On diffuse interface models of binary mixtures of compressible fluids

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Basic fields in diffuse interface modeling

Density

$$\varrho = \varrho(t, x)$$

Bulk velocity

$$\mathbf{u} = \mathbf{u}(t, x)$$

Phase variable - concentration difference

$$c = c(t, x)$$

Free energy, pressure, chemical potential

$$f = f(\varrho, c), \;\; p(\varrho, c) = \varrho^2 \frac{\partial f(\varrho, c)}{\partial \varrho}, \;\; \mu(\varrho, c) \approx \frac{\partial f(\varrho, c)}{\partial c}$$

Model by Anderson, McFadden and Wheeler

Mass conservation - equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equation

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, c)$$

$$= \mathrm{div}_x \mathbb{S}(c, \nabla_x \mathbf{u}) - \mathrm{div}_x \left(\nabla_x c \otimes \nabla_x c - \frac{|\nabla_x c|^2}{2} \mathbb{I} \right)$$

$$\mathbb{S}(c, \nabla_x \mathbf{u}) = \nu(c) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta(c) \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Cahn–Hilliard equation

$$\partial_t(\varrho c) + \mathrm{div}_x(\varrho c \mathbf{u}) = \Delta \mu, \;\; \varrho \mu = \varrho \frac{\partial f(\varrho, c)}{\partial c} - \Delta c$$

Model by Lowengrub and Truskinovsky

Mass conservation - equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equation

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, c)$$
$$= -\mathrm{div}_x \left(\varrho \nabla_x c \otimes \nabla_x c - \varrho \frac{|\nabla_x c|^2}{2} \mathbb{I} \right)$$

Cahn–Hilliard equation

$$\partial_t(\varrho c) + \mathrm{div}_x(\varrho c \mathbf{u}) = \Delta \mu, \ \ \varrho \mu = \varrho \frac{\partial f(\varrho, c)}{\partial c} - \mathrm{div}_x(\varrho \nabla_x c)$$

Existence of weak solution - viscous case

Basic assumptions

$$p(\varrho, c) = p_e(\varrho) + \varrho H(c),\ \ p_e(\varrho) \approx \varrho^{\gamma},\ \ \gamma > \frac{3}{2}$$

Global-in-time weak solutions, H.Abels, EF [Indiana Univ. Math. J. 2008]

The model by Anderson, McFadden and Wheeler (viscous model) admits global-in-time weak solutions for any finite energy initial data

Existence of weak solutions - inviscid case

Basic assumption

$$f(\varrho, c) = H(c) + \log(\varrho)\left(\alpha_1 \frac{1-c}{2} + \alpha_2 \frac{1+c}{2}\right)$$

Global-in-time weak solutions, EF [DCDS(S) 2016]

The model by Lowengrub and Truskinovsky (inviscid model) admits *infinitely many* global-in-time weak solutions for any initial data

$$\varrho_0,\ \mathbf{u}_0,\ c_0 \in C^3,\ \varrho_0 > 0.$$

The solutions satisfy $\varrho > 0$ (no-vacuum)

Total energy - dissipative solutions

Total energy

$$E(\varrho, \mathbf{u}, c) = \int \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{2}\varrho|\nabla_x c|^2 + \varrho f(\varrho, c) \right] \mathrm{d}x$$

Dissipative solutions

$$E(t) \text{ non-increasing}$$

$$E(t) \leq E_0 \text{ for all} t > 0$$

Existence of dissipative solutions

Theorem [EF, IM Preprint 2-2015 (to appear)]

Let $\varrho_0 \in C^3$ be given.
Then for a dense (in L^2) set of $c_0 \in C^3$, there exists $u_0 \in L^\infty$ such that the model by Lowengrub and Truskinovsky (inviscid model) admits *infinitely many* global-in-time *dissipative* weak solutions

Abstract formulation

Variable coefficients "Euler system"

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{H}[\mathbf{v}]) \odot (\mathbf{v} + \mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} + \mathbb{M}[\mathbf{v}] \right) = 0$$

$$\mathrm{div}_x \mathbf{v} = 0,$$

Kinetic energy

$$\frac{1}{2} \frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} = E[\mathbf{v}]$$

Data

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Abstract operators

Boundedness

b maps bounded sets in $L^\infty((0,T)\times\Omega;R^N)$ on bounded sets in $C_b(Q,R^M)$

Continuity

$$b[\mathbf{v}_n] \to b[\mathbf{v}] \text{ in } C_b(Q;R^M) \text{ (uniformly for } (t,x)\in Q \text{)}$$

whenever

$$\mathbf{v}_n \to \mathbf{v} \text{ in } C_{\rm weak}([0,T];L^2(\Omega;R^N))$$

Causality

$\mathbf{v}(t,\cdot) = \mathbf{w}(t,\cdot)$ for $0\le t\le\tau\le T$ implies $b[\mathbf{v}] = b[\mathbf{w}]$ in $[(0,\tau]\times\Omega]$

Subsolutions

Field equations, differential constraints

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{F} = 0, \ \mathrm{div}_x \mathbf{v} = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Non-linear constraint

$$\mathbf{v} \in C(Q; R^N), \ \mathbb{F} \in C(Q; R^{N \times N}_{\mathrm{sym},0}),$$

$$\frac{N}{2} \lambda_{\max} \left[\frac{(\mathbf{v} + \mathbf{H}[\mathbf{v}]) \otimes (\mathbf{v} + \mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} - \mathbb{F} + \mathbb{M}[\mathbf{v}] \right] < E[\mathbf{v}]$$

Subsolution relaxation

Algebraic inequality

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} \leq \frac{N}{2}\lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{H}[\mathbf{v}])\otimes(\mathbf{v}+\mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} - \mathbb{F} + \mathbb{M}[\mathbf{v}]\right]$$

$$< E[\mathbf{v}]$$

Solutions

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} = E[\mathbf{v}]$$

$$\Rightarrow$$

$$\mathbb{F} = \frac{(\mathbf{v}+\mathbf{H}[\mathbf{v}])\odot(\mathbf{v}+\mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} + \mathbb{M}[\mathbf{v}]$$

Oscillatory lemma

Hypotheses:

$$U \subset R \times R^N, \ N = 2,3 \text{ bounded open set}$$

$$\tilde{\mathbf{h}} \in C(U; R^N), \ \tilde{\mathbb{H}} \in C(U; R^{N\times N}_{\mathrm{sym},0}), \ \tilde{e}, \ \tilde{r} \in C(U), \ \tilde{r} > 0, \ \tilde{e} \le \overline{e} \text{ in } U$$

$$\frac{N}{2}\lambda_{\max}\left[\frac{\tilde{\mathbf{h}} \otimes \tilde{\mathbf{h}}}{\tilde{r}} - \tilde{\mathbb{H}}\right] < \tilde{e} \text{ in } U.$$

Conclusion:

$$\mathbf{w}_n \in C^\infty_c(U; R^N),\ \mathbb{G}_n \in C^\infty_c(U; R^{N\times N}_{\mathrm{sym},0}),\ n = 0, 1, \dots$$

$$\partial_t \mathbf{w}_n + \mathrm{div}_x \mathbb{G}_n = 0,\ \mathrm{div}_x \mathbf{w}_n = 0 \text{ in } R \times R^N,$$

$$\frac{N}{2}\lambda_{\max}\left[\frac{(\tilde{\mathbf{h}} + \mathbf{w}_n) \otimes (\tilde{\mathbf{h}} + \mathbf{w}_n)}{\tilde{r}} - (\tilde{\mathbb{H}} + \mathbb{G}_n)\right] < \tilde{e} \text{ in } U,$$

$$\mathbf{w}_n \to 0 \text{ weakly in } L^2(U; R^N)$$

$$\liminf_{n\to\infty} \int_U \frac{|\mathbf{w}_n|^2}{\tilde{r}}\ \mathrm{d}x\mathrm{d}t \geq \Lambda(\overline{e}) \int_U \left(\tilde{e} - \frac{1}{2}\frac{|\tilde{\mathbf{h}}|^2}{\tilde{r}}\right)^2\ \mathrm{d}x\mathrm{d}t$$

Basic ideas of proof

Localization

Localizing the result of DeLellis and Széhelyhidi to "small" cubes by means of scaling arguments

Linearization

Replacing all continuous functions by their means on any of the "small" cubes

Eliminating singular sets

Applying Whitney's decomposition lemma to the non-singular sets (e.g. out of the vacuum $\{h = 0\}$)

Energy and other coefficients depending on solutions

Applying compactness of the abstract operators in C

Results

Result (A)

The set of subsolutions is non-empty $\Rightarrow$ there exists infinitely many weak solutions of the problem with the same initial data

Initial energy jump

$$\frac{1}{2}\frac{|\mathbf{v_0} + \mathbf{H}[\mathbf{v_0}]|^2}{h[\mathbf{v_0}]} \boxed{<} \liminf_{t\to 0} \frac{1}{2}\frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]}$$

Result (B)

The set of subsolutions is non-empty $\Rightarrow$ there exists a dense set of times such that the values $\mathbf{v}(t)$ give rise to non-empty subsolution set with

$$\frac{1}{2}\frac{|\mathbf{v_0} + \mathbf{H}[\mathbf{v_0}]|^2}{h[\mathbf{v_0}]} \boxed{=} \liminf_{t\to 0} \frac{1}{2}\frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]}$$