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SOLVABILITY OF AN UNSATURATED POROUS MEDIA FLOW PROBLEM WITH THERMOMECHANICAL INTERACTION*

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Abstract. A PDE system consisting of the momentum balance, mass balance, and energy balance equations for displacement, capillary pressure, and temperature as a model for unsaturated fluid flow in a porous viscoelastoplastic solid is shown to admit a solution under appropriate assumptions on the constitutive behavior. The problem involves two hysteresis operators accounting for plastic and capillary hysteresis.

Key words. porous media, hysteresis, thermomechanical interactions

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1. Introduction.

In a deformable porous solid filled with two immiscible fluids (water and air, say), two sources of hysteresis are observed: the solid itself is subject to irreversible plastic deformations, and the fluid flow exhibits capillary hysteresis, which is often explained by the surface tension on the interfaces between the two fluids. A lot of work has been devoted to this phenomenon; see, e.g., [1, 2, 3, 4, 10, 11, 13]. Mathematical analysis of various mechanical porous media models with capillary hysteresis and without temperature effects has been carried out in [6, 7, 22, 23]. A PDE system for elastoplastic porous media flow with thermal interaction was derived in [5], but the existence of solutions was proved only for the isothermal case.

Here, we focus on the qualitative analysis of the model derived in [5], assuming in addition that the temperature dependence of the heat conductivity coefficient at infinity has asymptotics controlled similarly to those in the phase transition model considered in [21]. This is indeed a purely theoretical assumption, which is, however, compatible with thermodynamic principles. In practical applications, arbitrarily high temperatures are never observed, but the complexity of the model is such that the occurrence of unbounded temperatures in the equations cannot be a priori excluded, so that for mathematical consistency some hypothesis on the behavior in the whole temperature range is necessary. We borrow here some techniques employed in [21] and [5] in order to prove existence of a weak solution for the initial boundary value problem associated with the PDE system coupling the momentum balance (cf. (2.4)), featuring, in particular, a thermal expansion term depending on the temperature field, with a mass balance (cf. (2.5)) ruling the evolution of the capillary pressure, and an

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energy balance (cf. (2.6)) displaying, in particular, quadratic dissipative terms on the right-hand side.

The main mathematical difficulties are related to the low regularity of the temperature field, mainly due to the presence of the high order dissipative terms in the internal energy balance. This is the reason we need to employ a key-estimate (cf. (6.3)), already exploited in [9] and more recently in [21] for the analysis of non-isothermal phase transition models. Roughly speaking, since the test of the internal energy balance by the temperature θ is not allowed, we test by a suitable negative power of θ and use the growth condition of the heat conductivity κ in Hypothesis 4.1(ii). Another key point in our proof is the L^∞ estimate we get on the pressure, which entails a bound in a proper negative Sobolev space for the time derivative of the absolute temperature, which turns out to be another fundamental ingredient in order to pass to the limit in our approximation scheme.

The structure of the paper is as follows. The model from [5] is briefly summarized in section 2. In section 3 we recall the definitions and main results of the theory of hysteresis operators that are used here. Section 4 contains the mathematical hypotheses and statements of the main results. In section 5 we regularize the problem by adding a small parameter δ accounting for "micromovements" and a large cut-off parameter R to control the nonlinearities, and solve the regularized problem by the standard Faedo–Galerkin method. In section 6 we let δ tend to 0 and R to ∞ and prove that, in the limit, we obtain a solution to the original problem.

2. The model. Consider a domain $\Omega \subset \mathbb{R}^3$ filled with a deformable solid matrix material with pores containing a mixture of liquid and gas. We state the balance laws in referential (Lagrangian) coordinates, assume the deformations to be small, and denote for $x \in \Omega$ and time $t \in [0, T]$ the following:

$u(x,t)$	displacement vector of the referential particle x at time t,
$\varepsilon(x,t) = \nabla_s u(x,t)$	linear strain tensor, $(\nabla_s u)_{ij} := \frac{1}{2}\big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\big)$,
$\sigma(x,t)$	stress tensor,
$p(x,t)$	capillary pressure,
$\theta(x,t)$	absolute temperature,
$A(x,t)$	relative gas content.

For the stress σ and gas content A we assume the empirical constitutive relations

$$\sigma = \mathbf{B}\varepsilon_t + P[\varepsilon] + (p - \beta(\theta - \theta_c))\mathbf{1}, \tag{2.1}$$

$$A = G[p]\,, \tag{2.2}$$

where $\mathbf{1}$ is the Kronecker tensor. The meaning of (2.1) is that the total stress tensor in the solid can be decomposed into four components: the viscous one $\mathbf{B}\varepsilon_t$ with a constant symmetric positive definite fourth order viscosity tensor $\mathbf{B}$, the elastoplastic one $P[\varepsilon]$ characterized in terms of a hysteresis operator P defined below in subsection 3.1, the term $p\mathbf{1}$ of pressure interaction between the solid and the liquid, and the thermal expansion term $\beta(\theta - \theta_c)\mathbf{1}$, where $\beta \in \mathbb{R}$ is the relative solid-liquid thermal expansion coefficient and $\theta_c > 0$ is a fixed referential temperature. According to the studies carried out in [10, 11], the pressure-saturation hysteresis, as in Figure 1, is represented by a Preisach hysteresis operator G defined below in subsection 3.2. We will see that both hysteresis operators P and G admit hysteresis potentials V_P (clockwise) and V_G (counterclockwise) and dissipation operators D_P, D_G such that for all absolutely continuous inputs ε, p the energy balance equations

$$P[\varepsilon] : \varepsilon_t - V_P[\varepsilon]_t = \|D_P[\varepsilon]_t\|_*, \qquad G[p]_t p - V_G[p]_t = |D_G[p]_t| \tag{2.3}$$

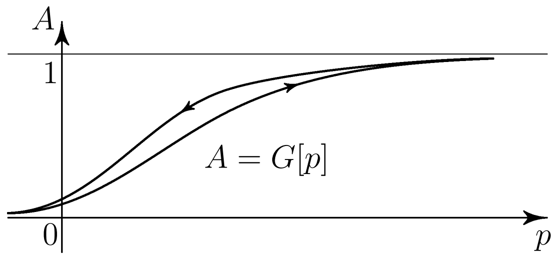

FIG. 1. *Pressure-saturation hysteresis diagram.*

hold almost everywhere, where $\|\cdot\|_*$ is a seminorm in the space $\mathbb{R}^{3\times 3}_{\mathrm{sym}}$ of symmetric 3×3 tensors, and the subscript $\{\}_t$ denotes the partial derivative with respect to t; that is, for example,

$$V_P[\varepsilon]_t(x,t) = \frac{\partial}{\partial t}(V_P[\varepsilon](x,t)),$$

etc. Each of the two identities in (2.3) can indeed be interpreted as an energy balance. The left-hand side is the difference between the power supplied to the system and the increment of the potential energy; the right-hand side is the dissipation rate.

We assume the heat conductivity $\kappa(\theta)$ depending on θ, and as in [5] we obtain the system of momentum balance (2.4), mass balance (2.5) based on the Darcy law, and energy balance equations (2.6) in the form

$$\rho_S u_{tt} = \mathrm{div}\,(\mathbf{B}\nabla_s u_t + P[\nabla_s u]) + \nabla p - \beta\nabla\theta + g, \tag{2.4}$$

$$G[p]_t = \mathrm{div}\, u_t + \frac{1}{\rho_L}\,\mathrm{div}\,(\mu(p)\nabla p), \tag{2.5}$$

$$\begin{aligned} c_0\theta_t = {} & \mathrm{div}\,(\kappa(\theta)\nabla\theta) + \|D_P[\nabla_s u]_t\|_* + |D_G[p]_t| + \mathbf{B}\nabla_s u_t : \nabla_s u_t \\ & + \frac{1}{\rho_L}\mu(p)|\nabla p|^2 - \beta\theta\,\mathrm{div}\, u_t, \end{aligned} \tag{2.6}$$

where $c_0 > 0$ is a constant specific heat; ρ_S, ρ_L are the mass densities of the solid and liquid, respectively; $\mathbf{B}$ is a positive definite viscosity matrix; $\beta \in \mathbb{R}$ is the relative thermal expansion coefficient; and g is a given volume force (gravity, e.g.). The term $\mathrm{div}\, u_t$ in (2.5) accounts for the modeling hypothesis that the increment of the pore volume, due to the deformation of the solid matrix, gives available space for the penetrating liquid, and $\mu(p)\nabla p$ is the liquid mass flux.

Our concern is to describe the principal physical phenomena and insist on the thermodynamic consistency of the model, still keeping the complexity of the presentation within reasonable limits. This is why not all possible constitutive relations between the physical fields are taken into account. In particular, we assume that the material is homogeneous and that the parameters of the model have a simple form.

We complement the system with initial conditions

$$u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \quad p(x,0) = p^0(x), \quad \theta(x,0) = \theta^0(x), \tag{2.7}$$

and boundary conditions

$$\left.\begin{aligned} u &= 0 \\ \tfrac{1}{\rho_L}\mu(p)\nabla p\cdot n &= \gamma_p(x)(p^* - p) \\ \kappa(\theta)\nabla\theta\cdot n &= \gamma_\theta(x)(\theta^* - \theta) \end{aligned}\right\} \quad \text{on } \partial\Omega, \tag{2.8}$$

where p^* is a given outer pressure, θ^* is a given outer temperature, and $\gamma_p, \gamma_\theta : \partial\Omega \to [0,\infty)$ are given smooth functions representing the permeability and the heat conductivity of the boundary.

3. Hysteresis operators. We recall here the basic concepts of the theory of hysteresis operators that are needed in what follows.

3.1. The operator $\boldsymbol{P}$. In (2.1), P stands for the elastoplastic part σ^{ep} of the stress tensor $\sigma \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$. We proceed as in [18] and assume that σ^{ep} can be represented as the sum $\sigma^{ep} = \sigma^e + \sigma^p$ of an elastic component σ^e and plastic component σ^p. While σ^e obeys the classical linear elasticity law

$$\sigma^e = \mathbf{A}^e \varepsilon \tag{3.1}$$

with a constant symmetric positive definite fourth order elasticity tensor $\mathbf{A}^e$, for the description of the behavior of σ^p, we split also the strain tensor ε into the sum $\varepsilon = \varepsilon^e + \varepsilon^p$ of the elastic strain ε^e and plastic strain ε^p and assume

$$\sigma^p = \mathbf{A}^p \varepsilon^e, \tag{3.2}$$

again with a constant symmetric positive definite fourth order elasticity tensor $\mathbf{A}^p$, and for a given time evolution $\varepsilon(t)$ of the strain tensor, $t \in [0,T]$, we require σ^p to satisfy the constraint

$$\sigma^p(t) \in Z \quad \forall t \in [0,T], \tag{3.3}$$

where $Z \subset \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ is the domain of admissible plastic stress components. We assume that it has the form

$$Z = Z_0 \oplus \mathrm{Lin}\{\mathbf{1}\}, \tag{3.4}$$

where $\mathrm{Lin}\{\mathbf{1}\}$ is the one-dimensional space spanned by the Kronecker tensor $\mathbf{1}$ and Z_0 is a bounded convex closed subset with 0 in its interior of the orthogonal complement $\mathrm{Lin}\{\mathbf{1}\}^\perp$ of $\mathrm{Lin}\{\mathbf{1}\}$ (the *deviatoric space*). The boundary ∂Z of Z is the *yield surface*. The time evolution of ε^p is governed by the *flow rule*

$$\varepsilon^p_t : (\sigma^p - \tilde{\sigma}) \ge 0 \quad \forall \tilde{\sigma} \in Z, \tag{3.5}$$

which implies that

$$\varepsilon^p_t : \sigma^p = M_{Z^*}(\varepsilon^p_t), \tag{3.6}$$

where M_{Z^*} is the Minkowski functional of the polar set Z^* to Z. The physical interpretation of (3.5) is the *maximal dissipation principle*. Geometrically, it states that the plastic strain rate ε^p_t points in the outward normal direction to the yield surface at the point σ^p. Indeed, if σ^p is in the interior of Z, then $\varepsilon^p_t = 0$.

It follows from (3.4) that there exist $\sigma^p_0 \in Z_0$ (plastic stress deviator) and $c \in \mathbb{R}$ (pressure) such that $\sigma^p = \sigma^p_0 - c\mathbf{1}$. On the other hand, setting $\tilde{\sigma} = \sigma^p_0 - \rho\mathbf{1}$ in (3.5) for an arbitrary $\rho \in \mathbb{R}$, we obtain $\varepsilon^p_t : \mathbf{1} = 0$ (in other words, no volume changes occur during plastic deformation), so that

$$M_{Z^*}(\varepsilon^p_t) = \varepsilon^p_t : \sigma^p_0 \le \mathrm{diam}(Z_0)|\varepsilon^p_t|. \tag{3.7}$$

We can eliminate the internal variables $\varepsilon^e, \varepsilon^p$ and write (3.5) in the form

$$(\varepsilon_t - (\mathbf{A}^p)^{-1}\sigma^p_t) : (\sigma^p - \tilde{\sigma}) \ge 0 \quad \forall \tilde{\sigma} \in Z. \tag{3.8}$$

We now define a new scalar product $\langle\cdot,\cdot\rangle_{\mathbf{A}^p}$ in $\mathbb{R}^{3\times3}_{\mathrm{sym}}$ by the formula $\langle\xi,\eta\rangle_{\mathbf{A}^p} = (\mathbf{A}^p)^{-1}\xi : \eta$ for generic tensors ξ,η and rewrite (3.8) as

$$\langle \mathbf{A}^p\varepsilon_t - \sigma^p_t, \sigma^p - \tilde\sigma\rangle_{\mathbf{A}^p} \ge 0 \quad \forall\tilde\sigma\in Z. \tag{3.9}$$

We prescribe a canonical initial condition for σ^p, namely

$$\sigma^p(0) = \mathrm{Proj}_Z(\mathbf{A}^p\varepsilon(0)), \tag{3.10}$$

where Proj_Z is the orthogonal projection $\mathbb{R}^{3\times3}_{\mathrm{sym}} \to Z$ with respect to the scalar product $\langle\cdot,\cdot\rangle_{\mathbf{A}^p}$ and is characterized by the variational inequality

$$x = \mathrm{Proj}_Z(u) \iff x\in Z, \qquad \langle u-x, x-y\rangle_{\mathbf{A}^p} \ge 0 \quad \forall y\in Z. \tag{3.11}$$

We list here some properties of the variational problem (3.3), (3.9), (3.10). The proof can be found in [15, Chapter I].

Proposition 3.1. *For every $\varepsilon \in W^{1,1}(0,T;\mathbb{R}^{3\times3}_{\mathrm{sym}})$ there exists a unique $\sigma^p \in W^{1,1}(0,T;\mathbb{R}^{3\times3}_{\mathrm{sym}})$ satisfying* (3.3), (3.9), (3.10). *The solution mapping*

$$P_0 : W^{1,1}(0,T;\mathbb{R}^{3\times3}_{\mathrm{sym}}) \to W^{1,1}(0,T;\mathbb{R}^{3\times3}_{\mathrm{sym}}) : \varepsilon\mapsto\sigma^p$$

has the following properties:

(i) *For all $\varepsilon\in W^{1,1}(0,T;\mathbb{R}^{3\times3}_{\mathrm{sym}})$ we have $|P_0[\varepsilon]_t| \le |\varepsilon_t|$ a.e.; $P_0 : W^{1,1}(0,T;\mathbb{R}^{3\times3}_{\mathrm{sym}}) \to W^{1,1}(0,T;\mathbb{R}^{3\times3}_{\mathrm{sym}})$ is strongly continuous and admits an extension to a strongly continuous mapping $C([0,T];\mathbb{R}^{3\times3}_{\mathrm{sym}}) \to C([0,T];\mathbb{R}^{3\times3}_{\mathrm{sym}})$.*

(ii) *There exists a constant $C>0$ such that for every $\varepsilon_1,\varepsilon_2 \in W^{1,1}(0,T;\mathbb{R}^{3\times3}_{\mathrm{sym}})$ and every $t\in[0,T]$ we have*

$$|P_0[\varepsilon_1](t) - P_0[\varepsilon_2](t)| \le C\left(|\varepsilon_1(0)-\varepsilon_2(0)| + \int_0^t |(\varepsilon_1)_t(\tau) - (\varepsilon_2)_t(\tau)|\,\mathrm{d}\tau\right). \tag{3.12}$$

(iii) *For all $\varepsilon\in W^{1,1}(0,T;\mathbb{R}^{3\times3}_{\mathrm{sym}})$, the energy balance equation*

$$P_0[\varepsilon] : \varepsilon_t - \frac{1}{2}\left((\mathbf{A}^p)^{-1}P_0[\varepsilon] : P_0[\varepsilon]\right)_t = M_{Z^*}\left((\varepsilon - (\mathbf{A}^p)^{-1}P_0[\varepsilon])_t\right) \tag{3.13}$$

is satisfied a.e. in $(0,T)$, where M_{Z^*} is the Minkowski functional of the polar set Z^* to Z.

With the above notation, we define the operator P in (2.1) by the formula

$$P[\varepsilon] = \mathbf{A}^e\varepsilon + P_0[\varepsilon], \tag{3.14}$$

and the first energy identity in (2.3) holds with the choice

$$V_P[\varepsilon] = \frac{1}{2}\mathbf{A}^e\varepsilon:\varepsilon + \frac{1}{2}(\mathbf{A}^p)^{-1}P_0[\varepsilon]:P_0[\varepsilon], \quad D_P[\varepsilon] = \varepsilon - (\mathbf{A}^p)^{-1}P_0[\varepsilon], \quad \|\cdot\|_* = M_{Z^*}(\cdot). \tag{3.15}$$

3.2. The operator G. Similarly as in (3.14), the operator G is considered as a sum

$$G[p] = f(p) + G_0[p], \tag{3.16}$$

where f is a monotone function satisfying Hypothesis 4.1(iii) below and G_0 is a Preisach operator that we briefly describe here.

The construction of the Preisach operator G_0 is also based on a variational inequality of the type (3.9). More precisely, for a given input function $p \in W^{1,1}(0,T)$ and a memory parameter $r > 0$, we define the function $\xi_r(t)$ as the solution of the variational inequality

$$\begin{cases} |p(t) - \xi_r(t)| \le r & \forall t \in [0,T], \\ (\xi_r)_t(p(t) - \xi_r(t) - z) \ge 0 & \forall z \in [-r,r] \quad \text{for almost every } t \in (0,T), \end{cases} \tag{3.17}$$

with a prescribed initial condition $\xi_r(0) \in [p(0) - r, p(0) + r]$.

This is indeed a scalar version of (3.9) with Z replaced by the interval $[-r,r]$, ε replaced by p, and σ^p replaced by $p - \xi_r$. Here, we consider the whole continuous family of variational inequalities (3.17) parameterized by $r > 0$. We introduce the memory state space

$$\Lambda = \{\lambda \in W^{1,\infty}(0,\infty) : |\lambda'(r)| \le 1 \text{ a.e.}\} \tag{3.18}$$

and its subspace

$$\Lambda_K = \{\lambda \in \Lambda : \lambda(r) = 0 \text{ for } r \ge K\}. \tag{3.19}$$

We fix $K > 0$ and an initial state $\lambda_{-1} \in \Lambda_K$, and choose the initial condition as

$$\xi_r(0) = \max\{p(0) - r, \min\{\lambda_{-1}(r), p(0) + r\}\}. \tag{3.20}$$

We have indeed for all $r > 0$ the natural bound

$$|\xi_r(0)| \le \max\{|p(0)|, K\}. \tag{3.21}$$

The mapping $\mathfrak{p}_r : W^{1,1}(0,T) \to W^{1,1}(0,T)$, which with each $p \in W^{1,1}(0,T)$ associates the solution $\xi_r = \mathfrak{p}_r[p] \in W^{1,1}(0,T)$ of (3.17), (3.20), is called the *play*. This concept goes back to [14], and the proof of the following statements can be found, e.g., in [15, Chapter II].

PROPOSITION 3.2. *For each $r > 0$, the mapping $\mathfrak{p}_r : W^{1,1}(0,T) \to W^{1,1}(0,T)$ is Lipschitz continuous and admits a Lipschitz continuous extension to $\mathfrak{p}_r : C[0,T] \to C[0,T]$ in the sense that for every $p_1, p_2 \in C[0,T]$ and every $t \in [0,T]$ we have*

$$|\mathfrak{p}_r[p_1](t) - \mathfrak{p}_r[p_2](t)| \le \max_{\tau \in [0,t]} |p_1(\tau) - p_2(\tau)|. \tag{3.22}$$

Moreover, for each $p \in W^{1,1}(0,T)$, the energy balance equation

$$\mathfrak{p}_r[p]_t p - \frac{1}{2}\left(\mathfrak{p}_r^2[p]\right)_t = |r \mathfrak{p}_r[p]_t| \tag{3.23}$$

and the identity

$$\mathfrak{p}_r[p]_t p_t = (\mathfrak{p}_r[p]_t)^2 \tag{3.24}$$

hold a.e. in $(0,T)$.

PROPOSITION 3.3. *Let $\lambda_{-1} \in \Lambda_K$ be given, and let $\{\mathfrak{p}_r : r > 0\}$ be the family of play operators. Then for every $p \in C[0,T]$ and every $t \in [0,T]$ we have the following:*

(i) *$\mathfrak{p}_r[p](t) = 0$ for $r \ge K^*(t) := \max\{K, \max_{\tau \in [0,t]} |p(\tau)|\}$.*

(ii) *The function $r \mapsto \mathfrak{p}_r[p](t)$ belongs to $\Lambda_{K^*(t)}$.*

Given a nonnegative function $\rho \in L^1((0,\infty)\times\mathbb{R})$ (the *Preisach density*), we define the Preisach operator G_0 as a mapping that with each $p \in C[0,T]$ associates the integral

$$G_0[p](t) = \int_0^\infty \int_0^{\mathfrak{p}_r[p](t)} \rho(r,v)\,\mathrm{d}v\,\mathrm{d}r. \tag{3.25}$$

For our purposes, we adopt the following hypothesis on the Preisach density.

HYPOTHESIS 3.4. *There exists a function $\rho^* \in L^1(0,\infty)$ such that for almost every $v \in \mathbb{R}$ we have $0 \le \rho(r,v) \le \rho^*(r)$, $\int_0^\infty r\rho^*(r)\,\mathrm{d}r < \infty$, and we set*

$$C_\rho = \int_0^\infty \int_{-\infty}^\infty \rho(r,v)\,\mathrm{d}v\,\mathrm{d}r, \qquad C_\rho^* = \int_0^\infty \rho^*(r)\,\mathrm{d}r. \tag{3.26}$$

For the reader who is more familiar with the original Preisach construction in [20] based on nonideal relays, let us just point out that for integrable densities the variational setting in (3.25) is equivalent, as shown in [16].

From (3.23), (3.24), and (3.25) we immediately deduce the Preisach energy identity

$$G_0[p]_t p - V_0[p]_t = |D_0[p]_t| \quad \text{a.e.}, \tag{3.27}$$

provided we define the Preisach potential V_0 and the dissipation operator D_0 by the integrals

$$V_0[p](t) = \int_0^\infty \int_0^{\mathfrak{p}_r[p](t)} v\rho(r,v)\,\mathrm{d}v\,\mathrm{d}r, \qquad D_0[p](t) = \int_0^\infty \int_0^{\mathfrak{p}_r[p](t)} r\rho(r,v)\,\mathrm{d}v\,\mathrm{d}r. \tag{3.28}$$

The second identity in (2.3) then holds with the choice

$$V_G[p] = pf(p) - \int_0^p f(z)\,\mathrm{d}z + V_0[p], \qquad D_G[p] = D_0[p]. \tag{3.29}$$

A straightforward computation shows that G_0 (and, consequently, G) is Lipschitz continuous in $C[0,T]$. Indeed, using (3.22) and Hypothesis 3.4, we obtain for $p_1, p_2 \in C[0,T]$ and $t \in [0,T]$ that

$$|G_0[p_2](t) - G_0[p_1](t)| = \left| \int_0^\infty \int_{\mathfrak{p}_r[p_1](t)}^{\mathfrak{p}_r[p_2](t)} \rho(v,r)\,\mathrm{d}v\,\mathrm{d}r \right| \le C_\rho^* \max_{\tau\in[0,t]} |p_2(\tau) - p_1(\tau)|. \tag{3.30}$$

We similarly get, using (3.21), bounds for the initial time $t = 0$, namely,

$$|G_0[p](0)| = \left| \int_0^\infty \int_0^{\mathfrak{p}_r[p](0)} \rho(v,r)\,\mathrm{d}v\,\mathrm{d}r \right| \le \min\{C_\rho, C_\rho^* \max\{|p(0)|, K\}\}, \tag{3.31}$$

$$|V_0[p](0)| = \left| \int_0^\infty \int_0^{\mathfrak{p}_r[p](0)} v\rho(v,r)\,\mathrm{d}v\,\mathrm{d}r \right| \le C_\rho \max\{|p(0)|, K\}. \tag{3.32}$$

The Preisach operator admits also a family of "nonlinear" energies. As a consequence of (3.23), we have for almost every t the inequality

$$\mathfrak{p}_r[p]_t (p - \mathfrak{p}_r[p]) \ge 0, \tag{3.33}$$

and hence

$$\mathfrak{p}_r[p]_t(h(p) - h(\mathfrak{p}_r[p])) \ge 0 \tag{3.34}$$

for every nondecreasing function $h : \mathbb{R} \to \mathbb{R}$. Hence, for every absolutely continuous input p, a counterpart of (3.27) in the form

$$G_0[p]_t h(p) - V_h[p]_t \ge 0 \quad \text{a.e.} \tag{3.35}$$

holds with a modified potential

$$V_h[p](t) = \int_0^\infty \int_0^{\mathfrak{p}_r[p](t)} h(v)\rho(r,v)\,\mathrm{d}v\,\mathrm{d}r. \tag{3.36}$$

This is related to the fact that, for every absolutely continuous nondecreasing function $\hat{h} : \mathbb{R} \to \mathbb{R}$, the mapping $G_{\hat{h}} := G_0 \circ \hat{h}$ is also a Preisach operator; see [17].

4. Main results. We denote

$$X_q = W^{1,q}(\Omega) \ \text{ for } \ q > 1, \qquad X_2^0 = \{\phi \in W^{1,2}(\Omega;\mathbb{R}^3) : \phi\big|_{\partial\Omega} = 0\}, \tag{4.1}$$

and reformulate problem (2.4)–(2.6) in variational form for all test functions $\phi \in X_2^0$, $\psi \in X_2$, and $\zeta \in X_{q^*}$ for a suitable $q^* > 2$ (to be specified later) as follows:

$$\int_\Omega (\rho_S u_{tt}\cdot\phi + (\mathbf{B}\nabla_s u_t + P[\nabla_s u]) : \nabla_s\phi + (p-\beta\theta)\operatorname{div}\phi)\,\mathrm{d}x = \int_\Omega g\cdot\phi\,\mathrm{d}x, \tag{4.2}$$

$$\int_\Omega ((G[p]_t - \operatorname{div} u_t)\psi + \frac{1}{\rho_L}\mu(p)\nabla p\cdot\nabla\psi)\,\mathrm{d}x = \int_{\partial\Omega} \gamma_p(x)(p^*-p)\psi\,\mathrm{d}s(x), \tag{4.3}$$

$$\begin{aligned}&\int_\Omega \big((c_0\theta_t - \|D_P[\nabla_s u]_t\|_* - |D_G[p]_t|)\zeta + \kappa(\theta)\nabla\theta\cdot\nabla\zeta\big)\,\mathrm{d}x\\ &- \int_\Omega (\mathbf{B}\nabla_s u_t : \nabla_s u_t + \frac{1}{\rho_L}\mu(p)|\nabla p|^2 - \beta\theta\operatorname{div} u_t)\zeta\,\mathrm{d}x = \int_{\partial\Omega} \gamma_\theta(x)(\theta^*-\theta)\zeta\,\mathrm{d}s(x).\end{aligned} \tag{4.4}$$

HYPOTHESIS 4.1. *We assume that Ω is a bounded connected domain with $C^{1,1}$ boundary. We fix an arbitrary final time $T > 0$, a constant $\bar\theta > 0$, and functions $p^* \in W^{1,\infty}(\partial\Omega\times(0,T))$, $\theta^* \in L^\infty(\partial\Omega\times(0,T))$ such that $\theta^*(x,t) \ge \bar\theta$, $g \in L^2(\Omega\times(0,T))$, $\gamma_\theta \in L^\infty(\partial\Omega)$, $\gamma_p \in W^{1,\infty}(\partial\Omega)$, $\gamma_\theta \ge 0$, $\gamma_p \ge 0$ a.e., $\int_{\partial\Omega}\gamma_p(x)\,\mathrm{d}s(x) > 0$, $\beta \in \mathbb{R}$, $c_0 > 0$. The coefficients ρ_S, ρ_L are constant and positive, and $\mathbf{B}$ is the isotropic symmetric positive definite fourth order tensor of the form*

$$\mathbf{B}_{ijkl} = 2\eta\delta_{ik}\delta_{jl} + \omega\delta_{kl}\delta_{ij} \tag{4.5}$$

with constants $\eta > 0$, $\omega > 0$. The nonlinearities in (4.2)–(4.4) satisfy the following conditions:

(i) *$\mu : \mathbb{R} \to [\mu_0,\mu_1]$ is a C^1 function, $0 < \mu_0 < \mu_1$ are fixed constants, and we set*

$$M(p) = \int_0^p \mu(p')\,\mathrm{d}p'. \tag{4.6}$$

(ii) *$\kappa : \mathbb{R} \to (0,\infty)$ is a C^1 function, $\kappa(0) > 0$, and there exist constants $0 < a < b < \frac{16}{5} + \frac{6}{5}a$ such that*

$$\liminf_{\theta\to\infty} \frac{\kappa(\theta)}{\theta^{1+a}} > 0, \qquad \limsup_{\theta\to\infty} \frac{\kappa(\theta)}{\theta^{1+b}} < \infty.$$

(iii) *$G[p] = f(p) + G_0[p]$, where G_0 is the Preisach operator from subsection* 3.2 *with an initial memory state $\lambda_{-1} \in \Lambda_K$ for some $K \geq \sup|p^*|$. The dissipation operator D_G associated with G is defined in* (3.28)–(3.29)*, and $f : \mathbb{R} \to (0, f_1)$ for some $f_1 > 0$ is a C^1 function such that there exist $0 < f_2 < f_3$ with the property*

$$f_2 \leq f'(p)(1 + p^2) \leq f_3 \quad \forall p \in \mathbb{R}\,. \tag{4.7}$$

(iv) *The operator $P : C([0,T]; \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \to C([0,T]; \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ has the form* (3.14) *with P_0 defined in Proposition* 3.1*, and with dissipation operator D_P defined in* (3.15).

We prescribe initial conditions (2.7) *with $u^0 \in X_2^0 \cap W^{2,2}(\Omega; \mathbb{R}^3)$, $u^1 \in X_2^0$, $p^0 \in X_2 \cap L^\infty(\Omega)$, $|p^0(x)| \leq K$ a.e., $\theta^0 \in L^\infty(\Omega)$, $\theta^0(x) \geq \bar{\theta} > 0$ a.e.*

Condition (ii) in Hypothesis 4.1 is a slight generalization of Hypothesis (I) of [21]. Note also that the function μ is assumed here to be more regular than in the situation of [5]. These assumptions serve to achieve the required regularity of solutions. The $C^{1,1}$ smoothness of $\partial\Omega$ and condition (4.5) are chosen in order to guarantee the full $W^{2,2}$ regularity of the elliptic operators in (4.2)–(4.4), and the connectedness of Ω is used in the argument leading to (6.19). The growth condition (4.7) is purely technical and plays a substantial role in the Moser iteration argument in subsection 6.7.

The main result of this paper reads as follows.

THEOREM 4.2. *Let Hypotheses* 3.4 *and* 4.1 *hold. Then there exist $q^* > 2$, which will be specified in formula* (6.78) *below, and a solution (u, p, θ) to* (4.2)–(4.4) *and* (2.7) *with the properties $u_t \in L^2(0, T; X_2^0 \cap W^{2,2}(\Omega; \mathbb{R}^3))$, $u_{tt} \in L^2(\Omega \times (0,T))$, $p \in L^\infty(\Omega \times (0,T))$, $M(p) \in L^2(0, T; W^{2,2}(\Omega))$ with $M(p)$ given by* (4.6)*, $p_t \in L^2(\Omega \times (0,T))$, $\theta \in L^z(\Omega \times (0,T))$ for every $z < 8 + 3a$, $\nabla\theta \in L^2(\Omega \times (0,T); \mathbb{R}^3)$, $\theta_t \in L^2(0, T; W^{-1,q^*}(\Omega))$.*

We first regularize the problem, prove the existence of a solution for the regularized system, derive estimates independent of the regularization parameters, and pass to the limit.

5. Regularization. We choose regularizing parameters $R > K$ with K from Hypothesis 3.4 and $\delta > 0$ with the intention to let $R \to \infty$ and $\delta \to 0$, and define mappings $Q_R : \mathbb{R} \to [0, R]$ and $K_R : \mathbb{R} \to \mathbb{R}$ by the formulas

$$Q_R(z) = \max\{0, \min\{z, R\}\}, \quad K_R(z) = \max\{z - R, \min\{0, z + R\}\} \ \text{ for } \ z \in \mathbb{R}. \tag{5.1}$$

Let $\mathcal{B} : W^{2,2}(\Omega; \mathbb{R}^3) \cap X_2^0 \to L^2(\Omega; \mathbb{R}^3)$ denote the mapping

$$\mathcal{B}v = -\operatorname{div} \mathbf{B}\nabla_s v. \tag{5.2}$$

It follows from a vector counterpart of [12, Lemma 9.17] (cf. also the methods proposed in [19, p. 260, Lemma 3.2]) that $|\mathcal{B}v|_2$ is an equivalent norm for v in $W^{2,2}(\Omega) \cap X_2^0$; that is, there exist positive constants $C_1 < C_2$ such that for every $v \in W^{2,2}(\Omega; \mathbb{R}^3) \cap X_2^0$ we have

$$C_1 \|v\|_{W^{2,2}(\Omega)} \leq |\mathcal{B}v|_2 \leq C_2 \|v\|_{W^{2,2}(\Omega)}. \tag{5.3}$$

We set $\kappa_R(\theta) := \kappa(Q_R(\theta))$ and replace (4.2)–(4.4) by the system

$$\begin{aligned}
&\int_\Omega (\rho_S u_{tt}\cdot\phi + \delta\mathcal{B}u_{tt}\cdot\mathcal{B}\phi + (\mathbf{B}\nabla_s u_t + P[\nabla_s u]) : \nabla_s\phi)\,\mathrm{d}x \\
&\qquad + \int_\Omega (p - \beta Q_R(\theta))\operatorname{div}\phi\,\mathrm{d}x = \int_\Omega g\cdot\phi\,\mathrm{d}x,
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
&\int_\Omega \left(\big((K_R(p) + G[p])_t - \operatorname{div} u_t\big)\psi + \frac{1}{\rho_L}\mu(p)\nabla p\cdot\nabla\psi \right)\mathrm{d}x \\
&\quad = \int_{\partial\Omega} \gamma_p(x)(p^* - p)\psi\,\mathrm{d}s(x),
\end{aligned} \tag{5.5}$$

$$\begin{aligned}
&\int_\Omega \big((c_0\theta_t - \|D_P[\nabla_s u]_t\|_* - |D_G[p]_t|)\zeta + \kappa_R(\theta)\nabla\theta\cdot\nabla\zeta\big)\,\mathrm{d}x \\
&\qquad - \int_\Omega \left(\mathbf{B}\nabla_s u_t{:}\nabla_s u_t + \frac{1}{\rho_L}\mu(p)Q_R(|\nabla p|^2) - \beta Q_R(\theta)\operatorname{div} u_t \right)\zeta\,\mathrm{d}x \\
&\quad = \int_{\partial\Omega} \gamma_\theta(x)(\theta^* - \theta)\zeta\,\mathrm{d}s(x)
\end{aligned} \tag{5.6}$$

with test functions $\phi \in W^{2,2}(\Omega;\mathbb{R}^3)\cap X_2^0$, $\psi,\zeta \in X_2$ and initial conditions (2.7).

PROPOSITION 5.1. *In addition to Hypotheses* 3.4 *and* 4.1, *assume* $u^1 \in X_2^0 \cap W^{2,2}(\Omega;\mathbb{R}^3)$. *Then there exists a solution* (u,p,θ) *to* (5.4)–(5.6) *with the properties* $u_{tt} \in L^2(0,T;W^{2,2}(\Omega;\mathbb{R}^3)\cap X_2^0)$, $p \in L^2(0,T;W^{1,2}(\Omega))$, $M(p) \in L^2(0,T;W^{2,2}(\Omega))$ *with* $M(p)$ *given by* (4.6), $p_t \in L^2(\Omega\times(0,T))$, $\theta \in L^2(\Omega\times(0,T))$, $\nabla\theta \in L^2(\Omega\times(0,T))$, $\theta_t \in L^2(0,T;X_2^*)$, *where* X_2^* *is the dual of* X_2.

System (5.4)–(5.6) for each fixed $R > 0$ and $\delta > 0$ will be solved by Faedo–Galerkin approximations. This explains the motivation for all the regularizations. The Faedo–Galerkin technique does not allow for any other test functions but a linear combination of the basis elements. To get the nonlinear terms under control, we have to regularize the degenerate term $G[p]$, truncate the quadratic nonlinearities, and add the space-time regularization $\delta\mathcal{B}^2 u_{tt}$.

We choose $\mathcal{E} = \{e_k; k = 1,2,\dots\}$ in $L^2(\Omega;\mathbb{R}^3)$ and $\mathcal{W} = \{w_j; j = 0,1,2,\dots\}$ in $L^2(\Omega)$ to be the complete orthonormal systems of eigenfunctions defined by

$$\mathcal{B}e_k = \lambda_k e_k \ \text{ in } \Omega, \quad e_k\big|_{\partial\Omega} = 0, \qquad -\Delta w_j = \mu_j w_j \ \text{ in } \Omega, \quad \nabla w_j\cdot n\big|_{\partial\Omega} = 0, \tag{5.7}$$

with $\mu_0 = 0$, $\lambda_k > 0, \mu_j > 0$ for $j,k \geq 1$, and put for $n \in \mathbb{N}$

$$u^{(n)}(x,t) = \sum_{k=1}^n u_k(t)e_k(x), \qquad \theta^{(n)}(x,t) = \sum_{j=0}^n \theta_j(t)w_j(x) \tag{5.8}$$

with coefficients $u_k : [0,T] \to \mathbb{R}$, $\theta_j : [0,T] \to \mathbb{R}$ which will be determined as the solution of the system

$$(\rho_S+\delta\lambda_k^2)\ddot{u}_k + \lambda_k\dot{u}_k+\int_\Omega P[\nabla_s u^{(n)}] : \nabla_s e_k \,\mathrm{d}x$$
$$+\int_\Omega (p^{(n)} - \beta Q_R(\theta^{(n)}))\,\mathrm{div}\, e_k \,\mathrm{d}x = \int_\Omega g\cdot e_k \,\mathrm{d}x, \tag{5.9}$$

$$\int_\Omega \big((K_R(p^{(n)}) + G[p^{(n)}])_t - \,\mathrm{div}\, u_t^{(n)}\big)\psi \,\mathrm{d}x$$
$$+\frac{1}{\rho_L}\int_\Omega \mu(p^{(n)})\nabla p^{(n)}\cdot\nabla\psi \,\mathrm{d}x = \int_{\partial\Omega}\gamma_p(x)(p^* - p^{(n)})\psi \,\mathrm{d}s(x), \tag{5.10}$$

$$c_0\dot{\theta}_j + \int_\Omega \big(-|D_G[p^{(n)}]_t|w_j+\kappa_R(\theta^{(n)})\nabla\theta^{(n)}\cdot\nabla w_j\big)\,\mathrm{d}x$$
$$+\int_\Omega \big(\beta Q_R(\theta^{(n)})\,\mathrm{div}\, u_t^{(n)} - \|D_P[\nabla_s u^{(n)}]_t\|_*\big)w_j \,\mathrm{d}x \tag{5.11}$$
$$-\int_\Omega \left(\mathbf{B}\nabla_s u_t^{(n)}{:}\nabla_s u_t^{(n)}+\frac{1}{\rho_L}\mu(p^{(n)})Q_R(|\nabla p^{(n)}|^2)\right) w_j \,\mathrm{d}x = \int_{\partial\Omega}\gamma_\theta(x)(\theta^* - \theta^{(n)})w_j \,\mathrm{d}s(x)$$

for $k = 1,\dots,n$ and $j = 0,1,\dots,n$, and for all $\psi\in X_2$. We prescribe initial conditions

$$u_k(0) = \int_\Omega u^0(x)\cdot e_k(x)\,\mathrm{d}x, \qquad \dot{u}_k(0) = \int_\Omega u^1(x)\cdot e_k(x)\,\mathrm{d}x, \tag{5.12}$$

$$\theta_j(0) = \int_\Omega \theta^0(x)w_j(x)\,\mathrm{d}x, \qquad p^{(n)}(x,0) = p^0(x). \tag{5.13}$$

This is an ODE system (5.9), (5.11) coupled with a standard PDE with hysteresis (5.10). We do not decompose $p^{(n)}$ into a series, because we want to test (5.10) in Estimate 2 below by a nonlinear expression. System (5.9)–(5.11) has a strong solution in a maximal interval of existence, which coincides with the whole interval $[0,T]$, provided that we prove that the solution remains bounded in the maximal interval of existence.

Set $\mathcal{E}_n = \{e_k; k = 1,2,\dots,n\}$ and $\mathcal{W}_n = \{w_j; j = 0,1,2,\dots,n\}$. Then (5.9)–(5.11) can be equivalently written as

$$\int_\Omega \Big(\rho_S u_{tt}^{(n)}\cdot\phi + (\mathbf{B}\nabla_s u_t^{(n)}+P[\nabla_s u^{(n)}]) : \nabla_s\phi\Big)\,\mathrm{d}x$$
$$+\delta\int_\Omega \mathcal{B}u_{tt}^{(n)}\cdot\mathcal{B}\phi\,\mathrm{d}x + \int_\Omega (p^{(n)} - \beta Q_R(\theta^{(n)}))\,\mathrm{div}\,\phi\,\mathrm{d}x = \int_\Omega g\cdot\phi\,\mathrm{d}x, \tag{5.14}$$

$$\int_\Omega ((K_R(p^{(n)}) + G[p^{(n)}])_t - \,\mathrm{div}\, u_t^{(n)})\psi \,\mathrm{d}x$$
$$+\int_\Omega \frac{1}{\rho_L}\mu(p^{(n)})\nabla p^{(n)}\cdot\nabla\psi\,\mathrm{d}x = \int_{\partial\Omega}\gamma_p(x)(p^* - p^{(n)})\psi\,\mathrm{d}s(x), \tag{5.15}$$

$$\int_\Omega \big((c_0\theta_t^{(n)} - |D_G[p^{(n)}]_t|)\zeta + \kappa_R(\theta^{(n)})\nabla\theta^{(n)}\cdot\nabla\zeta\big)\,\mathrm{d}x$$
$$+\int_\Omega \Big(\beta Q_R(\theta^{(n)})\,\mathrm{div}\, u_t^{(n)} - \|D_P[\nabla_s u^{(n)}]_t\|_*\Big)\zeta\,\mathrm{d}x$$
$$-\int_\Omega \left(\mathbf{B}\nabla_s u_t^{(n)}{:}\nabla_s u_t^{(n)} + \frac{1}{\rho_L}\mu(p^{(n)})Q_R(|\nabla p^{(n)}|^2)\right)\zeta\,\mathrm{d}x = \int_{\partial\Omega}\gamma_\theta(x)(\theta^* - \theta^{(n)})\zeta\,\mathrm{d}s(x) \tag{5.16}$$

with test functions $\phi \in \operatorname{Span} \mathcal{E}_n$, $\zeta \in \operatorname{Span} \mathcal{W}_n$, and $\psi \in X_2$.

We now derive a series of estimates. By C we denote any positive constant depending only on the data, by C_R any constant depending on the data and on R, and by $C_{R,\delta}$ any constant depending on the data, on R, and on δ, all independent of the dimension n of the Galerkin approximation.

To simplify the presentation, we introduce now the notation $|\cdot|_q$ for the norm in $L^q(\Omega)$, and $\|\cdot\|_q$ for the norm in $L^q(\Omega \times (0,T))$. We will systematically use the Gagliardo–Nirenberg inequality in the form

$$|w|_q \le C(|w|_s + |w|_s^{1-\gamma}|\nabla w|_r^{\gamma}), \qquad \gamma = \frac{\frac{1}{s} - \frac{1}{q}}{\frac{1}{3} + \frac{1}{s} - \frac{1}{r}}, \tag{5.17}$$

which holds for every $w \in W^{1,r}(\Omega)$ and every $\frac{1}{s} > \frac{1}{q} > \frac{1}{r} - \frac{1}{3}$. For the proof, see, e.g., [8, section 15].

5.1. Estimate 1. We test (5.14) with $\phi = u_t^{(n)}$ and (5.15) with $\psi = p^{(n)}$ and sum up the results to obtain

$$\begin{aligned}\int_\Omega \Bigg(&\rho_S u_{tt}^{(n)} \cdot u_t^{(n)} + \delta \mathcal{B} u_{tt}^{(n)} \cdot \mathcal{B} u_t^{(n)} + (\mathbf{B}\nabla_s u_t^{(n)} + P[\nabla_s u^{(n)}]) : \nabla_s u_t^{(n)} \\ &+ (K_R(p^{(n)}) + G[p^{(n)}])_t p^{(n)} + \frac{1}{\rho_L}\mu(p^{(n)})|\nabla p^{(n)}|^2\Bigg)\,\mathrm{d}x \\ = \int_\Omega &(\beta Q_R(\theta^{(n)}) \operatorname{div} u_t^{(n)} + g \cdot u_t^{(n)})\,\mathrm{d}x + \int_{\partial\Omega} \gamma_p(x) p^{(n)}(p^* - p^{(n)})\,\mathrm{d}s(x).\end{aligned} \tag{5.18}$$

Note that $(K_R(p^{(n)}) + f(p^{(n)}))_t p^{(n)} = F_R(p^{(n)})_t$, where $F_R(p)$ is a function bounded below and above by a positive multiple of p^2. Integrating (5.18) in time from 0 to t, using the energy identity (3.27), and neglecting lower order positive terms on the left-hand side, we obtain for all $t \in (0,T)$ the estimate

$$\begin{aligned}&|u_t^{(n)}(t)|_2^2 + \delta|\mathcal{B}u_t^{(n)}(t)|_2^2 + |\nabla_s u^{(n)}(t)|_2^2 + \|\nabla_s u_t^{(n)}\|_2^2 \\ &\quad + |p^{(n)}(t)|_2^2 + \|\nabla p^{(n)}\|_2^2 + \int_0^T \int_{\partial\Omega} \gamma_p(x)|p^{(n)}|^2\,\mathrm{d}s(x)\,\mathrm{d}t \le C_R.\end{aligned} \tag{5.19}$$

5.2. Estimate 2. In (5.15) we choose $\psi = M(p^{(n)})_t$ with M given by (4.6). By (3.24)–(3.25), we have that $G_0[p^{(n)}]_t M(p^{(n)})_t = G_0[p^{(n)}]_t p_t^{(n)} \mu(p^{(n)}) \ge 0$. By Hypothesis 4.1(i) and (iii), we thus have

$$(K_R(p^{(n)}) + G[p^{(n)}]_t) M(p^{(n)})_t \ge \mu_0 \min\left\{1, \frac{f_2}{1+R^2}\right\} |p_t^{(n)}|^2,$$

and we obtain for all $t \in (0,T)$ the estimate

$$\|p_t^{(n)}\|_2^2 + |\nabla p^{(n)}(t)|_2^2 + \int_{\partial\Omega} \gamma_p(x)|p^{(n)}(t)|^2\,\mathrm{d}s(x) \le C_R. \tag{5.20}$$

Using (5.19) and (5.20) in (5.15), together with standard regularity results, we see that

$$\|M(p^{(n)})\|_{L^2(0,T;W^{2,2}(\Omega))} \le C_R. \tag{5.21}$$

5.3. Estimate 3. Choosing $\phi = u_{tt}^{(n)}$ in (5.14) and integrating by parts the term $P[\nabla_s u^{(n)}] : \nabla_s u_{tt}^{(n)}$ yields

$$\begin{aligned} &\int_\Omega \left(\rho_S |u_{tt}^{(n)}|^2 + \delta |\mathcal{B} u_{tt}^{(n)}|^2 + \mathbf{B} \nabla_s u_t^{(n)} : \nabla_s u_{tt}^{(n)} + \frac{\partial}{\partial t}\big(P[\nabla_s u^{(n)}] : \nabla_s u_t^{(n)}\big)\right) \mathrm{d}x \\ &= \int_\Omega \left(P[\nabla_s u^{(n)}]_t : \nabla_s u_t^{(n)} + (\beta Q_R(\theta^{(n)}) - p^{(n)}) \operatorname{div} u_{tt}^{(n)} + g \cdot u_{tt}^{(n)}\right) \mathrm{d}x. \end{aligned} \tag{5.22}$$

We now integrate in time again and use Proposition 3.1(i), estimate (5.20), as well as the Gronwall argument, to conclude for all $t \in (0,T)$ that

$$\|u_{tt}^{(n)}\|_2^2 + \delta \|\mathcal{B} u_{tt}^{(n)}\|_2^2 + |\nabla_s u_t^{(n)}(t)|_2^2 \le C_{R,\delta}. \tag{5.23}$$

5.4. Estimate 4. We choose $\zeta = \theta^{(n)}$ in (5.16). By (3.28)–(3.29), (3.24), and Hypothesis 3.4, there exists a constant $C > 0$ such that

$$|D_G[p^{(n)}]_t| \le C |p_t^{(n)}| \quad \text{a.e.} \tag{5.24}$$

Similarly, by (3.15), (3.7), and Proposition 3.1(i), we have

$$\|D_P[\nabla_s u^{(n)}]_t\|_* \le C |\nabla_s u_t^{(n)}| \quad \text{a.e.} \tag{5.25}$$

The only superlinear term in (5.16) is $\mathbf{B}\nabla_s u_t^{(n)} : \nabla_s u_t^{(n)}$, which has to be estimated in the norm of $L^2(\Omega \times (0,T))$; that is, $\nabla_s u_t^{(n)}$ has to be estimated in $L^4(\Omega \times (0,T))$. This will be done using the Gagliardo–Nirenberg inequality (5.17), which yields for every $t \in (0,T)$ that

$$|\nabla_s u_t^{(n)}(t)|_4 \le C(|\nabla_s u_t^{(n)}(t)|_2 + |\nabla_s u_t^{(n)}(t)|_2^{1/4} |\mathcal{B} u_t^{(n)}(t)|_2^{3/4}) \le C_{R,\delta} \tag{5.26}$$

by virtue of (5.19) and (5.3). We thus obtain

$$|\theta^{(n)}(t)|_2^2 + \|\nabla \theta^{(n)}\|_2^2 + \int_0^T \int_{\partial\Omega} \gamma_\theta(x) |\theta^{(n)}|^2 \, \mathrm{d}s(x) \, \mathrm{d}t \le C_{R,\delta} \tag{5.27}$$

for all $t \in (0,T)$. Finally, let $\zeta \in L^2(0,T;X_2)$ be arbitrary, $\zeta(x,t) = \sum_{j=0}^\infty \zeta_j(t) w_j(x)$. We test (5.16) with $\zeta = \zeta_j(t)$ and obtain, using the previous estimates, that

$$\int_0^T \int_\Omega \theta_t^{(n)} \zeta \, \mathrm{d}x \, \mathrm{d}t \le C_{R,\delta} \|\zeta\|_{L^2(0,T;X_2)} \tag{5.28}$$

or, in other words,

$$\|\theta_t^{(n)}\|_{L^2(0,T;X_2^*)} \le C_{R,\delta}. \tag{5.29}$$

5.5. Passage to the limit as $n \to \infty$. We keep fixed for the moment the regularization parameters R and δ, and let n tend to ∞. By a standard argument based on compact anisotropic embeddings (see [8]), we infer, passing to a subsequence if necessary, that there exist functions (u, p, θ) such that the following convergences take place:

$$
\begin{array}{rcllr}
u_{tt}^{(n)} & \to & u_{tt} & \text{weakly in} & L^2(0,T;W^{2,2}(\Omega;\mathbb{R}^3)\cap X_2^0)\,, \\
\nabla_s u_t^{(n)} & \to & \nabla_s u_t & \text{strongly in} & L^4(\Omega;C([0,T];\mathbb{R}^{3\times 3}_{\mathrm{sym}}))\,, \\
\nabla_s u^{(n)} & \to & \nabla_s u & \text{strongly in} & L^4(\Omega;C([0,T];\mathbb{R}^{3\times 3}_{\mathrm{sym}}))\,, \\
P[\nabla_s u^{(n)}] & \to & P[\nabla_s u] & \text{strongly in} & L^4(\Omega;C([0,T];\mathbb{R}^{3\times 3}_{\mathrm{sym}}))\,, \\
\|D_P[\nabla_s u^{(n)}]_t\|_* & \to & \|D_P[\nabla_s u]_t\|_* & \text{strongly in} & L^2(\Omega;C[0,T])\,, \\
p^{(n)} & \to & p & \text{strongly in} & L^4(\Omega;C[0,T])\,, \\
p_t^{(n)} & \to & p_t & \text{weakly in} & L^2(\Omega\times(0,T))\,, \\
K_R(p^{(n)})_t & \to & K_R(p)_t & \text{weakly in} & L^2(\Omega\times(0,T))\,, \\
G[p^{(n)}]_t & \to & G[p]_t & \text{weakly in} & L^2(\Omega\times(0,T))\,, \\
|D_G[p^{(n)}]_t| & \to & |D_G[p]_t| & \text{weakly in} & L^2(\Omega\times(0,T))\,, \\
\nabla p^{(n)} & \to & \nabla p & \text{strongly in} & L^2(\Omega\times(0,T);\mathbb{R}^3)\,, \\
Q_R(|\nabla p^{(n)}|^2) & \to & Q_R(|\nabla p|^2) & \text{strongly in} & L^2(\Omega\times(0,T))\,, \\
\gamma_p p^{(n)} & \to & \gamma_p p & \text{strongly in} & L^2(\partial\Omega\times[0,T])\,, \\
\theta^{(n)} & \to & \theta & \text{strongly in} & L^2(\Omega\times(0,T))\,, \\
\theta_t^{(n)} & \to & \theta_t & \text{weakly in} & L^2(0,T;X_2^*)\,, \\
\nabla\theta^{(n)} & \to & \nabla\theta & \text{weakly in} & L^2(\Omega\times(0,T);\mathbb{R}^3)\,, \\
\gamma_\theta\theta^{(n)} & \to & \gamma_\theta\theta & \text{strongly in} & L^2(\partial\Omega\times[0,T])\,.
\end{array}
$$

The convergences of the hysteresis terms $P[\nabla_s u^{(n)}]$, $G[p^{(n)}]_t$, $\|D_P[\nabla_s u^{(n)}]_t\|_*$, and $|D_G[p^{(n)}]_t|$ follow indeed from (3.14), (3.12), (3.16), (3.30), and (2.3). We can therefore let n tend to ∞ in (5.4)–(5.6) and conclude that the limit (u,p,θ) satisfies the conditions of Proposition 5.1.

6. Proof of Theorem 4.2. In this section, we show that a sequence of solutions to (5.4)–(5.6) converges to a solution to (4.2)–(4.4) as $R\to\infty$ and $\delta\to 0$. To this end, we fix sequences $\{R_i\},\{\delta_i\}$ for $i\in\mathbb{N}$ such that

$$\lim_{i\to\infty} R_i = \infty, \qquad \lim_{i\to\infty}\delta_i = 0. \tag{6.1}$$

Because of the presence of the highest order term $\delta\mathcal{B}^2 u_{tt}$ in the regularized system, we have to choose a sequence $\{u_i^1\}$ of initial conditions in $X_2^0\cap W^{3,2}(\Omega;\mathbb{R}^3)$ such that

$$\lim_{i\to\infty}\|u_i^1-u^1\|_{X_2^0}=0, \qquad \lim_{i\to\infty}\delta_i\|u_i^1\|^2_{W^{3,2}(\Omega;\mathbb{R}^3)}=0. \tag{6.2}$$

We further denote by $(u^{(i)},p^{(i)},\theta^{(i)})$ the solutions (u,p,θ) to problem (5.4)–(5.6) corresponding to the choice $R=R_i$, $\delta=\delta_i$, $u^1=u_i^1$. The next step is to derive some properties of the sequence $(u^{(i)},p^{(i)},\theta^{(i)})$ independent of i.

6.1. Positivity of temperature. We first observe, using also Korn's inequality, that there exists a constant $C>0$ such that for every nonnegative test function $\zeta\in X_2$ we have by virtue of (5.6) that

$$\int_\Omega\left(c_0\theta_t^{(i)}\zeta+\kappa_{R_i}(\theta^{(i)})\nabla\theta^{(i)}\cdot\nabla\zeta\right)\mathrm{d}x \ge -C\int_\Omega Q_{R_i}(\theta^{(i)})^2\zeta\,\mathrm{d}x+\int_{\partial\Omega}\gamma_\theta(x)(\theta^*-\theta^{(i)})\zeta\,\mathrm{d}s(x). \tag{6.3}$$

Let $v(t)$ be the solution of the ODE

$$c_0\dot v(t) = -Cv^2(t), \qquad v(0)=\bar\theta; \tag{6.4}$$

that is,

$$v(t) = \left(\frac{C}{c_0}t + \frac{1}{\bar{\theta}}\right)^{-1}. \tag{6.5}$$

For every nonnegative test function $\zeta \in X_2$ we have in particular

$$\int_\Omega (c_0 v_t \zeta + \kappa_{R_i}(\theta^{(i)})\nabla v \cdot \nabla \zeta)\,\mathrm{d}x \le -C\int_\Omega v^2\zeta\,\mathrm{d}x + \int_{\partial\Omega} \gamma_\theta(x)(\theta^* - v)\zeta\,\mathrm{d}s(x). \tag{6.6}$$

Note that the boundary term on the right-hand side of (6.6) is positive by the assumption $\theta^* \ge \bar{\theta}$ in Hypothesis 4.1. Subtracting (6.3) from (6.6), we obtain

$$\begin{aligned} &\int_\Omega (c_0(v-\theta^{(i)})_t\zeta + \kappa_{R_i}(\theta^{(i)})\nabla(v-\theta^{(i)})\cdot\nabla\zeta)\,\mathrm{d}x \\ &\quad \le C\int_\Omega (Q^2_{R_i}(\theta^{(i)}) - v^2)\zeta\,\mathrm{d}x + \int_{\partial\Omega} \gamma_\theta(x)(\theta^{(i)} - v)\zeta\,\mathrm{d}s(x). \end{aligned} \tag{6.7}$$

We now choose any smooth convex function $F : \mathbb{R} \to \mathbb{R}$ such that $F(s) = 0$ for $s \le 0$, $F(s) > 0$ for $s > 0$, and we test (6.7) by $\zeta = F'(v - \theta^{(i)})$. We have in all cases

$$(Q^2_{R_i}(\theta^{(i)}) - v^2)F'(v-\theta^{(i)}) \le 0 \quad \text{a.e.},$$

and hence

$$c_0\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega F(v-\theta^{(i)})\,\mathrm{d}x \le 0, \tag{6.8}$$

and we conclude for every $i \in \mathbb{N}$ that

$$\theta^{(i)}(x,t) \ge v(t) \quad \text{a.e.} \tag{6.9}$$

We now pass to a series of estimates independent of i. To simplify the presentation, we occasionally omit the indices $\{\}^{(i)}$ in the computations in subsections 6.2–6.9 below, and write simply (u, p, θ) instead of $(u^{(i)}, p^{(i)}, \theta^{(i)})$ whenever there is no risk of confusion. As before, the symbol C denotes any constant independent of i.

6.2. Estimate 5. Test (5.4) by $\phi = u_t = u_t^{(i)}$, (5.5) by $\psi = p = p^{(i)}$, and (5.6) by $\zeta = 1$. Summing up the three resulting equations, we obtain by virtue of (2.3) that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(c_0\theta + \frac{\rho_S}{2}|u_t|^2 + \frac{\delta_i}{2}|\mathcal{B}u_t|^2 + V_P[\nabla_s u] + V_G[p] + \hat{K}_{R_i}(p)\right)\,\mathrm{d}x \\ &\qquad + \frac{1}{\rho_L}\int_\Omega \mu(p)\big(|\nabla p|^2 - Q_{R_i}(|\nabla p|^2)\big)\,\mathrm{d}x \\ &\quad = \int_\Omega g\cdot u_t\,\mathrm{d}x + \int_{\partial\Omega} (\gamma_p(x)(p^* - p)p + \gamma_\theta(x)(\theta^* - \theta))\,\mathrm{d}s(x), \end{aligned} \tag{6.10}$$

where we set $\hat{K}_R(p) = \int_0^p K_R'(p')p'\,\mathrm{d}p'$ for $p \in \mathbb{R}$ and $R_i > 0$. Integrating in time and using (3.15) and (6.2), we get for every $t \in (0,T)$ the estimate

$$\begin{aligned} &\int_\Omega \left(\theta^{(i)} + |u_t^{(i)}|^2 + |\nabla_s u^{(i)}|^2\right)(x,t)\,\mathrm{d}x \\ &\qquad + \int_0^T\int_{\partial\Omega} \left(\gamma_\theta(x)\theta^{(i)} + \gamma_p(x)|p^{(i)}|^2\right)(x,t)\,\mathrm{d}s(x)\,\mathrm{d}t \le C. \end{aligned} \tag{6.11}$$

6.3. Estimate 6. We set $\hat\theta^{(i)} := Q_{R_i}(\theta^{(i)})$ and test (5.6) by $\zeta = (\hat\theta^{(i)})^{-a}$ with a from Hypothesis 4.1(ii), and observe, omitting the index $^{(i)}$ for simplicity, that

$$
\begin{aligned}
&\int_\Omega \hat\theta^{-a}\left(\|D_P[\nabla_s u]_t\|_* + |D_G[p]_t| + \mathbf{B}\nabla_s u_t{:}\nabla_s u_t + \frac{1}{\rho_L}\mu(p)Q_{R_i}(|\nabla p|^2)\right)\mathrm{d}x\\
&\quad + a\int_\Omega \kappa(\hat\theta)\hat\theta^{-1-a}|\nabla\hat\theta|^2\,\mathrm{d}x \qquad (6.12)\\
&= \beta\int_\Omega Q_{R_i}(\theta)\hat\theta^{-a}\operatorname{div} u_t\,\mathrm{d}x - \int_{\partial\Omega}\gamma_\theta(x)\hat\theta^{-a}(\theta^* - \theta)\,\mathrm{d}s(x) + \frac{c_0}{1-a}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \hat\theta^{1-a}\,\mathrm{d}x.
\end{aligned}
$$

Integrating in time and using (6.9), (6.11), and Hypothesis 4.1(ii), we obtain in particular

$$
\int_0^T\int_\Omega (\hat\theta^{-a}|\operatorname{div} u_t|^2 + |\nabla\hat\theta|^2)\,\mathrm{d}x\,\mathrm{d}t \le C\left(1 + \int_0^T\int_\Omega \hat\theta^{1-a}|\operatorname{div} u_t|\,\mathrm{d}x\,\mathrm{d}t\right). \tag{6.13}
$$

The integral on the right-hand side can be estimated by Hölder's inequality,

$$
\int_0^T\int_\Omega \hat\theta^{1-a}|\operatorname{div} u_t|\,\mathrm{d}x\,\mathrm{d}t \le \left(\int_0^T\int_\Omega \hat\theta^{-a}|\operatorname{div} u_t|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\int_0^T\int_\Omega \hat\theta^{2-a}\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2},
$$

which entails that

$$
\int_0^T\int_\Omega |\nabla\hat\theta|^2\,\mathrm{d}x\,\mathrm{d}t \le C\left(1 + \int_0^T\int_\Omega \hat\theta^{2-a}\,\mathrm{d}x\,\mathrm{d}t\right). \tag{6.14}
$$

Applying the Gagliardo–Nirenberg inequality (5.17) with $s = 1$, $r = 2$, and $q = 2 - a$ and using (6.11), we estimate the right-hand side of (6.14) from above by $C(1 + \|\nabla\hat\theta\|_2^{(1-a)6/5})$. Thus, for $H := \|\nabla\hat\theta\|_2^2$, inequality (6.14) has the form $H \le C(H^\omega + 1)$ with $\omega = (1-a)3/5 < 1$, and $H^\omega \le \omega\delta H + (1-\omega)\delta^{-\omega/(1-\omega)}$. Choosing, for example, $\delta = 1/C$, we obtain

$$
\|\nabla\hat\theta^{(i)}\|_2 \le C. \tag{6.15}
$$

Using (5.17) again with $s = 1$, $r = 2$, and $q = 8/3$, we obtain

$$
\|\hat\theta^{(i)}\|_{8/3} \le C. \tag{6.16}
$$

6.4. Estimate 7. Test (5.4) by $\phi = u_t = u_t^{(i)}$ and (5.5) by $\psi = p = p^{(i)}$. The sum of the two equations yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{\rho_S}{2}|u_t|^2 + \frac{\delta_i}{2}|\mathcal{B}u_t|^2 + \hat K_R(p)\right)\mathrm{d}x\\
&\qquad + \int_\Omega\left((\mathbf{B}\nabla_s u_t + P[\nabla_s u]) : \nabla_s u_t + G[p]_t p + \frac{1}{\rho_L}\mu(p)|\nabla p|^2\right)\mathrm{d}x\\
&= \int_\Omega(\beta Q_{R_i}(\theta)\operatorname{div} u_t + g\cdot u_t)\,\mathrm{d}x + \int_{\partial\Omega}\gamma_p(x)p(p^* - p)\,\mathrm{d}s(x). \qquad (6.17)
\end{aligned}
$$

By (2.3) and (3.29), we have $G[p]_t p \ge V_G[p]_t$, $V_G[p](x,t) \ge 0$, and $V_G[p](x,0) \le C|p(x,0)|^2 \le C$ a.e. Integrating in time, taking into account (6.2) and the previous estimates, we get

$$
\|\nabla_s u_t^{(i)}\|_2 + \|\nabla p^{(i)}\|_2 + \delta_i|\mathcal{B}u_t^{(i)}(t)|_2 + \int_0^T\int_{\partial\Omega}\gamma_p(x)|p^{(i)}|^2\,\mathrm{d}s(x) \le C \tag{6.18}
$$

for all $t \in (0,T)$. Consequently, as γ_p does not identically vanish on $\partial\Omega$ by Hypothesis 4.1, we also have by the Poincaré inequality that

$$\|p^{(i)}\|_{L^2(0,T;W^{1,2}(\Omega))} \le C. \tag{6.19}$$

6.5. Estimate 8. Test (5.4) by $\phi = u_{tt} = u_{tt}^{(i)}$. Integrating by parts the term $P[\nabla_s u] : \nabla_s u_{tt}$ yields

$$\begin{aligned}\int_\Omega (\rho_S |u_{tt}|^2 + \delta_i |\mathcal{B} u_{tt}|^2)\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\mathbf{B}\nabla_s u_t + 2P[\nabla_s u]) : \nabla_s u_t\,\mathrm{d}x \\ \le C\int_\Omega (|g| + |\nabla p| + |\nabla Q_{R_i}(\theta)|)|u_{tt}|\,\mathrm{d}x + \int_\Omega P[\nabla_s u]_t : \nabla_s u_t\,\mathrm{d}x.\end{aligned} \tag{6.20}$$

From Proposition 3.1(i), (6.15), Korn's inequality, and (6.18) it follows that

$$\|u_{tt}^{(i)}\|_2^2 + \delta_i \|\mathcal{B} u_{tt}^{(i)}\|_2^2 + |\nabla_s u_t^{(i)}(t)|_2^2 \le C \tag{6.21}$$

for every $t \in (0,T)$.

6.6. Estimate 9. We rewrite (5.4) in the form

$$\int_\Omega \big((\rho_S u_{tt} + \mathcal{B} u_t)\cdot\phi + \delta_i \mathcal{B} u_{tt}\cdot\mathcal{B}\phi\big)\,\mathrm{d}x = \int_\Omega (f+h)\cdot\phi\,\mathrm{d}x \tag{6.22}$$

for all $\phi \in W^{2,2}(\Omega;\mathbb{R}^3) \cap X_2^0$, where

$$f = g - \beta\nabla Q_{R_i}(\theta) + \nabla p, \qquad h = \operatorname{div} P[\nabla_s u]. \tag{6.23}$$

We have $f \in L^2(\Omega;\mathbb{R}^3)$ by (6.15), (6.18), and Hypothesis 4.1. To estimate h in L^2, we use (3.12) and proceed as follows. Let E_l, $l = 1,2,3$, be the lth coordinate vector, let $(x,t) \in \Omega \times (0,T)$ be an arbitrary Lebesgue point of $\partial_{x_l} P[\nabla_s u]$, and let $s_0 \in \mathbb{R}$ be sufficiently small such that $x + sE_l \in \Omega$ for $|s| < s_0$. By (3.12) and (3.14) we have

$$\begin{aligned}|P[\nabla_s u](x+sE_l,t) - P[\nabla_s u](x,t)| \le C\bigg(&|\nabla_s u^0(x+sE_l) - \nabla_s u^0(x)| \\ &+ \int_0^t |\nabla_s u_t(x+sE_l,\tau) - \nabla_s u_t(x,\tau)|\,\mathrm{d}\tau\bigg),\end{aligned} \tag{6.24}$$

so that in the limit as $s \to 0$ we have

$$\left|\frac{\partial}{\partial x_l} P[\nabla_s u](x,t)\right| \le C\left(\left|\frac{\partial}{\partial x_l}\nabla_s u^0(x)\right| + \int_0^t \left|\frac{\partial}{\partial x_l}\nabla_s u_t(x,\tau)\right|\mathrm{d}\tau\right) \quad \text{a.e.} \tag{6.25}$$

and

$$|h(t)|_2 \le C\left(1 + \int_0^t |\mathcal{B} u_t(\tau)|\,\mathrm{d}\tau\right). \tag{6.26}$$

Consider now the Fourier expansion of $u = u^{(i)}$ in the form

$$u(x,t) = \sum_{k=1}^\infty u_k(t) e_k(x), \tag{6.27}$$

similar to (5.8), with coefficients

$$u_k(t) = \int_\Omega u(t)\cdot e_k(x)\,\mathrm{d}x. \tag{6.28}$$

It follows, e.g., from (6.18) that the series

$$\mathcal{B}u_t(x,t) = \sum_{k=1}^{\infty} \lambda_k \dot u_k(t) e_k(x) \tag{6.29}$$

is strongly convergent in $L^2(\Omega;\mathbb{R}^3)$.

We now test (6.22) by $\phi = \mathcal{B}u_t^{(n)}$, where $u^{(n)}$ is as in (5.8) with coefficients $u_k(t)$ given by (6.28). Then

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{\rho_S}{2}\mathbf{B}\nabla_s u_t^{(n)} : \nabla_s u_t^{(n)} + \frac{\delta_i}{2}\mathbf{B}\nabla_s\mathcal{B}u_t^{(n)} : \nabla_s\mathcal{B}u_t^{(n)}\right)\mathrm{d}x + |\mathcal{B}u_t^{(n)}(t)|_2^2\\ &\quad\le C\left(1+|f(t)|_2 + \int_0^t |\mathcal{B}u_t(\tau)|_2\,\mathrm{d}\tau\right)|\mathcal{B}u_t^{(n)}(t)|_2,\end{aligned} \tag{6.30}$$

and hence,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{\rho_S}{2}\mathbf{B}\nabla_s u_t^{(n)} : \nabla_s u_t^{(n)} + \frac{\delta_i}{2}\mathbf{B}\nabla_s\mathcal{B}u_t^{(n)} : \nabla_s\mathcal{B}u_t^{(n)}\right)\mathrm{d}x + |\mathcal{B}u_t^{(n)}(t)|_2^2\\ &\quad\le C\left(1+|f(t)|_2^2 + \int_0^t |\mathcal{B}u_t(\tau)|_2^2\,\mathrm{d}\tau\right).\end{aligned} \tag{6.31}$$

By (6.2), we can integrate this inequality from 0 to t, pass to the limit as $n\to\infty$, and use Gronwall's argument to obtain in particular that

$$\|\mathcal{B}u_t\|_2 = \|\mathcal{B}u_t^{(i)}\|_2 \le C. \tag{6.32}$$

The next computation based on (5.17) is to check that

$$\|u_t^{(i)}\|_{L^r(0,T;C(\bar\Omega;\mathbb{R}^3))} \le C \quad \text{for every } r\in[1,4). \tag{6.33}$$

Indeed, we choose any $\alpha\in[0,1/6)$ and set $\frac{1}{q} = \frac{1}{3}-\alpha$. By (6.21), (6.32), and (5.17) we have

$$|\partial_{x_j}u_t^{(i)}(t)|_q \le C\left(|\partial_{x_j}u_t^{(i)}(t)|_2 + |\partial_{x_j}u_t^{(i)}(t)|_2^{1-\gamma}|\partial_{x_j}\nabla u_t^{(i)}(t)|_2^{\gamma}\right), \qquad \gamma = \frac{\frac12-\frac1q}{\frac13}. \tag{6.34}$$

Then $\partial_{x_j}u_t^{(i)}$ is bounded in $L^p(0,T;L^q(\Omega;\mathbb{R}^3))$ for $p\gamma=2$; that is,

$$\left|\partial_{x_j}u_t^{(i)}\right|_{L^p(0,T;L^q(\Omega;\mathbb{R}^3))} \le C \quad \text{for } q = \frac{3}{1-3\alpha},\ p = \frac{4}{6\alpha+1}. \tag{6.35}$$

By (6.21), $u_t^{(i)}$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$, and by (5.17) for $\alpha>0$ we have

$$|u_t^{(i)}(t)|_\infty \le C\left(|u_t^{(i)}(t)|_2 + |u_t^{(i)}(t)|_2^{1-\hat\gamma}|\nabla u_t^{(i)}(t)|_q^{\hat\gamma}\right), \qquad \hat\gamma = \frac{\frac12}{\frac56-\frac1q}. \tag{6.36}$$

We then obtain (6.33) for $r\hat\gamma = p$; that is,

$$r = 4\frac{1+2\alpha}{1+6\alpha}\in[1,4). \tag{6.37}$$

6.7. Estimate 10. In this subsection, we prove the following statement.

PROPOSITION 6.1. *Let Hypothesis* 4.1 *hold. Then there exists a constant* $C^* > 0$ *such that*

$$|p^{(i)}(x,t)| \le C^* \quad a.e.$$

Note that, by (6.33), we have

$$U \in L^3(0,T), \quad \text{where we set } \; U(t) := 1 + \sup_{x\in\Omega} |u_t^{(i)}(x,t)| \,. \tag{6.38}$$

As a preliminary step before we pass to the proof of Proposition 6.1, we prove the following auxiliary result for $p = p^{(i)}$.

LEMMA 6.2. *There exist constants* $c > 0$ *and* $C > 0$ *independent of* m *such that for every* $m \ge 1$ *and every* $t \in [0,T]$ *we have*

$$\begin{aligned} &\int_\Omega |p(x,t)|^{2m}\,\mathrm{d}x + c\int_0^t\int_\Omega |\nabla(p|p|^{m-1})|^2\,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad \le C(1+m^2)\int_0^t U^2(\tau)\left(K^{2m} + \int_\Omega |p(x,\tau)|^{2m}\,\mathrm{d}x\right)\mathrm{d}\tau. \end{aligned} \tag{6.39}$$

Proof. We choose an arbitrary $Q > 0$ and $m \ge 1$, and test (5.5) by $h_{Q,m}(p)$, where we set

$$h_{Q,m}(p) = \begin{cases} p|p|^{2m} & \text{for } \; |p| < Q, \\ Q^{2m+1} + (2m+1)(p-Q)Q^{2m} & \text{for } \; p \ge Q, \\ -Q^{2m+1} + (2m+1)(p+Q)Q^{2m} & \text{for } \; p \le -Q. \end{cases} \tag{6.40}$$

We have $h_{Q,m}(p) \in L^2(0,T;W^{1,2}(\Omega))$ by virtue of (6.19); hence this is an admissible test function. By (3.35) we have $G_0[p]_t h_{Q,m}(p) \ge V_{h_{Q,m}}[p]_t$ and $K_R(p)p_t h_{Q,m}(p) = \hat{K}_{R,Q,m}(p)_t$, with $\hat{K}_{R,Q,m}(p) = \int_0^p K_R(p')h_{Q,m}(p')\,\mathrm{d}p'$. We thus have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\hat{K}_{R,Q,m}(p) + V_{h_{Q,m}}[p])\,\mathrm{d}x \\ &\qquad + \int_\Omega f'(p)h_{Q,m}(p)p_t\,\mathrm{d}x + \mu_0(2m+1)\int_\Omega |\nabla p|^2\min\{|p|,Q\}^{2m}\,\mathrm{d}x \\ &\le -(2m+1)\int_\Omega u_t\min\{|p|,Q\}^{2m}\nabla p\,\mathrm{d}x + \int_{\partial\Omega}\gamma(x)(p^*-p)h_{Q,m}(p)\,\mathrm{d}s(x), \end{aligned} \tag{6.41}$$

together with $\hat{K}_{R,Q,m}(p)(x,0) = 0$ by assumption $|p^0(x)| \le K$ in Hypothesis 4.1, and

$$\int_\Omega V_{h_{Q,m}}[p](x,0)\,\mathrm{d}x \le CK^{2m} \tag{6.42}$$

as a consequence of (3.36) and (3.21), with C independent of Q and m. We estimate the right-hand side of (6.41) as follows:

$$\begin{aligned} &-(2m+1)\int_\Omega u_t\min\{|p|,Q\}^{2m}\nabla p\,\mathrm{d}x \\ &\le (2m+1)U(t)\left(\int_\Omega \min\{|p|,Q\}^{2m}\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega |\nabla p|^2\min\{|p|,Q\}^{2m}\,\mathrm{d}x\right)^{1/2} \\ &\le \frac{\mu_0}{2}(2m+1)\int_\Omega |\nabla p|^2\min\{|p|,Q\}^{2m}\,\mathrm{d}x + \frac{2m+1}{2\mu_0}U^2(t)\int_\Omega \min\{|p|,Q\}^{2m}\,\mathrm{d}x. \end{aligned}$$

For the boundary term we have

$$\int_{\partial\Omega} \gamma(x)(p^*-p)h_{Q,m}(p)\,\mathrm{d}s(x) \le \frac{1}{2m+2}\int_{\partial\Omega}\gamma(x)\left(|p^*|^{2m+2} - \min\{|p|,Q\}^{2m+2}\right)\,\mathrm{d}s(x).$$

On the left-hand side of (6.41) we have

$$\int_0^t f'(p)h_{Q,m}(p)p_t(x,\tau)\,\mathrm{d}\tau = F_{Q,m}(p(x,t)) - F_{Q,m}(p(x,0)),$$

where we set $F_{Q,m}(p) = \int_0^p f'(z)h_{Q,m}(z)\,\mathrm{d}z$. We claim that for every $p\in\mathbb{R}$ we have

$$\frac{f_3}{2m+2}|p|^{2m} \ge F_{Q,m}(p) \ge \frac{f_2}{4m}(\min\{|p|,Q\}^{2m}-1). \tag{6.43}$$

The upper bound is easy. We have for $z>0$ that $f'(z)h_{Q,m}(z)\le f_3 z|z|^{2m}$ and similarly for $z<0$, and it suffices to integrate. To get the lower bound, set

$$\hat F_{Q,m}(p) = F_{Q,m}(p) - \frac{f_2\min\{|p|,Q\}^{2m}}{4m}.$$

Then for $p>Q$ we have $\hat F'_{Q,m}(p) = f'(p)h_{Q,m}(p) > 0$. For $p\in(0,Q)$ we have

$$\hat F'_{Q,m}(p) = f'(p)p|p|^{2m} - \frac{f_2 p|p|^{2m-1}}{2} \ge \frac{f_2 p|p|^{2m-1}}{2(1+p^2)}(p^2-1),$$

and hence the minimum of $\hat F_{Q,m}(p)$ is attained at $p=1$, with $\hat F_{Q,m}(1)\ge -\frac{f_2}{4m}$, which is exactly (6.43). The case $p<0$ is symmetric.

Summarizing the above estimates, we obtain by integrating (6.41) from 0 to t that

$$\begin{aligned}
&\int_\Omega \min\{|p|,Q\}^{2m}(x,t)\,\mathrm{d}x + \frac{\mu_0}{f_2}2m(2m+1)\int_0^t\int_\Omega |\nabla p|^2\min\{|p|,Q\}^{2m}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad + \frac{2m}{f_2(m+1)}\int_{\partial\Omega}\gamma(x)\min\{|p|,Q\}^{2m+2}\,\mathrm{d}s(x)\\
&\le C(1+m^2)\int_0^t U^2(\tau)\left(K^{2m} + \int_\Omega \min\{|p|,Q\}^{2m}(x,\tau)\,\mathrm{d}x\right)\mathrm{d}\tau.
\end{aligned} \tag{6.44}$$

In particular, the function $w_m(t) := \int_\Omega \min\{|p|,Q\}^{2m}(x,t)\,\mathrm{d}x$ satisfies the inequality

$$w_m(t) \le C(1+m^2)\int_0^t U^2(\tau)\left(K^{2m} + w_m(\tau)\right)\mathrm{d}\tau,$$

and by Gronwall's argument (note that $U^2\in L^1(0,T)$ by (6.38)) there exists a constant $C(m)$ depending on m and independent of Q such that $\sup_{t\in[0,T]} w_m(t)\le C(m)$. Hence, we can let Q tend to ∞ in (6.44) and obtain

$$\begin{aligned}
&\int_\Omega |p|^{2m}(x,t)\,\mathrm{d}x + \frac{\mu_0}{f_2}2m(2m+1)\int_0^t\int_\Omega |\nabla p|^2|p|^{2m}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad + \frac{2m}{f_2(m+1)}\int_{\partial\Omega}\gamma(x)|p|^{2m+2}\,\mathrm{d}s(x)\\
&\le C(1+m^2)\int_0^t U^2(\tau)\left(K^{2m} + \int_\Omega |p|^{2m}(x,\tau)\,\mathrm{d}x\right)\mathrm{d}\tau.
\end{aligned} \tag{6.45}$$

In particular,

$$p \in L^\infty(0,T;L^q(\Omega)) \quad \forall q \geq 1, \tag{6.46}$$

but the norm of p in this space still depends on q. Note that

$$\begin{aligned}\int_0^t \int_\Omega |\nabla p|^2 |p|^{2m} \,\mathrm{d}x\,\mathrm{d}\tau &\geq \int_0^t \int_\Omega |\nabla p|^2 |p|^{2m-2} \,\mathrm{d}x\,\mathrm{d}\tau - \int_0^t \int_\Omega |\nabla p|^2 \,\mathrm{d}x\,\mathrm{d}\tau \\ &\geq \int_0^t \int_\Omega |\nabla p|^2 |p|^{2m-2} \,\mathrm{d}x\,\mathrm{d}\tau - C \end{aligned} \tag{6.47}$$

by virtue of (6.18). Indeed, it suffices to split the integration domain into the parts where $p \geq 1$ and $p < 1$. Using (6.47), we rewrite (6.44) as

$$\begin{aligned}&\int_\Omega |p(x,t)|^{2m} \,\mathrm{d}x + \left(\frac{\mu_0}{f_2}\right) \frac{2(2m+1)}{m} \int_0^t \int_\Omega |\nabla(p|p|^{m-1})|^2 \,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad + \frac{2m}{f_2(m+1)} \int_{\partial\Omega} \gamma(x)|p|^{2m+2} \,\mathrm{d}s(x) \\ &\leq C(1+m^2) \int_0^t U^2(\tau) \left(K^{2m} + \int_\Omega |p(x,\tau)|^{2m} \,\mathrm{d}x\right) \mathrm{d}\tau. \end{aligned} \tag{6.48}$$

Setting $c := 4\mu_0/f_2$, we obtain (6.39), and Lemma 6.2 is proved. ∎

Proof of Proposition 6.1. Set $w := p|p|^{m-1}$ for $p = p^{(i)}$. Then (6.39) reads

$$|w(t)|_2^2 + c\int_0^t |\nabla w(\tau)|_2^2 \,\mathrm{d}\tau \leq C(1+m^2) \int_0^t U^2(\tau) \left(K^{2m} + |w(\tau)|_2^2\right) \mathrm{d}\tau. \tag{6.49}$$

We now choose $s = 3/4$ and invoke the Gagliardo–Nirenberg inequality (5.17) in the form

$$|w(\tau)|_2 \leq C \left(|w(\tau)|_{2s} + |w(\tau)|_{2s}^{1-\gamma} |\nabla w(\tau)|_2^\gamma\right), \qquad \gamma = \frac{\frac{1}{2s} - \frac{1}{2}}{\frac{1}{2s} - \frac{1}{6}} = \frac{1}{3}.$$

Hölder's inequality enables us to rewrite (6.49) as

$$\begin{aligned}&|w(t)|_2^2 + c\int_0^t |\nabla w(\tau)|_2^2 \,\mathrm{d}\tau \\ &\quad \leq C(1+m^2) \int_0^t U^2(\tau) \left(K^{2m} + |w(\tau)|_{2s}^2\right) \mathrm{d}\tau \\ &\qquad + C(1+m^2) \int_0^t U^2(\tau) |w(\tau)|_{2s}^{4/3} |\nabla w(\tau)|_2^{2/3} \,\mathrm{d}\tau \\ &\quad \leq C(1+m^2) \int_0^t U^2(\tau) \left(K^{2m} + |w(\tau)|_{2s}^2\right) \mathrm{d}\tau \\ &\qquad + C(1+m^2) \left(\int_0^t U^3(\tau) |w(\tau)|_{2s}^2 \,\mathrm{d}\tau\right)^{2/3} \left(\int_0^t |\nabla w(\tau)|_2^2 \,\mathrm{d}\tau\right)^{1/3}, \end{aligned}$$

and we conclude that

$$|w(t)|_2^2 \leq C(1+m^3) \int_0^t U^3(\tau) \left(K^{2m} + |w(\tau)|_{2s}^2\right) \mathrm{d}\tau \tag{6.50}$$

or, in terms of p,

$$|p(t)|_{2m}^{2m} \le C(1+m^3)\int_0^t U^3(\tau)\left(K^{2m} + |p(\tau)|_{2sm}^{2m}\right) \mathrm{d}\tau. \tag{6.51}$$

We have $U \in L^3(0,T)$ by (6.38). If now V is a constant such that $\max\{K, |p(\tau)|_{2sm}\} \le V$, then

$$\tilde{V} := \max\{K, |p(t)|_{2m}\} \le \left(C(1+m^3)\right)^{1/2m} V. \tag{6.52}$$

We now define $m_k := s^{-k} = (4/3)^k$ and set

$$V_k := \max\left\{K, \sup_{\tau\in(0,T)} |p(\tau)|_{2m_k}\right\}.$$

By (6.52) we have for all $k = 2, 3, \dots$ that

$$V_k \le \left(C\left(1+\left(\frac{4}{3}\right)^{3k}\right)\right)^{((3/4)^k)/2} V_{k-1}, \tag{6.53}$$

and hence

$$\log V_k - \log V_{k-1} \le \frac{1}{2}\left(\frac{3}{4}\right)^k \log\left(C\left(1+\left(\frac{4}{3}\right)^{3k}\right)\right). \tag{6.54}$$

The right-hand side of (6.54) is a convergent series, and V_1 is finite by virtue of (6.46), so we can conclude that the sequence $\{V_k\}$ is uniformly bounded independently of i, which is what we wanted to prove. ∎

6.8. Estimate 11. Test (5.5) by $\psi = M(p)_t$, with $M(p)$ given by (4.6). From (3.24) it follows that $(K_R(p) + G[p])_t M(p)_t \ge f'(p)\mu(p)p_t^2$. We have $\mu(p) \ge \mu_0 > 0$ by Hypothesis 4.1(i). Furthermore, by Proposition 6.1 we have $|p| \le C^*$; hence $f'(p) \ge f_2/(1+(C^*)^2) > 0$ by Hypothesis 4.1(iii). We conclude that there exists a constant $c > 0$ such that for every $t \in (0,T)$ we have

$$\begin{aligned} c\int_\Omega p_t^2(x,t)\,\mathrm{d}x &+ \frac{1}{2\rho_L}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla M(p)|^2(x,t)\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega}\gamma_p(x)\hat{M}(p)(x,t)\,\mathrm{d}s(x)\\ &\le \int_\Omega (|\operatorname{div} u_t||M(p)_t|)(x,t)\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega}\gamma_p(x)(M(p)p^*)(x,t)\,\mathrm{d}s(x)\\ &\quad - \int_{\partial\Omega}\gamma_p(x)(M(p)p_t^*)(x,t)\,\mathrm{d}s(x) \end{aligned} \tag{6.55}$$

with $\hat{M}(p) = \int_0^p p'\mu(p')\,\mathrm{d}p'$. We integrate in time from 0 to t. By Hypothesis 4.1, p^* and p_t^* are bounded functions. Since $\hat{M}$ is bounded below by a multiple of p^2 and M has linear growth, we can estimate the boundary terms by Hölder's inequality, e.g.,

$$\left|\int_{\partial\Omega}\gamma_p(x)(M(p)p^*)\,\mathrm{d}s(x)\right| \le C\left(\int_{\partial\Omega}\gamma_p(x)\hat{M}(p)\,\mathrm{d}s(x)\right)^{1/2}\left(\int_{\partial\Omega}\gamma_p(x)|p^*|^2\,\mathrm{d}s(x)\right)^{1/2},$$

so that the boundary term on the left-hand side of (6.55) is dominant. Hence, using also (6.21), we have for all $t \in (0,T)$ that

$$\|p_t^{(i)}\|_2^2 + |\nabla p^{(i)}(t)|_2^2 + \int_{\partial\Omega}\gamma_p(x)|p^{(i)}|^2(x,t)\,\mathrm{d}s(x) \le C. \tag{6.56}$$

The term $G[p]_t$ is of the order of p_t by (3.24)–(3.25). By comparison, in (5.5), we thus obtain that

$$\|M(p^{(i)})\|_{L^2(0,T;W^{2,2}(\Omega))} \leq C. \tag{6.57}$$

6.9. Estimate 12. We have $\nabla M(p) = \mu(p)\nabla p$, $M(p)_t = \mu(p)p_t$, so that $\nabla M(p)$ and ∇p are integrable with the same powers. By the same argument based on (6.56)–(6.57), we have similarly as in (6.35) that

$$\|\nabla p^{(i)}\|_{L^p(0,T;L^q(\Omega))} \leq C \quad \text{for } q = \frac{3}{1-3\alpha},\ p = \frac{4}{6\alpha+1} \tag{6.58}$$

for every $\alpha \in [0, 1/6)$. In particular, for $\alpha = 1/30$, we have $\partial_{x_i} u_t, \partial_{x_i} p \in L^{10/3}(\Omega \times (0,T))$. The two hysteresis dissipation terms are estimated as in (5.24)–(5.25). As in Estimate 6, we set $\hat\theta := Q_{R_i}(\theta)$ and rewrite (5.6) in the form

$$\int_\Omega \left(c_0\theta_t\zeta + \kappa(\hat\theta)\nabla\theta\cdot\nabla\zeta\right) \mathrm{d}x + \int_{\partial\Omega} \gamma_\theta(x)(\theta-\theta^*)\zeta \,\mathrm{d}s(x) = \int_\Omega (A(x,t) + B(x,t)\hat\theta)\zeta \,\mathrm{d}x, \tag{6.59}$$

with $A \in L^{5/3}(\Omega\times(0,T))$, $B \in L^{10/3}(\Omega\times(0,T))$ bounded independently of i. We test (6.59) with $\zeta = \hat\theta^r$ (with r to be chosen later) and obtain, using Hölder's inequality for every $t \in (0,T)$, that

$$\begin{aligned} &\frac{1}{r+1}\int_\Omega \hat\theta^{r+1}(x,t)\,\mathrm{d}x + r\int_0^T\int_\Omega \hat\theta^{r-1}\kappa(\hat\theta)|\nabla\hat\theta|^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad + \int_0^T\int_{\partial\Omega}\gamma_\theta(x)\hat\theta^{r+1}(x,\tau)\,\mathrm{d}s(x)\,\mathrm{d}\tau \\ &\quad \leq C\left(1 + \|\hat\theta\|_{5r/2}^r + \|\hat\theta\|_{10(r+1)/7}^{r+1}\right) \end{aligned} \tag{6.60}$$

with, by Hypothesis 4.1(ii),

$$\hat\theta^{r-1}\kappa(\hat\theta)|\nabla\hat\theta|^2 \geq \frac{1}{C}\hat\theta^{r+a}|\nabla\hat\theta|^2. \tag{6.61}$$

We already have estimate (6.16). Assume that for some $z \geq 8/3$ we have proved

$$\|\hat\theta\|_z \leq C_0(z) \tag{6.62}$$

with some $C_0(z) > 0$. For this value of z we choose in (6.60)

$$r = r(z) = \begin{cases} (7z/10) - 1 & \text{for } z \in [8/3, 10/3], \\ 2z/5 & \text{for } z > 10/3. \end{cases} \tag{6.63}$$

Then $\|\hat\theta\|_{5r/2}^r + \|\hat\theta\|_{10(r+1)/7}^{r+1} \leq C(1 + \|\hat\theta\|_z^{r+1})$, and we have by virtue of (6.60)–(6.61) that

$$\frac{1}{r+1}\int_\Omega \hat\theta^{r+1}(x,t)\,\mathrm{d}x + r\int_0^T\int_\Omega \hat\theta^{r+a}|\nabla\hat\theta|^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \leq C\left(1 + C_0(z)^{r+1}\right). \tag{6.64}$$

Set

$$p = \frac{r+a}{2} + 1, \quad s = \frac{r+1}{p}, \quad w = \hat\theta^p. \tag{6.65}$$

Then (6.64) can be written as

$$\frac{1}{r+1}\int_\Omega w^s(x,t)\,\mathrm{d}x + r\int_0^T\int_\Omega |\nabla w|^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \le C\left(1+C_0(z)^{r+1}\right). \tag{6.66}$$

For $s<q<6$ we have by virtue of the Gagliardo–Nirenberg inequality (5.17) that

$$|w(\tau)|_q \le C\left(|w(\tau)|_s + |w(\tau)|_s^{1-\gamma}|\nabla w(\tau)|_2^\gamma\right), \qquad \gamma = \frac{\frac{1}{s}-\frac{1}{q}}{\frac{1}{s}-\frac{1}{6}}. \tag{6.67}$$

If q is chosen in such a way that $q\gamma = 2$, that is,

$$q = \frac{2}{3}s + 2, \tag{6.68}$$

then it follows from (6.67) and Young's inequality that

$$\begin{aligned}\|w\|_q &\le C\left(\sup_{\tau\in[0,T]}|w(\tau)|_s + \sup_{\tau\in[0,T]}|w(\tau)|_s^{1-\gamma}\|\nabla w\|_2^{2/q}\right)\\ &\le C\left(\sup_{\tau\in[0,T]}|w(\tau)|_s + \|\nabla w\|_2\right).\end{aligned} \tag{6.69}$$

By (6.66), we have

$$\begin{aligned}\sup_{\tau\in[0,T]}|w(\tau)|_s &\le C\left((1+r)\big(1+C_0(z)^{r+1}\big)\right)^{1/s},\\ \|\nabla w\|_2 &\le C\big(1+C_0(z)^{r+1}\big)^{1/2} \le C\big(1+C_0(z)^{r+1}\big)^{1/s}\end{aligned}$$

(note that $s<2$), so that

$$\|w\|_q \le C\left((1+r)\big(1+C_0(z)^{r+1}\big)\right)^{1/s}. \tag{6.70}$$

Setting

$$\hat z = pq = \frac{5}{3}r(z) + \frac{8}{3} + a,$$

we have by (6.70) that

$$\|\hat\theta\|_{\hat z} = \|w\|_q^{1/p} \le C\left((1+r)\big(1+C_0(z)^{r+1}\big)\right)^{1/(ps)}, \tag{6.71}$$

and using the identity $ps = r+1$, we obtain the implication

$$\|\hat\theta\|_z \le C_0(z) \Longrightarrow \|\hat\theta\|_{\hat z} \le \hat C_0(z), \qquad \hat C_0(z) = C(1+C_0(z)). \tag{6.72}$$

The sequence

$$z_k = \frac{5}{3}r(z_{k-1}) + \frac{8}{3} + a, \qquad z_0 = \frac{8}{3},$$

converges to $z_\infty = 8+3a$. After finitely many iterations we obtain

$$\|\hat\theta\|_z \le C \quad \text{for every } z < 8+3a. \tag{6.73}$$

Consequently, by (6.60)–(6.64), we have for $t \in (0,T)$ that

(6.74)
$$\int_\Omega \hat\theta^{r+1}(x,t)\,\mathrm{d}x + r\int_0^T\int_\Omega \hat\theta^{r-1}\kappa(\hat\theta)|\nabla\hat\theta|^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^T\int_{\partial\Omega}\gamma_\theta\hat\theta^{r+1}(x,\tau)\,\mathrm{d}s(x)\,\mathrm{d}\tau \le C$$

whenever the inequality

$$r < \frac{16}{5} + \frac{6}{5}a \tag{6.75}$$

holds. We now come back to (6.59), which we test by θ and obtain, using the estimate (6.74), that

(6.76)
$$\int_\Omega \theta^2(x,t)\,\mathrm{d}\,x + \int_0^T\int_\Omega \kappa(\hat\theta)|\nabla\theta|^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^T\int_{\partial\Omega}\gamma_\theta\theta^2(x,\tau)\,\mathrm{d}s(x)\,\mathrm{d}\tau \le C.$$

This enables us to derive an upper bound for the integral $\int_\Omega \kappa(\hat\theta)\nabla\theta\cdot\nabla\zeta\,\mathrm{d}x$, which we need for getting an estimate for θ_t from (6.59). We have by Hölder's inequality and Hypothesis 4.1(ii) that

$$\begin{aligned}\int_\Omega |\kappa(\hat\theta)\nabla\theta\cdot\nabla\zeta|\,\mathrm{d}x &= \int_\Omega |\kappa^{1/2}(\hat\theta)\nabla\theta\cdot\kappa^{1/2}(\hat\theta)\nabla\zeta|\,\mathrm{d}x\\ &\le C\left(\int_\Omega \kappa(\hat\theta)|\nabla\theta|^2\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega \hat\theta^{1+b}|\nabla\zeta|^2\,\mathrm{d}x\right)^{1/2}.\end{aligned}\tag{6.77}$$

We now choose $\hat q > 1$ and r satisfying (6.75) such that $(1+b)\hat q = r$. This is indeed possible by Hypothesis 4.1(ii). Choosing now

$$q^* = \frac{2\hat q}{\hat q - 1}, \tag{6.78}$$

we obtain from Hölder's inequality that

$$\int_\Omega \hat\theta^{1+b}|\nabla\zeta|^2\,\mathrm{d}x \le \left(\int_\Omega \hat\theta^{1+r}dx\right)^{1/\hat q}\left(\int_\Omega |\nabla\zeta|^{q^*}\right)^{2/q^*} \le C\left(\int_\Omega |\nabla\zeta|^{q^*}\right)^{2/q^*} \tag{6.79}$$

by virtue of (6.74). Equation (6.77) then yields

$$\int_\Omega |\kappa(\hat\theta)\nabla\theta\cdot\nabla\zeta|\,\mathrm{d}x \le C\left(\int_\Omega \kappa(\hat\theta)|\nabla\theta|^2\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega |\nabla\zeta|^{q^*}\,\mathrm{d}x\right)^{1/q^*}. \tag{6.80}$$

Hence, by (6.76),

$$\int_0^T\int_\Omega |\kappa_{R_i}(\theta^{(i)})\nabla\theta^{(i)}\cdot\nabla\zeta|\,\mathrm{d}x\,\mathrm{d}t \le C\|\zeta\|_{L^2(0,T;W^{1,q^*}(\Omega))}. \tag{6.81}$$

By (6.35) and (6.58) for $\alpha = 0$, the functions A and B in (6.59) satisfy uniform bounds $A \in L^2(0,T;L^{3/2}(\Omega))$, $B \in L^4(0,T;L^3(\Omega))$ independent of i, so that testing with $\zeta \in L^2(0,T;W^{1,q^*}(\Omega))$ is admissible. We thus obtain from (6.59) that

$$\int_0^T\int_\Omega \theta_t^{(i)}\zeta\,\mathrm{d}x\,\mathrm{d}t \le C\|\zeta\|_{L^2(0,T;W^{1,q^*}(\Omega))}. \tag{6.82}$$

6.10. Passage to the limit as $i \to \infty$. In the system (5.4)–(5.6) with $\delta = \delta_i$, $R = R_i$, and $(u, p, \theta) = (u^{(i)}, p^{(i)}, \theta^{(i)})$, we fix test functions $\phi \in W^{2,2}(\Omega; \mathbb{R}^3) \cap X_2^0$, $\psi \in X_2$, and $\zeta \in L^2(0, T; W^{1,q^*}(\Omega))$ with q^* from (6.81). The term $\delta_i \mathcal{B} u_{tt}^{(i)}$ in (5.4) converges to 0 in L^2 by (6.21), and the regularization $K_{R_i}(p)$ vanishes for $R_i > C^*$ by Proposition 6.1.

By (6.21) and (6.32), the sequence $\{\nabla_s u_t^{(i)}\}$ is precompact in $L^4(0, T; L^q(\Omega; \mathbb{R}^{3\times 3}_{\mathrm{sym}}))$ for every $q \in [1, 3)$. Similarly, by (6.56)–(6.57), $\{\nabla p^{(i)}\}$ is precompact in $L^4(0, T; L^q(\Omega; \mathbb{R}^3))$ for every $q \in [1, 3)$, and $\{p^{(i)}\}$ is precompact in $L^q(\Omega; C[0, T])$ for every $q > 1$ by virtue of Proposition 6.1. Finally, by (6.74), (6.76), and (6.82), both $\{\theta^{(i)}\}$ and $\{\hat{\theta}^{(i)}\}$ are precompact in $L^q(\Omega \times (0, T))$ for suitable q. Hence, we can select a subsequence and pass to the weak limit in the linear terms in (5.4)–(5.6), and to the strong limit in all nonlinear nonhysteretic terms. Obviously, the limits of $\theta^{(i)}$ and $\hat{\theta}^{(i)}$ coincide, and if $|\nabla p^{(i)}|^2 \to |\nabla p|^2$ in $L^2(0, T; L^{q/2}(\Omega))$ strongly, then $Q_{R_i}(|\nabla p^{(i)}|^2) \to |\nabla p|^2$ in $L^2(0, T; L^{q/2}(\Omega))$ strongly. By the same argument as in subsection 5.5, we show that the hysteresis terms $G[p^{(i)}]_t$, $\|D_P[\nabla_s u^{(i)}]_t\|_*$, $|D_G[p^{(i)}]_t|$ converge weakly in $L^2(\Omega \times (0, T))$, and that the limit as $i \to \infty$ yields a solution to (4.2)–(4.4) with $\phi \in W^{2,2}(\Omega; \mathbb{R}^3) \cap X_2^0$. By density we conclude that $\phi \in X_2^0$ is an admissible test function, which completes the proof of Theorem 4.2.

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