]}H$ is also "almost" wandering in the following sense:

Proposition 1. Let $\alpha, \beta \in \mathbb{Z}_+^n$, $\operatorname{supp}\alpha \cap \operatorname{supp}\beta = \emptyset$. Suppose that $\max\{\alpha_j, \beta_j : j = 1, \dots, n\} \geq 2$. Then

$$\left\langle U^\alpha D_{[1,n]}h, U^\beta D_{[1,n]}g\right\rangle = 0$$

for all $h, g \in H$.

Proof. Without loss of generality we may assume that $\alpha_j \geq 2$ for some $j \in [1, n]$. We have

$$\begin{aligned}&\left\langle U^\alpha D_{[1,n]}h, U^\beta D_{[1,n]}g\right\rangle \\ =&\left\langle U^\alpha\big(U_j D_{[1,n]\setminus\{j\}} - D_{[1,n]\setminus\{j\}}T_j\big)h, U^\beta\big(U_j D_{[1,n]\setminus\{j\}} - D_{[1,n]\setminus\{j\}}T_j\big)g\right\rangle \\ =&\left\langle U^\alpha D_{[1,n]\setminus\{j\}}h, U^\beta D_{[1,n]\setminus\{j\}}g\right\rangle - \left\langle U^{\alpha-e_j}D_{[1,n]\setminus\{j\}}T_j h, U^\beta D_{[1,n]\setminus\{j\}}g\right\rangle \\ &- \left\langle U^{\alpha+e_j}D_{[1,n]\setminus\{j\}}h, U^\beta D_{[1,n]\setminus\{j\}}T_j g\right\rangle + \left\langle U^\alpha D_{[1,n]\setminus\{j\}}T_j h, U^\beta D_{[1,n]\setminus\{j\}}T_j g\right\rangle = 0\end{aligned}$$

by (3). $\square$

Let $\alpha \in \mathbb{Z}_+^n$ and $\max\{\alpha_j : 1 \le j \le n\} \le 1$. Then $U^\alpha = U_F$, where $F = \operatorname{supp}\alpha$. For $F, G \subset [1,n]$, $F \cap G = \emptyset$ the spaces $U_F D_{[1,n]} H$ and $U_G D_{[1,n]} H$ are not orthogonal in general. However, we can express their "angle".

Lemma 2. Let $F, G, A \subset [1,n]$, $F \cap G = \emptyset$, $h, g \in H$. Then

$$\big\langle U_F D_A h, U_G D_A g\big\rangle = \sum_{\substack{F_1 \subset F\cap A\\ G_1 \subset G \cap A}} (-1)^{|F_1|+|G_1|} \big\langle U_{F\setminus F_1} D_{A\setminus(F\cup G)} T_{F_1} h, U_{G\setminus G_1} D_{A\setminus(F\cup G)} T_{G_1} g\big\rangle.$$

Proof. The statement is trivial if $(F\cup G)\cap A = \emptyset$. We prove it by induction on $|(F\cup G)\cap A|$.

Let $(F \cup G) \cap A \neq \emptyset$ and suppose that the statement is true for all F', G', A' with $F' \cap G' = \emptyset$ and $|(F' \cup G') \cap A'| < |(F \cup G) \cap A|$. Without loss of generality we may assume that $F \cap A \neq \emptyset$. Let $j \in F \cap A$. We have

$$\begin{aligned}
&\langle U_F D_A h, U_G D_A g\rangle = \Big\langle U_F\big(U_j D_{A\setminus\{j\}} - D_{A\setminus\{j\}} T_j\big) h, U_G\big(U_j D_{A\setminus\{j\}} - D_{A\setminus\{j\}} T_j\big) g\Big\rangle\\
=&\big\langle U_F D_{A\setminus\{j\}} h, U_G D_{A\setminus\{j\}} g\big\rangle - \big\langle U_F D_{A\setminus\{j\}} T_j h, U_G U_j D_{A\setminus\{j\}} g\big\rangle\\
&\quad - \big\langle U_F U_j D_{A\setminus\{j\}} h, U_G D_{A\setminus\{j\}} T_j g\big\rangle + \big\langle U_F D_{A\setminus\{j\}} T_j h, U_G D_{A\setminus\{j\}} T_j g\big\rangle\\
=&\big\langle U_F D_{A\setminus\{j\}} h, U_G D_{A\setminus\{j\}} g\big\rangle - \big\langle U_{F\setminus\{j\}} D_{A\setminus\{j\}} T_j h, U_G D_{A\setminus\{j\}} g\big\rangle.
\end{aligned}$$

By the induction assumption this is equal to

$$\begin{aligned}
&\sum_{\substack{F_1 \subset (F\cap A)\setminus\{j\}\\ G_1 \subset (G\cap A)\setminus\{j\}}} (-1)^{|F_1|+|G_1|} \big\langle U_{F\setminus F_1} D_{A\setminus(F\cup G\cup\{j\})} T_{F_1} h, U_{G\setminus G_1} D_{A\setminus(F\cup G\cup\{j\})} T_{G_1} g\big\rangle\\
-&\sum_{\substack{F_1 \subset (F\cap A)\setminus\{j\}\\ G_1 \subset (G\cap A)\setminus\{j\}}} (-1)^{|F_1|+|G_1|} \big\langle U_{F\setminus(F_1\cup\{j\})} D_{A\setminus(F\cup G\cup\{j\})} T_{F_1} T_j h, U_{G\setminus G_1} D_{A\setminus(F\cup G\cup\{j\})} T_{G_1} g\big\rangle\\
=&\sum_{\substack{F_1 \subset F\cap A\\ G_1 \subset G\cap A}} (-1)^{|F_1|+|G_1|} \big\langle U_{F\setminus F_1} D_{A\setminus(F\cup G)} T_{F_1} h, U_{G\setminus G_1} D_{A\setminus(F\cup G)} T_{G_1} g\big\rangle.
\end{aligned}$$

$\square$

Lemma 3. Let $R, S, B \subset [1,n]$ be mutually disjoint sets, $h, g \in H$. Then

$$\big\langle U_R D_B h, U_S D_B g\big\rangle = \sum_{A\subset B} \langle T_{R\cup A} h, T_{G\cup A}\rangle \sum_{\substack{C_1, C_2 \subset A\\ C_1\cap C_2 = \emptyset}} (-1)^{|C_1|+|C_2|} c_{R\cup C_1, S\cup C_2}.$$

Proof. We have

$$\begin{aligned}
&\langle U_R D_B h, U_S D_B g\rangle = \sum_{B_1, B_2 \subset B} (-1)^{|B\setminus B_1| + |B\setminus B_2|} \big\langle U_R U_{B_1} T_{B\setminus B_1} h, U_S U_{B_2} T_{B\setminus B_2} g\big\rangle\\
=&\sum_{B_1, B_2 \subset B} (-1)^{|B_1|+|B_2|} \big\langle U_R U_{B_1\setminus B_2} T_{B\setminus B_1} h, U_S U_{B_2\setminus B_1} T_{B\setminus B_2} g\big\rangle\\
=&\sum_{B_1, B_2 \subset B} (-1)^{|B_1|+|B_2|} \big\langle T_R T_{B\setminus(B_1\cap B_2)} h, T_S T_{B\setminus(B_1\cap B_2)} g\big\rangle c_{R\cup(B_1\setminus B_2), S\cup(B_2\setminus B_1)}\\
=&\sum_{A\subset B} \langle T_{R\cup A} h, T_{S\cup A} g\rangle \sum_{\substack{B_1, B_2 \subset B\\ A = B\setminus(B_1\cap B_2)}} (-1)^{|B_1|+|B_2|} c_{R\cup(B_1\setminus B_2), S\cup(B_2\setminus B_1)}.
\end{aligned}$$

Setting $C_1 = B_1 \setminus B_2 = B_1 \cap A$ and $C_2 = B_2 \setminus B_1 = B_2 \cap A$ we have

$$\langle U_R D_B h, U_S D_B g\rangle = \sum_{A\subset B} \langle T_{R\cup A} h, T_{G\cup A}\rangle \sum_{\substack{C_1,C_2\subset A\\ C_1\cap C_2=\emptyset}} (-1)^{|C_1|+|C_2|} c_{R\cup C_1, S\cup C_2}.$$

□

Proposition 4. Let $F, G \subset [1,n]$, $F \cap G = \emptyset$. Let $h, g \in H$. Then

$$\langle U_F D_{[1,n]} h, U_G D_{[1,n]} g\rangle = \sum_{A\subset[1,n]\setminus(F\cup G)} \langle T_{F\cup A} h, T_{G\cup A} g\rangle \tilde{r}_{F,G,A},$$

where

$$\tilde{r}_{F,G,A} = \sum_{\substack{F'\subset F\\ G'\subset G}} \sum_{\substack{C_1,C_2\subset A\\ C_1\cap C_2=\emptyset}} (-1)^{|C_1|+|C_2|+|F\setminus F'|+|G\setminus G'|} c_{F'\cup C_1, G'\cup C_2}$$

(note that the subsets $F, G, A \subset \{1,\dots,n\}$ are mutually disjoint).

Proof. We have

$$\begin{aligned}
\langle U_F D_{[1,n]} h, U_G D_{[1,n]} g\rangle &= \sum_{\substack{F_1\subset F\\ G_1\subset G}} (-1)^{|F_1|+|G_1|} \big\langle U_{F\setminus F_1} D_{[1,n]\setminus(F\cup G)} T_{F_1} h, U_{G\setminus G_1} D_{[1,n]\setminus(F\cup G)} T_{G_1} g\big\rangle \\
&= \sum_{\substack{F_1\subset F\\ G_1\subset G}} (-1)^{|F_1|+|G_1|} \sum_{A\subset[1,n]\setminus(F\cup G)} \big\langle T_{(F\setminus F_1)\cup A} T_{F_1} h, T_{(G\setminus G_1)\cup A} T_{G_1} g\big\rangle \\
&\qquad \sum_{\substack{C_1,C_2\subset A\\ C_1\cap C_2=\emptyset}} (-1)^{|C_1|+|C_2|} c_{(F\setminus F_1)\cup C_1, (G\setminus G_1)\cup C_2} \\
&= \sum_{A\subset[1,n]\setminus(F\cup G)} \langle T_{F\cup A} h, T_{G\cup A} g\rangle \tilde{r}_{F,G,A},
\end{aligned}$$

where

$$\tilde{r}_{F,G,A} = \sum_{\substack{F'\subset F\\ G'\subset G}} \sum_{\substack{C_1,C_2\subset A\\ C_1\cap C_2=\emptyset}} (-1)^{|C_1|+|C_2|+|F\setminus F'|+|G\setminus G'|} c_{F'\cup C_1, G'\cup C_2}.$$

□

Theorem 5. Let $T = (T_1,\dots,T_n) \in B(H)^n$ have a dilation $U = (U_1,\dots,U_n) \in B(K)^n$ satisfying (2). Then

$$\sum_{\substack{F,G,A\subset[1,n]\\ \text{mut.disjoint}}} \tilde{r}_{F,G,A} \langle T_{F\cup A} h, T_{G\cup A} h\rangle \geq 0$$

for all $h \in H$.

Proof. Let $h \in H$ and $N \in \mathbb{N}$. Consider the element

$$x = \sum_{\substack{\alpha\in\mathbb{Z}_+^n\\ \alpha\leq(N,\dots,N)}} U^\alpha D_{[1,n]} h \in K.$$

Then

$$0 \leq N^{-n}\|x\|^2 = N^{-n} \sum_{\substack{\alpha,\beta\in\mathbb{Z}_+^n\\ \alpha,\beta\leq(N,\dots,N)}} \langle U^{(\alpha-\beta)_+} D_{[1,n]} h, U^{(\beta-\alpha)_+} D_{[1,n]} h\rangle.$$

Setting $\gamma = \min\{\alpha, \beta\}$ one gets

$$0 \le N^{-n} \sum_{\substack{F,G \subset [1,n] \\ F \cap G = \emptyset}} \langle U_F D_{1,n]} h, U_G D_{[1,n]} h \rangle \cdot \operatorname{card} \left\{ \gamma \in \mathbb{Z}_+^n : \gamma + e_F, \gamma + e_G \le (N, \dots, N) \right\}$$

$$= N^{-n} \sum_{\substack{F,G \subset [1,n] \\ F \cap G = \emptyset}} \langle U_F D_{[1,n]} h, U_G D_{[1,n]} h \rangle \cdot N^{|F \cup G|} (N+1)^{n - |F \cup G|}.$$

Letting $N \to \infty$, we have

$$0 \le \sum_{\substack{F,G \subset [1,n] \\ F \cap G = \emptyset}} \langle U_F D_{[1,n]} h, U_G D_{[1,n]} h \rangle = \sum_{\substack{F,G,A \subset [1,n] \\ \text{mut.disjoint}}} \tilde{r}_{F,G,A} \langle T_{F \cup A} h, T_{G \cup A} h \rangle.$$

□

Instead of considering triples of pairwise disjoint subsets $F, G, A \subset \{1, \dots, n\}$ it is possible to simplify the notation by considering two sets $F \cup A$ and $G \cup A$ in a general position.

For $F, G \subset \{1, \dots, n\}$ let

$$r_{F,G} = \tilde{r}_{F \setminus G, G \setminus F, F \cap G} = \sum_{\substack{F' \subset F, G' \subset G \\ F' \cap G' = \emptyset}} (-1)^{|F| + |G| + |F' \cup G'|} c_{F',G'}. \tag{4}$$

Then the condition from the previous theorem becomes

$$\sum_{F,G \subset [1,n]} r_{F,G} \langle T_F h, T_G h \rangle \ge 0 \tag{5}$$

for all $h \in H$.

Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{T}^n$. The n-tuple $\varepsilon T = (\varepsilon_1 T_1, \dots, \varepsilon_n T_n)$ has dilation $\varepsilon U = (\varepsilon_1 U_1, \dots, \varepsilon_n U_n)$ satisfying (2). Thus we have

Theorem 6. Let $T = (T_1, \dots, T_n) \in B(H)^n$ have a dilation $U = (U_1, \dots, U_n) \in B(K)^n$ satisfying (2). Then

$$\sum_{F,G \subset [1,n]} r_{F,G} \big\langle (\varepsilon T)_F h, (\varepsilon T)_G h \big\rangle \ge 0$$

for all $\varepsilon \in \mathbb{T}^n$ and $h \in H$.

Note that $r_{F,G} = r_{G,F}$ for all subsets $F, G \subset \{1, \dots, n\}$. So one can write

$$\begin{aligned} & r_{F,G} \big\langle \varepsilon T)_F h, (\varepsilon T)_G h \big\rangle + r_{G,F} \big\langle (\varepsilon T)_G h, (\varepsilon T)_F h \big\rangle \\ =& 2 r_{F,G} \operatorname{Re} \big\langle (\varepsilon T)_F h, (\varepsilon T)_G h \big\rangle \end{aligned}$$

for all $\varepsilon \in \mathbb{T}^n$ and $h \in H$.

4 Sufficient conditions

We show that if the operators $T_1, \dots, T_n$ satisfy the vanishing condition $T_j^k \to 0$ in the strong operator topology for $j = 1, \dots, n$, then the condition in Theorem 6 is also sufficient.

Theorem 7. Let $T = (T_1, \dots, T_n) \in B(H)^n$ be a commuting n-tuple of operators satisfying $\mathrm{SOT} - \lim_{k\to\infty} T_j^k = 0$ for $j = 1, \dots, n$. Let $c_{F,G} \quad (F, G \subset \{1, \dots, \}, F \cap G = \emptyset)$ be real numbers satisfying $c_{\emptyset,\emptyset} = 1$ and $c_{G,F} = c_{F,G}$ for all F, G. The following statements are equivalent:

(i) T has a dilation $U = (U_1, \dots, U_n) \in B(K)^n$ such that

$$P_H U^\alpha h, U^\beta g|H = c_{\operatorname{supp}\alpha, \operatorname{supp}\beta} T^{*\beta} T^\alpha$$

for all $\alpha, \beta \in \mathbb{Z}_+^n$, $\operatorname{supp}\alpha \cap \operatorname{supp}\beta = \emptyset$;

(ii)

$$\sum_{F,G\subset[1,n]} r_{F,G} \big\langle (\varepsilon T)_F h, (\varepsilon T)_G h \big\rangle \geq 0 \tag{6}$$

for all $\varepsilon \in \mathbb{T}^n$ and $h \in H$.

Proof. The implication (i)⇒(ii) was proved in the previous section.

Let T satisfy (ii). It is sufficient to show that the function $\Phi : \mathbb{Z}^n \to B(H)$ defined by $\Phi(\alpha) = c_{\operatorname{supp}\alpha_+, \operatorname{supp}\alpha_-} T^{*\alpha_-} T^{\alpha_+}$ is a positive definite function on the group $\mathbb{Z}^n$, i.e., for each finite system $(h_\alpha)_{\alpha\in\Lambda}$ of vectors in H we have

$$\sum_{\alpha,\alpha'\in\Lambda} \langle \Phi(\alpha - \alpha') h_\alpha, h_{\alpha'} \rangle \geq 0,$$

see [4].

Let $(h_\alpha)_{\alpha\in\Lambda}$ be a finite system of vectors in H. Let $N \in \mathbb{N}$ satisfy $N > 2\max\{|\alpha_j| : \alpha \in \Lambda, j = 1, \dots, n\}$.

For $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{T}^n$ consider the vector

$$x_N(\varepsilon) = \sum_{\alpha\in\Lambda} \sum_{\substack{\beta\in\mathbb{Z}_+ \\ \beta\leq(N,\dots,N)}} \varepsilon^{\beta-\alpha} T^\beta h_\alpha.$$

Let m be the Lebesgue measure on $\mathbb{T}^n$. Using (6) we have

$$\begin{aligned} 0 &\leq \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \sum_{F,G\subset[1,n]} r_{F,G} \big\langle (\varepsilon T)_F x_N(\varepsilon), (\varepsilon T)_G x_N(\varepsilon) \big\rangle dm(\varepsilon) \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \sum_{F,G\subset[1,n]} r_{F,G} \sum_{\alpha,\alpha'\in\Lambda} \sum_{\substack{\beta,\beta'\in\mathbb{Z}_+^n \\ \beta,\beta'\leq(N,\dots,N)}} \varepsilon^{\beta-\alpha+e_F} \bar{\varepsilon}^{\beta'-\alpha'+e_G} \langle T_F T^\beta h_\alpha, T_G T^{\beta'} h_{\alpha'} \rangle dm(\varepsilon). \end{aligned}$$

All terms with $\beta - \alpha + \varepsilon_F \neq \beta' - \alpha' + e_G$ will disappear in the integration. For the remaining terms let $\gamma = \beta - \alpha + \varepsilon_F = \beta' - \alpha' + e_G$.

Thus we have

$$\begin{aligned} 0 &\leq \sum_{F,G\subset[1,n]} r_{F,G} \sum_{\alpha,\alpha'\in\Lambda} \sum_{\substack{\gamma:(0,\dots,0)\leq\gamma+\alpha-e_F\leq(N,\dots,N) \\ (0,\dots,0)\leq\gamma+\alpha'-e_G\leq(N,\dots,N)}} \langle T^{\gamma+\alpha} h_\alpha, T^{\gamma+\alpha'} h_{\alpha'} \rangle \\ &= \sum_{\alpha,\alpha'\in\Lambda} \langle Q_N(\alpha, \alpha') h_\alpha, h_{\alpha'} \rangle, \end{aligned}$$

where

$$Q_N(\alpha,\alpha') = \sum_{F,G\subset[1,n]} r_{F,G} \sum_{\substack{\gamma: e_F\le\gamma+\alpha\le(N,\dots,N)+e_F\\ e_G\le\gamma+\alpha'\le(N,\dots,N)+e_G}} T^{*\gamma+\alpha'}T^{\gamma+\alpha}.$$

Write $\tilde\alpha = \min\{\alpha,\alpha'\}$ and $\alpha = \tilde\alpha+\delta$, $\alpha' = \tilde\alpha+\delta'$. Then $\delta,\delta'\in\mathbb{Z}_+^n$ and $\operatorname{supp}\delta\cap\operatorname{supp}\delta' = \emptyset$. Setting $\eta = \gamma+\tilde\alpha$ we have

$$\begin{aligned} Q_N(\alpha,\alpha') &= \sum_{F,G\subset[1,n]} r_{F,G} \sum_{\substack{\eta: e_F\le\eta+\delta\le(N,\dots,N)+e_F\\ e_G\le\eta+\delta'\le(N,\dots,N)+e_G}} T^{*\eta+\delta'}T^{\eta+\delta} \\ &= \sum_{\substack{\eta\in\mathbb{Z}_+^n\\ \eta+\delta+\delta'\le(N+1,\dots,N+1)}} T^{*\eta+\delta'}T^{\eta+\delta}s_\eta, \end{aligned}$$

where

$$s_\eta = \sum_{\substack{\{j:\eta_j+\delta_j=N+1\}\subset F\subset\operatorname{supp}(\eta+\delta)\\ \{j:\eta_j+\delta'_j=N+1\}\subset G\subset\operatorname{supp}(\eta+\delta')}} r_{F,G}. \tag{7}$$

We need the following lemma:

Lemma 8. Let $\eta,\delta,\delta'\in\mathbb{Z}_+^n$, $\operatorname{supp}\delta\cap\operatorname{supp}\delta' = \emptyset$, $\max\{\eta_j+\delta_j, \eta_j+\delta'_j, j=1,\dots,n\}\le N+1$. Write for short $D = \operatorname{supp}(\eta+\delta)$, $D' = \operatorname{supp}(\eta+\delta')$, $E = \{j:\eta_j+\delta_j = N+1\}$, $E' = \{j:\eta_j+\delta'_j = N+1\}$. Then:

(i) if there exists $j\in\{1,\dots,n\}$ such that $j\in(D\cap D')\setminus(E\cup E')$ then

$$\sum_{\substack{E\subset F\subset D\\ E'\subset G\subset D'}} r_{F,G} = 0;$$

(ii) if $D\cap D' = \emptyset$ then

$$\sum_{\substack{F\subset D\\ G\subset D'}} r_{F,G} = c_{D,D'}.$$

Proof. (i) Using (4) we have

$$\sum_{\substack{E\subset F\subset D\\ E'\subset G\subset D'}} r_{F,G} = \sum_{\substack{M\subset D, L\subset D'\\ M\cap L=\emptyset}} c_{M,L}a_{M,L},$$

where

$$\begin{aligned} a_{M,L} &= \sum_{\substack{M\cup E\subset F\subset D\\ L\cup E'\subset G\subset D'}} (-1)^{|M\cup L|}(-1)^{|F|+|G|} \\ &= (-1)^{|M\cup L|}\Big(\sum_{M\cup E\subset F\subset D}(-1)^{|F|}\Big)\Big(\sum_{L\cup E'\subset G\subset D'}(-1)^{|G|}\Big). \end{aligned}$$

Let $j\in(D\cap D')\setminus(E\cup E')$. Since $M\cap L=\emptyset$, either $j\notin M$ or $j\notin L$. If $j\notin M$ then $\sum_{M\cup E\subset F\subset D}(-1)^{|F|} = 0$, and so $a_{M,L}=0$. Similarly, if $j\notin L$ then $\sum_{L\cup E'\subset G\subset D'}(-1)^{|G|} = 0$, and so $a_{M,L}=0$. Hence

$$\sum_{\substack{E\subset F\subset D\\ E'\subset G\subset D'}} r_{F,G} = 0.$$

(ii) Let $D\cap D'=\emptyset$. Again

$$\sum_{\substack{F\subset D\\ G\subset D'}} r_{F,G}=\sum_{\substack{M\subset D, L\subset D'\\ M\cap L=\emptyset}} c_{M,L}a_{M,L},$$

where

$$a_{M,L}=(-1)^{|M\cup L|}\Big(\sum_{M\subset F\subset D}(-1)^{|F|}\Big)\Big(\sum_{L\subset G\subset D')}(-1)^{|G|}\Big).$$

If $M\neq D$ then $\sum\limits_{M\subset F\subset D}(-1)^{|F|}=0$ and so $a_{M,L}=0$. Similarly, if $L\neq D'$ then $\sum\limits_{L\subset G\subset D'}(-1)^{|G|}=0$, and so $a_{M,L}=0$. If $M=D$ and $L=D'$ then $a_{M,L}=1$. So

$$\sum_{\substack{F\subset D\\ G\subset D'}} r_{F,G}=c_{D,D'}.$$

□

Continuation of the proof of Theorem 7: Recall that

$$Q_N(\alpha,\alpha')=\sum_{\substack{\eta\in\mathbb{Z}_+^n\\ \eta+\delta+\delta'\leq(N+1,\dots,N+1)}} T^{*\eta+\delta'}T^{\eta+\delta}s_\eta,$$

where

$$s_\eta=\sum_{\substack{\{j:\eta_j+\delta_j=N+1\}\subset F\subset\operatorname{supp}(\eta+\delta)\\ \{j:\eta_j+\delta'_j=N+1\}\subset G\subset\operatorname{supp}(\eta+\delta')}} r_{F,G}.$$

If there exists $j\in\operatorname{supp}\eta$ with $\max\{\eta_j+\delta_j,\eta_j+\delta'_j\}\leq N$, then $s_\eta=0$ by Lemma 8 (i). So

$$Q_N(\alpha,\alpha')=\sum T^{*\eta+\delta'}T^{\eta+\delta}s_\eta,$$

where the sum is taken over all $\eta\in\mathbb{Z}_+^n$ such that $\max\{\eta_j+\delta_j,\eta_j+\delta'_j\}=N+1$ for all $j\in\operatorname{supp}\eta$. Note that the number of nonzero terms in this sum does not depend on N (for N large enough). Moreover, the coefficients s_η are bounded independently of N. Since $T_j^N\to 0$ in the strong operator topology for $j=1,\dots,n$, we have

$$\lim_{N\to\infty}\langle Q_N(\alpha,\alpha')h_\alpha,h_{\alpha'}\rangle=s_{(0,\dots,0)}\langle T^{*\delta'}T^\delta h_\alpha,h_{\alpha'}\rangle=c_{\operatorname{supp}\delta,\operatorname{supp}\delta'}\langle T^{*\delta'}T^\delta h_\alpha,h_{\alpha'}\rangle.$$

Hence

$$\begin{aligned}0&\leq\lim_{N\to\infty}\sum_{\alpha,\alpha'\in\Lambda}\langle Q_N(\alpha,\alpha')h_\alpha,h_{\alpha'}\rangle\\ &=\sum_{\alpha,\alpha'\in\Lambda}c_{\operatorname{supp}(\alpha-\alpha')_+,\operatorname{supp}(\alpha-\alpha')_-}\langle T^{*(\alpha-\alpha')_-}T^{(\alpha-\alpha')_+}h_\alpha,h_{\alpha'}\rangle.\end{aligned}$$

Hence the function

$$\Phi(\alpha)=c_{\operatorname{supp}\alpha_+,\operatorname{supp}\alpha_-}T^{*\alpha_-}T^{\alpha_+}$$

defined on the group $\mathbb{Z}^n$ is positive definite and there exists a unitary dilation $U=(U_1,\dots,U_n)\in B(K)^n$ such that

$$\langle U^\alpha h,U^\beta g\rangle=c_{\operatorname{supp}\alpha,\operatorname{supp}\beta}\langle T^\alpha h,T^\beta g\rangle$$

for all $h,g\in H$ and $\alpha,\beta\in\mathbb{Z}_+^n$ with $\operatorname{supp}\alpha\cap\operatorname{supp}\beta=\emptyset$. □

Remark 9. Conditions $T_j^k \to 0 \quad$ (SOT) in Theorem 7 are necessary. Even in the classical case of regular unitary dilations condition (6) is not sufficient (for details see below).

If we do not assume that $T_j^k \to 0$, then it is possible to modify condition (6) in the following way:

Theorem 10. Let $c_{F,G} \quad (F, G \subset \{1, \dots, \}, F \cap G = \emptyset)$ be real numbers satisfying (1). Let $T = (T_1, \dots, T_n) \in B(H)^n$ be a commuting n-tuple of operators. Suppose that

$$\sum_{F,G \subset [1,n]} r_{F,G} \big\langle (zT)_F h, (zT)_G h \big\rangle \geq 0 \tag{8}$$

for all $h \in H, z = (z_1, \dots, z_n) \in \mathbb{D}^n$. Then T has a dilation $U = (U_1, \dots, U_n) \in B(K)^n$ such that

$$\langle U^\alpha h, U^\beta g \rangle = c_{\operatorname{supp}\alpha, \operatorname{supp}\beta} \cdot \langle T^\alpha h, T^\beta g \rangle$$

for all $h, g \in H$, $\alpha, \beta \in \mathbb{Z}_+^n$, $\operatorname{supp}\alpha \cap \operatorname{supp}\beta = \emptyset$.

Proof. Let $0 < r < 1$. The n-tuple $rT = (rT_1, \dots, rT_n)$ satisfies conditions of Theorem 7. Therefore the function $\Phi_r(\alpha) = c_{\operatorname{supp}\alpha_+, \operatorname{supp}\alpha_-} (rT)^{*\alpha_-} (rT)^{\alpha_+}$ is positive definite on the group $\mathbb{Z}^n$. Letting $r \to 1_-$ we get that the function

$$\Phi(\alpha) = c_{\operatorname{supp}\alpha_+, \operatorname{supp}\alpha_-} T^{*\alpha_-} T^{\alpha_+}$$

is positive definite on the group $\mathbb{Z}^n$. So T has a unitary dilation satisfying $\langle T^\alpha h, T^\beta g \rangle = c_{\operatorname{supp}\alpha, \operatorname{supp}\beta} \langle U^\alpha h, U^\beta g \rangle$ for all $h, g \in H, \alpha, \beta \in \mathbb{Z}_+^n, \operatorname{supp}\alpha \cap \operatorname{supp}\beta = \emptyset$. $\square$

Recall that a commuting n-tuple $T = (T_1, \dots, T_n) \in B(H)^n$ is polynomially bounded if there exists a constant $K > 0$ such that

$$\|p(T)\| \leq K \|p\|$$

for all polynomials p of n variables, where $\|p\| = \sup\{|p(z)| : z \in \mathbb{D}^n\}$.

Theorem 11. Let $c_{F,G} \quad (F, G \subset \{1, \dots, n\}, F \cap G = \emptyset)$ be real numbers satisfying $c_{\emptyset,\emptyset} = 1$, $c_{F,\emptyset} \neq 0$ and $c_{G,F} = c_{F,G}$ for all F, G. Let $T = (T_1, \dots, T_n) \in B(H)^n$ be a commuting n-tuple of operators having a unitary dilation U determined by the system $(c_{F,G})$. Then T is polynomially bounded.

Proof. Let $p(z) = \sum_{\alpha \in \mathbb{Z}_+^n}$ be a polynomial in n variables. For $F \subset \{1, \dots, n\}$ let

$$p_F(z) = \sum_{\substack{\alpha \in \mathbb{Z}_+^n \\ \operatorname{supp}\alpha \subset F}} a_\alpha z^\alpha.$$

Clearly $\|p_F\| \leq \|p\|$. We have

$$\sum_{F \subset [1,n]} p_F(U) \sum_{F \subset G \subset [1,n]} (-1)^{|G \setminus F|} c_{G,\emptyset}^{-1} = \sum_{F \subset [1,n]} \sum_{F' \subset F} \sum_{\operatorname{supp}\alpha = F'} a_\alpha U^\alpha \sum_{G : F \subset G \subset [1,n]} (-1)^{|G \setminus F|} c_{G,\emptyset}^{-1}$$

$$= \sum_{F' \subset [1,n]} \sum_{\operatorname{supp}\alpha = F'} a_\alpha U^\alpha \Big(\sum_{G : F' \subset G \subset [1,n]} c_{G,\emptyset}^{-1} \sum_{F : F' \subset F \subset G} (-1)^{|G \setminus F|} \Big)$$

$$= \sum_{F' \subset [1,n]} \sum_{\alpha : \operatorname{supp}\alpha = F'} a_\alpha U^\alpha c_{F',\emptyset}^{-1}.$$

So

$$\begin{aligned}\|p(T)\| &= \Big\| P_H \sum_{F'\subset[1,n]} \sum_{\operatorname{supp}\alpha=F'} a_\alpha U^\alpha c^{-1}_{\operatorname{supp}\alpha,\emptyset} | H \Big\| \\ &\le \Big\| \sum_{F'\subset[1,n]} \sum_{\operatorname{supp}\alpha=F'} a_\alpha U^\alpha c^{-1}_{\operatorname{supp}\alpha,\emptyset} \Big\| = \Big\| \sum_{F\subset[1,n]} p_F(U) \sum_{G:F\subset G\subset[1,n]} (-1)^{|G\setminus F|} c^{-1}_{G,\emptyset} \Big\| \\ &\le \|p\| \cdot \sum_{F\subset[1,n]} \Big| \sum_{G:F\subset G\subset[1,n]} (-1)^{|G\setminus F|} c^{-1}_{G,\emptyset} \Big|.\end{aligned}$$

Hence T is polynomially bounded with the polynomial bound

$$K \le \sum_{F\subset[1,n]} \Big| \sum_{G:F\subset G\subset[1,n]} (-1)^{|G\setminus F|} c^{-1}_{G,\emptyset} \Big|.$$

□

5 Examples

1. Let $n = 1$ and $\rho > 0$. Set $c_{\{1\},\emptyset} = c_{\emptyset,\{1\}} = \rho^{-1}$. We have $r_{\{1\},\emptyset} = r_{\emptyset,\{1\}} = c_{\{1\},\emptyset} - c_{\emptyset,\emptyset} = \frac{1}{\rho} - 1$ and $r_{\{1\},\{1\}} = 1 - \frac{2}{\rho}$. Clearly $r_{\emptyset,\emptyset} = 1$. Hence condition (6) becomes

$$\|h\|^2 + 2\Big(\frac{1}{\rho} - 1\Big)\operatorname{Re}\langle \varepsilon Th, h\rangle + \Big(1 - \frac{2}{\rho}\Big)\|Th\|^2 \ge 0$$

for all $h \in H, |\varepsilon| = 1$. Similarly condition (8) becomes

$$\|h\|^2 + 2\Big(\frac{1}{\rho} - 1\Big)\operatorname{Re}\langle zTh, h\rangle + \Big(1 - \frac{2}{\rho}\Big)\|zTh\|^2 \ge 0$$

for all $h \in H, z \in \mathbb{D}$, which is the well-known characterization of ρ-contractions.

The condition becomes simpler for either $\rho = 1$ or $\rho = 2$. For $\rho = 1$ it reduces to $\|h\|^2 - \|Th\|^2 \ge 0$, i.e., T is a contraction. For $\rho = 2$ it reduces to $\|h\|^2 - \operatorname{Re}\langle zTh, h\rangle \ge 0$, i.e., the numerical range of T is contained in the closed unit disc.

2. Let $n = 1$. The parameter $c_{\{1\},\emptyset}$ may be any real number, not only positive.

The case $c_{\{1\},\emptyset} = 0$ is rather trivial. In this case $r_{F,G} = (-1)^{|F|+|G|}$. Condition (8) then becomes

$$\|h\|^2 - 2\operatorname{Re}\langle zTh, h\rangle + \|zTh\|^2 \ge 0,$$

which is satisfied for any operator $T \in B(H)$. The corresponding dilation is $U = I_H \otimes S$ acting in the space $H \otimes \ell_2(\mathbb{Z})$, where S is the bilateral shift in $\ell_2(\mathbb{Z})$.

If $c_{\{1\},\emptyset} < 0$, then $r_{\{1\},\emptyset} = c_{\{1\},\emptyset} - 1$ and $r_{\{1\},\{1\}} = 1 - 2c_{\{1\},\emptyset}$. Thus (8) becomes

$$\|h\|^2 + 2(c_{\{1\},\emptyset} - 1)\operatorname{Re}\langle zTh, h\rangle + (1 - 2c_{\{1\},\emptyset})\|zTh\|^2 \ge 0.$$

This enables to define ρ-contractions for negative values of $\rho := c^{-1}_{\{1\},\emptyset}$.

3. Let $n \ge 1$ and $c_{F,G} = 0$ for all F, G with $F \cup G \ne \emptyset$. This case is again trivial. We have $r_{F,G} = (-1)^{|F|+|G|}$ and condition (8) becomes

$$\sum_{F,G\subset[1,n]} (-1)^{|F|+|G|} \big\langle (zT)_F h, (zT)_G h \big\rangle \ge 0$$

for all $h \in H$ and $z \in \mathbb{D}^n$. However, this condition is satisfied for any commuting n-tuple T since the left-hand side of the condition is equal to

$$\Big\| \sum_{F\subset[1,n]} (-1)^{|F|}(zT)_F h\Big\|^2.$$

4. Let $n \geq 1$ and $c_{F,G} = 1$ for all $F, G \subset [1,n]$, $F \cap G = \emptyset$. Then

$$\begin{aligned} r_{F,G} &= (-1)^{|F|+|G|} \sum_{\substack{F'\subset F, G'\subset G\\ F'\cap G'=\emptyset}} (-1)^{|F'\cup G'|} \\ &= (-1)^{|F|+|G|}\Big(\sum_{F_1\subset F\setminus G} (-1)^{|F_1|}\Big)\Big(\sum_{G_1\subset G\setminus F} (-1)^{|G_1|}\Big)\Big(\sum_{\substack{F_2,G_2\subset F\cap G\\ F_2\cap G_2=\emptyset}} (-1)^{|F_2\cup G_2|}\Big). \end{aligned}$$

If $F \neq G$ then either $F \setminus G \neq \emptyset$ or $G \setminus F \neq \emptyset$. In both cases $r_{F,G} = 0$.

Furthermore,

$$r_{F,F} = \sum_{\substack{F_2,G_2\subset F\\ F_2\cap G_2=\emptyset}} (-1)^{|F_2\cup G_2|} = (-1)^{|F|}.$$

Hence condition (6) becomes

$$\sum_{F\subset[1,n]} (-1)^{|F|}\|T_F h\|^2 \geq 0 \tag{9}$$

for all $h \in H$. So if T saisfies (9) and $T_j^k \to 0$ in the strong operator topology for all j, then T has the regular unitary dilation. However, condition (9) is satisfied for example if one of the operators T_j is an isometry and the remaining operators are arbitrary. The classical Brehmer conditions state that T has a regular unitary dilation if and only if

$$\sum_{F\subset B} (-1)^{|F|}\|T_F h\|^2 \geq 0 \tag{10}$$

for all $h \in H, B \subset [1,n]$. This is in fact equivalent to (8) which becomes

$$\sum_{F\subset[1,n]} (-1)^{|F|}\|(rT)_F h\|^2 \geq 0 \tag{11}$$

for all $h \in H, r \in [0,1]^n$. Indeed, if (11) is satisfied and $B \subset [1,n]$, then set $r_j = 1$ for all $j \in B$ and let $r_j = 0$ for all $j \notin B$. Thus one gets (10).

Conversely, suppose that T satisfy (10). Let $r_n \in [0,1]$ and $S := (T_1, \ldots, T_{n-1}, r_n T_n)$. Then

$$\sum_{F\subset[1,n]} (-1)^{|F|}\|S_F h\|^2 = \sum_{n\notin F\subset[1,n]} (-1)^{|F|}\|T_F h\|^2 + \sum_{n\in F\subset[1,n]} (-1)^{|F|} r_n^2 \|T_F h\|^2 = a + r_n^2 b,$$

where

$$a = \sum_{n\notin F\subset[1,n]} (-1)^{|F|}\|T_F h\|^2 \geq 0$$

and

$$b = \sum_{n\in F\subset[1,n]} (-1)^{|F|}\|T_F h\|^2 = \sum_{F\subset[1,n-1]} (-1)^{|F|+1}\|T_F T_n h\|^2 \leq 0.$$

Since T satisfies (10), we have $a+b \geq 0$, and so $a+r_n^2 b \geq 0$ for all $r_n \in [0,1]$. Thus (11) is satisfied for $r=(1,\dots,1,r_n)$. Inductively, we get that T satisfies (11) for any $r \in [0,1]^n$.

5. Let $\rho_1,\dots,\rho_n > 0$. Set $c_{F,G} = \prod_{j\in F\cup G} \rho_j^{-1} \quad (F,G \subset [1,n], F\cap G = \emptyset)$. Then

$$(-1)^{|F|+|G|} r_{F,G} = \sum_{\substack{F'\subset F, G'\subset G\\ F'\cap G'=\emptyset}} (-1)^{|F'\cup G'|} \prod_{j\in F'\cup G'} \rho_j^{-1}$$
$$= \Big(\sum_{F_1\subset F\setminus G} (-1)^{|F_1|} \prod_{j\in F_1} \rho_j^{-1}\Big)\Big(\sum_{G_1\subset G\setminus F} (-1)^{|G_1|} \prod_{j\in G_1} \rho_j^{-1}\Big)\Big(\sum_{\substack{F_2,G_2\subset F\cap G\\ F_2\cap G_2=\emptyset}} (-1)^{|F_2\cup G_2|} \prod_{j\in F_2\cup G_2} \rho_j^{-1}\Big)$$
$$= \prod_{j\in F\div G}\Big(1-\frac{1}{\rho_j}\Big)\cdot \prod_{j\in F\cap G}\Big(1-\frac{2}{\rho_j}\Big),$$

where $F\div G = (F\setminus G)\cup(G\setminus F)$ denotes the symmetrical difference of F and G. Hence

$$r_{F,G} = \prod_{j\in F\div G}\Big(\frac{1}{\rho_j}-1\Big)\cdot \prod_{j\in F\cap G}\Big(1-\frac{2}{\rho_j}\Big).$$

Conditions (6) and (8) then unify the characterizations of ρ-dilations of single contractions and of regular unitary dilations of n-tuples.

6. The previous conditions get simplified if $\rho_1=\dots=\rho_n=1$: this is the case of regular unitary dilations. Another interesting case is for $\rho_1=\dots=\rho_n=2$. In this case $r_{F,G}=0$ if $F\cap G\neq\emptyset$. If $F\cap G=\emptyset$ then

$$r_{F,G} = \prod_{j\in F\cup G}\Big(\frac{1}{\rho_j}-1\Big) = \Big(-\frac{1}{2}\Big)^{|F\cup G|}.$$

Condition (6) then becomes

$$\sum_{F,G\subset[1,n], F\cap G=\emptyset} 2^{-|F\cup G|}\big\langle(\varepsilon T)_F h, (\varepsilon T)_G h\big\rangle \geq 0$$

for all $h\in H$ and $\varepsilon\in\mathbb{T}^n$.

7. Let $\rho_1,\dots,\rho_n>0$, let $c_{F,G}=0$ if $F\neq\emptyset\neq G$, and $c_{F,\emptyset}=\prod_{j\in F}\rho_j^{-1}$. Then

$$(-1)^{|F|+|G|} r_{F,G} = \sum_{F'\subset F}(-1)^{|F'|}\prod_{j\in F'}\rho_j^{-1} + \sum_{G'\subset G}(-1)^{|G'|}\prod_{j\in G'}\rho_j^{-1} - 1$$
$$= \prod_{j\in F}\Big(1-\frac{1}{\rho_j}\Big) + \prod_{j\in G}\Big(1-\frac{1}{\rho_j}\Big) - 1.$$

Condition (6) becomes simpler if $\rho_1=\dots=\rho_n=1$. Then $r_{F,G}=(-1)^{|F|+|G|+1}$ if $F\neq\emptyset\neq G$, $r_{F,\emptyset}=0$ if $F\neq\emptyset$ and $r_{\emptyset,\emptyset}=1$. For details see Example 10 below.

8. Let $\rho>0$ and $c_{F,G}=\rho^{-1}$ for all $F,G\subset[1,n]$, $F\cap G=\emptyset$, $F\cup G\neq\emptyset$. Then

$$(-1)^{|F|+|G|} r_{F,G} = \rho^{-1}\sum_{\substack{F'\subset F, G'\subset G\\ F'\cap G'=\emptyset}}(-1)^{|F'|+|G'|} + (1-\rho^{-1}).$$

If $F \setminus G \neq \emptyset$ or $G \setminus F \neq \emptyset$ then $r_{F,G} = (1-\rho^{-1})(-1)^{|F|+|G|}$. If $F \neq \emptyset$ then $r_{F,F} = \rho^{-1}(-1)^{|F|} + (1-\rho^{-1})$. Then

$$\sum_{F,G\subset[1,n]} r_{F,G}\langle T_F h, T_G h\rangle = (1-\rho^{-1})\Big\|\sum_{F\subset[1,n]}(-1)^{|F|}T_F h\Big\|^2 + \rho^{-1}\sum_{F\subset[1,n]}(-1)^{|F|}\|T_F h\|^2.$$

Condition (6) then becomes

$$(1-\rho^{-1})\Big\|\sum_{F\subset[1,n]}(\varepsilon T)_F h\Big\|^2 + \rho^{-1}\sum_{F\subset[1,n]}(-1)^{|F|}\|T_F h\|^2 \geq 0$$

for all $h \in H$ and $\varepsilon \in \mathbb{T}^n$.

In particular, if $\rho = 1$ then this reduces to

$$\sum_{F\subset[1,n]}(-1)^{|F|}\|T_F h\|^2 \geq 0$$

for all $h \in H$, which is again condition (9) for regular unitary dilations.

9. Let $\rho > 0$ and $c_{F,\emptyset} = \rho^{-1}\quad (F \neq \emptyset)$, $c_{F,G} = 0\quad (F \neq \emptyset \neq G)$. Then

$$(-1)^{|F|+|G|}r_{F,G} = \rho^{-1}\sum_{F'\subset F}(-1)^{|F'|} + \rho^{-1}\sum_{G'\subset G}(-1)^{|G'|} + (1-2\rho^{-1}).$$

If $F \neq \emptyset \neq G$ then $(-1)^{|F|+|G|}r_{F,G} = 1-2\rho^{-1}$. If $F \neq \emptyset$ then $(-1)^{|F|}r_{F,\emptyset} = 1-\rho^{-1}$. Finally, $r_{\emptyset,\emptyset} = 1$ as in all cases.

Hence condition (6) becomes

$$\begin{aligned}\|h\|^2 &+ 2(1-\rho^{-1})\sum_{\emptyset\neq F\subset[1,n]}(-1)^{|F|}\mathrm{Re}\,\big\langle(\varepsilon T)_F h, h\big\rangle \\ &+ (1-2\rho^{-1})\sum_{\substack{F,G\subset[1,n]\\ F\neq\emptyset\neq G}}(-1)^{|F|+|G|}\big\langle(\varepsilon T)_F h, (\varepsilon T)_G h\big\rangle \geq 0,\end{aligned}$$

or, equivalently,

$$\|h\|^2 + 2(1-\rho^{-1})\mathrm{Re}\Big\langle\sum_{\emptyset\neq F\subset[1,n]}(\varepsilon T)_F h, h\Big\rangle + (1-2\rho^{-1})\Big\|\sum_{F\subset[1,n],F\neq\emptyset}(\varepsilon T)_F h\Big\|^2 \geq 0 \qquad (12)$$

for all $h \in H$ and $\varepsilon \in \mathbb{T}^n$. Clearly (12) is the multivariable analogy of the characterization of ρ-dilations of single operators.

10. Condition (12) becomes simpler for $\rho = 1$ and $\rho = 2$.

For $\rho = 1$ we have the following characterization:

Theorem 12. Let $T = (T_1, \dots, T_n) \in B(H)^n$ be a commuting n-tuple of operators satisfying $T_j^k \to 0\quad$ (SOT) for $j = 1, \dots, n$. The following conditions are equivalent:

(i) there exists a unitary dilation $U = (U_1, \dots, U_n) \in B(K)^n$, $K \supset H$ such that

$$T^\alpha = P_H U^\alpha | H \qquad (\alpha \in \mathbb{Z}_+^n)$$

and

$$P_H U^{*\beta}U^\alpha | H = 0 \qquad (\alpha, \beta \in \mathbb{Z}_+^n, \mathrm{supp}\,\alpha \cap \mathrm{supp}\,\beta = \emptyset, |\alpha+\beta| \neq 0);$$

(ii) $\Big\| \sum\limits_{F\subset[1,n],F\neq\emptyset} (\varepsilon T)_F h \Big\| \leq \|h\|$ for all $h \in H$ and $\varepsilon \in \mathbb{T}^n$.

If $\Big\| \sum\limits_{F\subset[1,n],F\neq\emptyset} (zT)_F h \Big\| \leq \|h\|$ for all $h \in H$ and $z \in \mathbb{D}^n$ then it is possible to omit the condition $T_j^k \to 0 \quad \text{(SOT)}$ for $j = 1, \ldots, n$.

For $\rho = 2$ condition (12) becomes

$$\|h\|^2 + \sum_{F\subset[1,n],F\neq\emptyset} \operatorname{Re} \big\langle (\varepsilon T)_F h, h \big\rangle \geq 0$$

for all $h \in H$ and $\varepsilon \in \mathbb{T}^n$. Equivalently,

$$\operatorname{Re} \big\langle (I + \varepsilon_1 T_1) \cdots (I + \varepsilon_n T_n) h, h \big\rangle \geq 0$$

for all $h \in H$ and $\varepsilon \in \mathbb{T}^n$. Similarly, (8) becomes

$$\operatorname{Re} \big\langle (I + z_1 T_1) \cdots (I + z_n T_n) h, h \big\rangle \geq 0$$

for all $h \in H$ and $z \in \mathbb{D}^n$.

This condition seems to be the proper generalization of operators with 2-dilation, i.e., with numerical radius ≤ 1.

Thus we have

Theorem 13. Let $T = (T_1, \ldots, T_n) \in B(H)^n$ be a commuting n-tuple of operators satisfying $T_j^k \to 0 \quad \text{(SOT)}$ for $j = 1, \ldots, n$. The following conditions are equivalent:

(i) there exists a unitary dilation $U = (U_1, \ldots, U_n) \in B(K)^n$, $K \supset H$ such that

$$T^\alpha = 2 P_H U^\alpha | H \qquad (\alpha \in \mathbb{Z}_+^n)$$

and

$$P_H U^{*\beta} U^\alpha | H = 0 \qquad (\alpha, \beta \in \mathbb{Z}_+^n, \operatorname{supp} \alpha \cap \operatorname{supp} \beta = \emptyset, |\alpha + \beta| \neq 0);$$

(ii) $\operatorname{Re} \big\langle (I + \varepsilon_1 T_1) \cdots (I + \varepsilon_n T_n) h, h \big\rangle \geq 0$ for all $h \in H$ and $\varepsilon \in \mathbb{T}^n$.

If $\operatorname{Re} \big\langle (I + z_1 T_1) \cdots (I + z_n T_n) h, h \big\rangle \geq 0$ for all $h \in H$ and $z \in \mathbb{D}^n$ then it is not necessary to assume the conditions $T_j^k \to 0 \quad \text{(SOT)}$ for the existence of the unitary dilation satisfying (i).

6 Concluding remarks.

Remark 14. It is possible to consider complex values of numbers $c_{F,G}$, such that $c_{\emptyset,\emptyset} = 1$ and $c_{G,F} = \overline{c_{F,G}}$ for all F, G. However, this more general setting does not bring anything new. One can show that $r_{G,F} = \overline{r_{F,G}}$. So conditions (6) and (8) remain unchanged if we replace the values $c_{F,G}$ by their real parts $\operatorname{Re} c_{F,G}$.

Remark 15. The vanishing conditions $T_j^k \to 0 \quad$ (SOT) appear frequently in the dilation theory and usually simplify the situation. In our situation, without this assumption we proved that conditions (8) are sufficient for the existence of the unitary dilation satisfying (2). We do not know whether (8) is also necessary. Equivalently, suppose that $T = (T_1, \dots, T_n)$ has a dilation U satisfying (2) and $r = (r_1, \dots, r_n) \in [0,1]^n$. Does it follow that $rT = (r_1 T_1, \dots, r_n T_n)$ has also a unitary dilation satisfying (2)? This is the case for ρ-dilations of single operators as well as for regular unitary dilations. We do not know if it is true in general in our setting.

Another possibility is to consider condition (6) for all subsets of $\{1, \dots, n\}$, i.e.,

$$\sum_{F,G \subset B} r_{F,G} \big\langle (\varepsilon T)_F h, (\varepsilon T)_G g \big\rangle \geq 0 \tag{13}$$

for all $B \subset \{1, \dots, n\}, h \in H, \varepsilon \in \mathbb{T}^n$. Such a condition is usually considered for regular unitary dilations. Condition (13) is clearly necessary. We do not know if it is also sufficient.

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